2018

THE ARNOLD HAGGER MATHEMATICS PRIZE COMPETITION

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WEDNESDAY 24th January

7.15pm - 8.45pm

Room M8

Calculators MAY be used

INTUS SI RECTE NE LABORA

The Arnold Hagger Mathematics Prize Competition 2018

* The thirteen questions may be answered in any order.
* Make your methods of solution clear by including all working and reasoning.
* The marks allocated to each question is shown - either **[5]** or **[10]** marks.
* **Calculators MAY be used**.

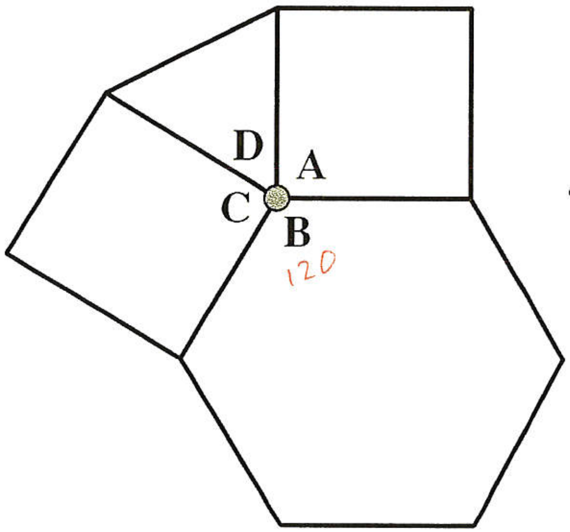
Question One : **A·B·O·U·T A P·O·I·N·T**

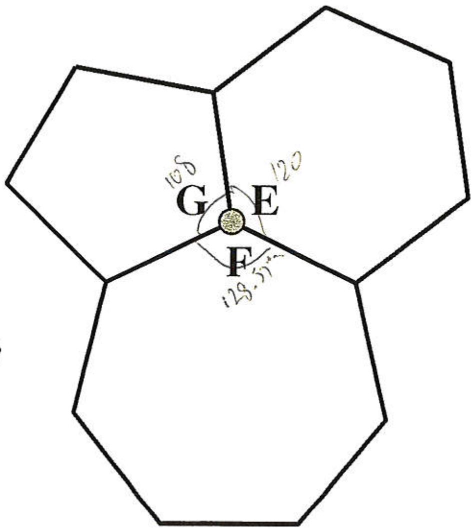
[10] (a) (i) On a flat surface, can a triangle, two squares and a hexagon be fitted perfectly about a point ?

(ii) Give a calculation to support your answer.

(b) (i) On a flat surface, can a pentagon, a hexagon and a septagon be fitted perfectly around a point ?

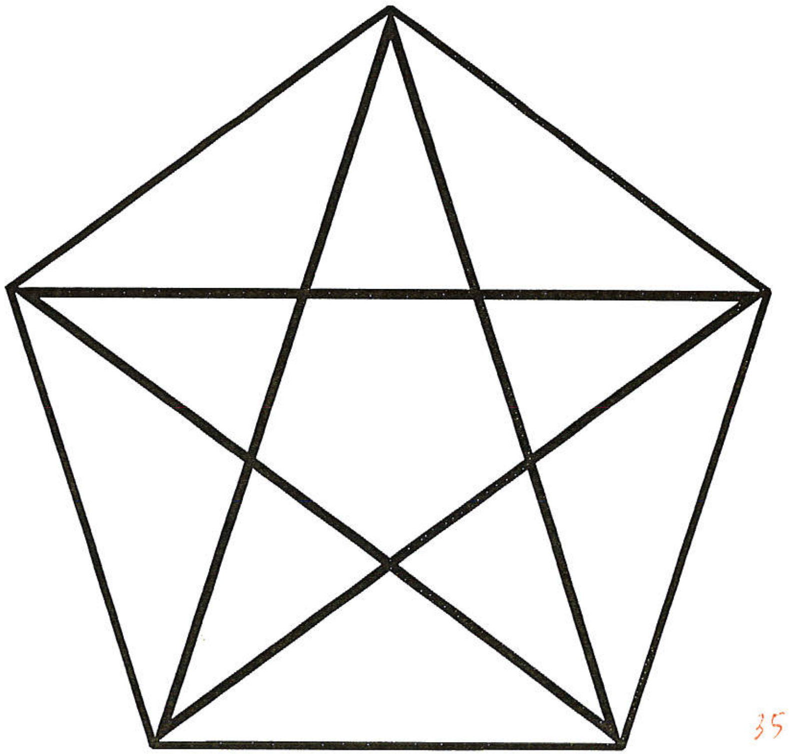
(ii) Give a calculation to support your answer.





