



# *Arnold Hagger Mathematics Prize*

*19<sup>th</sup> January 2005*

***90 minutes***

- Answer as many questions as you can.
- All answers must be written in the answer booklet provided.
- There are 100 marks altogether.
- You are not necessarily expected to finish the paper. Producing a few complete, elegant solutions is better than doing scraps from each question.
- A calculator may be used in any question, but will not be very useful.
- Standard geometrical instruments are also allowed.

Good luck!

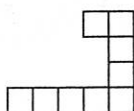
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1) A statue weighs 1000 kg plus half its weight. How much does it weigh altogether?  
[3 marks]

2) A television programme has an image which fits perfectly on a widescreen TV with a 16:9 rectangular screen. If the same programme is shown on a regular TV with a 4:3 screen, what proportion of the area of this screen is not used?  
[4 marks]

3) Mr Fussy likes his marmalade to be completely smooth (no lumpy bits) and entirely orange (no yellow bits). What a fussy old fusspot!  
One morning he opens a new jar. To his dismay, he finds lumpy bits spread out at random, taking up 30% of the marmalade. He also notices yellow bits (which may or may not be lumpy) spread out at random, taking up 40% of the marmalade.  
Once he has carefully weeded out all the bits he doesn't like, what percentage of the marmalade will be left for him to eat?  
[4 marks]

4) Prove that the shape below tessellates (i.e. that you can tile the plane without any overlaps or gaps using only this shape):



(You may find the squared paper provided useful.)

[4 marks]

5) A newspaper consists of a number of sheets folded in half, each sheet providing 4 pages. For example, a newspaper with 3 sheets would have 12 pages: pages 1, 2, 11, 12 would be on the outer sheet, pages 3, 4, 9, 10 on the middle sheet, and pages 5, 6, 7, 8 on the inside sheet.

If one sheet of a newspaper contains page 28 and at least one page in the sixties, how many pages could the entire newspaper have altogether?

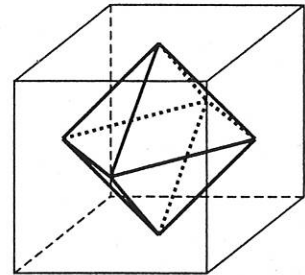
[5 marks]

- 6) Willy Wonka's bathtub has 3 taps: a dark chocolate tap, a milk chocolate tap, and a white chocolate tap. When only one tap is turned on full, Wonka knows that it takes 3 minutes to fill his bath with the dark tap, 4 minutes with the milk tap, and 6 minutes with the white tap.

How long does it take to fill his bath if all 3 taps are turned on full? Give your answer in minutes and seconds.

[5 marks]

- 7) A dot is placed at the centre of each face of a cube. A line is drawn between each pair dots lying on adjacent faces of the cube. The resulting solid formed by these lines is a regular octahedron, as shown in the diagram.



What proportion of the volume of the cube is taken up by the octahedron?

[6 marks]

- 8) a) Given that  $x + y$  is a factor of  $x^3 + y^3$ , factorise  $x^3 + y^3$ .  
 b) **Using your result in (a)**, find all the factors of  
 i) 65  
 ii) 1027  
 c) Prove carefully that there is only one prime number which can be expressed as the sum of two cubes of positive integers.

[10 marks]

- 9) Find the exact value of each of these recurring expressions:

a) 
$$\sqrt{12 + \sqrt{12 + \sqrt{12 + \sqrt{12 + \dots}}}}$$

[4 marks]

b) 
$$2 + \frac{15}{2 + \frac{15}{2 + \frac{15}{2 + \dots}}}$$

[4 marks]

- 10) In a most unusual court, a defendant may request one of two possible juries: a three-man jury or a one-man jury.

The three-man jury has two members, each of whom independently has probability  $p$  of making the correct decision, together with a third member who flips a fair coin for each decision. The majority verdict of the three members is used.

The one-man jury has one member who has probability  $p$  of making the correct decision.

Which jury has the higher probability of coming to the correct decision?

[5 marks]

- 11) Let the binary operation  $*$  be defined by:  $a * b = \frac{a+b}{2}$ .

For example,  $3 * 5 = \frac{3+5}{2} = 4$ .

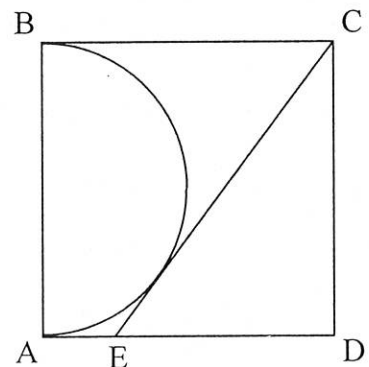
- Describe in words what  $*$  effectively does to 2 numbers  $a$  and  $b$ .
- Find a condition involving  $a, b, c$  such that  $(a * b) * c = \frac{a+b+c}{3}$ .
- Explain briefly why **no** expression (however long or complicated) involving only  $a, b, c, *$  and brackets, can be equal to  $\frac{a+b+c}{3}$  for **all** values of  $a, b, c$ .
- When does  $(a * b) * c = a * (b * c)$ ?

[9 marks]

- 12) A semicircle lies inside a square ABCD of side 2 units, with its diameter occupying the edge AB of the square.

Point E lies on edge AD so that the line CE is a tangent to the semicircle.

Find the area of triangle CDE.



[7 marks]

13) Two spheres of **radii**  $a$  and  $b$  lie inside an upright cylinder of **diameter**  $d$ . The spheres touch the base of the cylinder, the inside curved surface of the cylinder, and each other.

a) Find an expression for  $d$  in terms of  $a$  and  $b$ .

b) Hence show that  $\sqrt{d} - \sqrt{a} - \sqrt{b}$  is a constant, which you should determine.

[10 marks]

14) Find **all** solutions of the equation below, where  $x$  and  $y$  are integers.

$$\frac{14}{x} + \frac{7}{y} = 2$$

[10 marks]

15) After nearly half a century, Scrooge has reluctantly decided to buy a new top hat. He shops for a whole week in the tightest areas of London before discovering the cheapest one available is £5 from the Miser's Mall.

Given that he has accumulated an unlimited number of 1p, 2p and 5p coins, in how many ways can he pay for his new top hat using only these coins?

[10 marks]

**The End...**