SHREWSBURY SCHOOL

Mathematics Prize 1969

Attempt as many questions as you can, in any order. More credit will be given for complete questions than for a large number of fragments.

1. Prove that it is not possible tof ind a prime number n such that $n^2 + 16$ and $n^2 + 24$ are both prime numbers.
   (Hint: Consider remainders when divided by 5.)

2. Prove thatif three lines in three-dimensional space are such that any pair intersect, then the three lines either all pass through the same point or all lie in the same plane.

   Deduce the same property for four lines in three-dimensional space which are such that any pair intersect.

3. n brushers leave a party in no fit state to recognise their own overcoats. Let $t(n)$ be the number of ways in which they can all take the wrong overcoat. Show that

   $t(1) = 0$, $t(2) = 1$ and find $t(3)$ and $t(4)$.

   Prove that $t(5) = 5.4.3.2.1 - 5t(4) - 10t(3) - 10t(2) - 1$

   and hence find $t(5)$. Deduce that the probability, if there are 5 brushers, that none will take his own overcoat is 11/30.

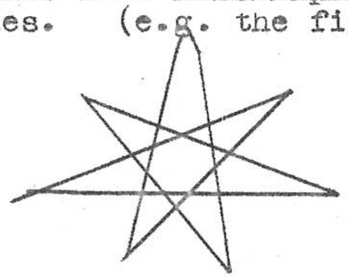
4. On Monday morning I said to a certain mathematics set:-
   'On some day this week, on or before Saturday, you will have a slip, but you won't know which day it is until the actual day of the slip. The slip will be during school hours.'
   On which day was the slip? You must explain your answer - carefully - no credit will be given for a guess.

5. Prove that any odd prime p may be expressed in one and only one way as the difference of two squares $m^2 - n^2$ where m and n are positive integers.

6. In a football league a team receives 2 points for a win, 1 point for a draw and no points if defeated. Seaside Rovers have gained 23 points and have won half as many games again as they have lost. If they have won more than a third of their games played, how many games have they played and how many games have they drawn?

7. A beetle crawls along the edges of a cube starting and finishing at a given vertex and traces a route such that he moves along no edge more than once, and misses out 4 edges. How many different such routes can he choose from? (No jumping or flying is allowed.)

8. If p and q are integers with no common factor, a regular polygon of p/q sides is obtained by drawing p equal lines in such a way that we go round a point in the centre q times in a continuous description of the p sides. (e.g. the figures show polygons of $5/2$ and $7/3$ sides.)



Prove that each angle at a vertex is $(2 - 4/n)$ right angles, where $n = p/q$, and deduce that the sum of the interior angles is $q(2n - 4)$ right angles.