Question Two : **T·R·I, T·R·I A·N·D T·R·I A·G·A·I·N.**

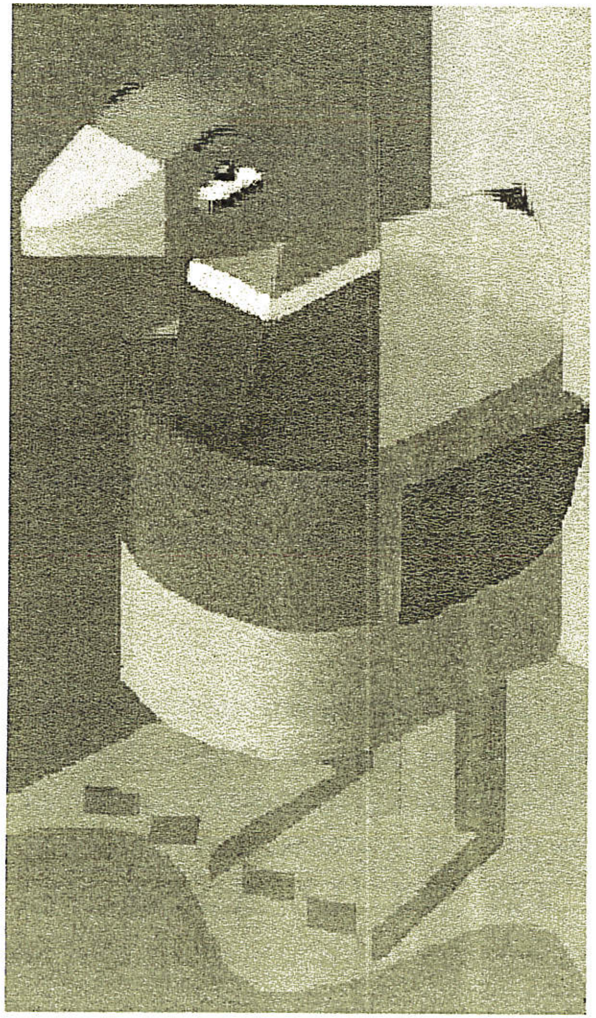
[5] How many triangles are in the diagram directly below ?



Question Three : **Q·U·A·C·K·E·R·S**

[5]

A normal duck has two legs.
A lame duck has one leg.
A sitting duck has no legs.
There are 66 ducks with a total of 64 legs.
The total number of normal ducks and lame ducks is twice the number of sitting ducks.
How many lame ducks are there ?



Question Four : **K·N·I·G·H·T·S A·N·D K·N·A·V·E·S**

[5]

A group of five people contain knights and knaves.
Knights always tell the truth.
Knaves always lie.

knighTs Always tell tHe TRUTH

They each give a statement;

Exactly 1 of us is a knave
Exactly 2 of us are knights
Exactly 3 of us are knaves
Exactly 4 of us are knights
Exactly 5 of us are knaves

How many of the five people are knights ?

