

Shrewsbury School

The Arnold Hagger Mathematics Prize

Wednesday February 12th 1997 at 7.15

Unlike GCSE and, to some extent "A" level, answers are marked largely on *elegance* and *mathematical coherence*. Try to justify everything you say, try to reason. Short, beautiful answers will appeal to your examiner more than long, contorted ones.

You are unlikely to score well if you write down only the answers.

The word *Prove* has a special significance in the hearts of mathematicians

Although the questions are in roughly ascending order of difficulty you may find you can score more rapidly by skipping certain questions out.

1) Say how many and which (if any) of the following 10 statements is/are true, giving reasons :

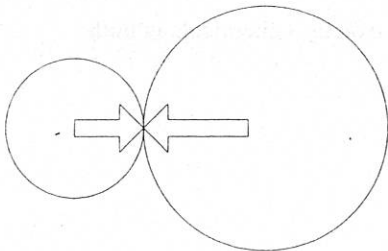
1. Exactly ONE statement on this list is true
2. Exactly TWO statements on this list are true
3. Exactly THREE statements on this list are true
4. Exactly FOUR statements on this list are true
5. Exactly FIVE statements on this list are true
6. Exactly SIX statements on this list are true
7. Exactly SEVEN statements on this list are true
8. Exactly EIGHT statements on this list are true
9. Exactly NINE statements on this list are true
10. Exactly TEN statements on this list are true

2) If $m_1a^2 + m_2b^2 = m_1c^2 + m_2d^2$ and $m_1a + m_2b = m_1c + m_2d$ find relationships between a,b,c and d

3) Cryptography is the science and study of secret writing.. One common system is to substitute letters in a *plaintext string* for others. Under the code $C = (qwerty)(uiop)(asd)(fgh)(jk)(lz)(xcv)(bnm)$, for instance, the string THISQUESTIONISEASY becomes YFODWIRDYOPMODRSDQ. The bracketed expression above is called a *representation* of the code.

- (i) Write down the representation of the code C^2 .
- (ii) Write down the representation of a code which leaves every letter unchanged.
- (iii) Is there a power of C for which the code leaves every plaintext string unchanged? If so, what is it?
- (iv) How would you try to decode messages written in this code if you don't know the representation? Is it secure?

4) Two gear wheels are meshed directly together. Each is marked with an arrow, initially pointing directly at the other gear wheel.



The smaller wheel rotates clockwise until the arrows point directly towards one another once more. If the larger wheel has 181 teeth, how many times will the smaller wheel have rotated?

5) Prove that in the square of an odd whole number (*integer*) expressed in base 10 the tens digit is always even - for example, $39^2 = 1521$

Please turn over

6) It is well known that $\sqrt{2}$ cannot be written as a fraction (i.e. $\frac{\text{whole number}}{\text{whole number}}$). Taking this as known,

prove that $\frac{\sqrt{2} + 3}{\sqrt{2} - 1}$ is also irrational

7) How many 7 digit numbers which are divisible by 25 can be formed from the digits 2,2,2,3,4,5 and 7 ?

8) One sleepless night I take my family silver out to bury it on a rectangular plot of farmland. I bury it (underground) exactly 21000m from one corner, 18000 metres from the opposite corner, and 6000m from a third corner. How far is it from the fourth corner?

9) Evaluate the following expression *exactly*

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

10) We write (a,b) to mean the greatest common divisor of a and b - that is, the largest number which divides into both a and b . So, for example, $(21,14) = 7$

Prove that $(a^2, b^2) = (a,b)^2$

11) Four points are placed at random inside an equilateral triangle of side 2. Show that there is a vertex of the triangle which has *two* points within a distance of one unit from the vertex.

12) Under the present calendar system, every fourth year is a leap year. There are three exceptions to this rule every 400 years. If a year number is divisible by 100 then it is a leap year if and only if it is divisible by 400. Thus, 1800, 1900 and 2100 are not leap years, but 2000 will be a leap year. The beginning of the 20th century, January 1st 1900, was a Monday. Show that although Sunday begins every week it will never begin a century. *[clearly, in questions of this nature, all the marks are for the clarity and accuracy of the reasoning!]*

13) A chemist is working with 1500 compounds, from which he selects two sample sets of 300 for testing.

- (i) How many chemicals would you expect to be in both sets?
- (ii) What is the probability of exactly 56 chemicals being in both sets?

A chemist is working with n chemicals, from which he selects two samples of s chemicals.

- (iii) What is the probability of an overlap of s chemicals (that is, exactly s chemicals in both collections) ?
- (iv) Write down and simplify an expression for the mean overlap.

MT

Monday, 10 February 1997

Please turn over...