Question Five : **S·O·C·K I·T T·O M·E**

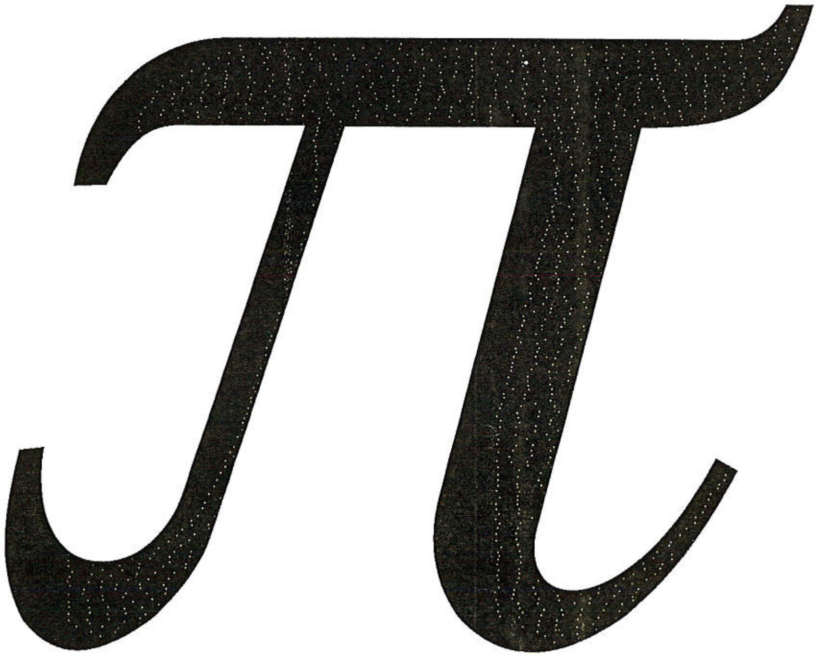
[5] In my sock box are sixteen socks, some red, some green. On randomly pulling out first one, then a second, (without replacing the first) the probability they're both green is $\frac{1}{12}$. How many of my sixteen socks are red ?

Question Six : **I P·R·E·F·E·R P·I**

[5] A number is said to be palindromic if it reads the same backwards as forwards.

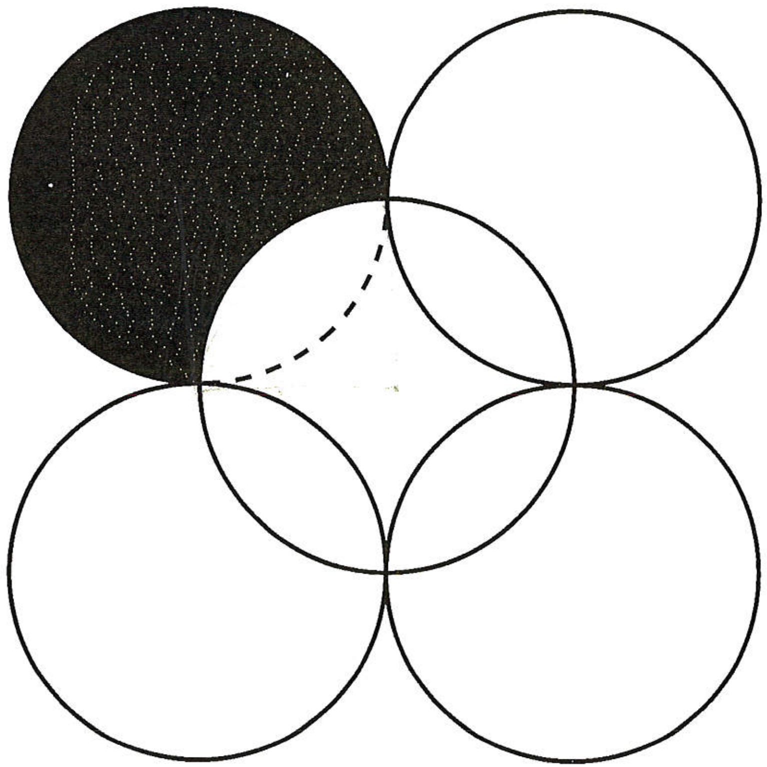
For example, the number 7891987 is palindromic.

Show that all 4-digit palindromic numbers are divisible by 11.



Question Seven : **E·C·L·I·P·S·E**

[10] Determine the area of the shaded shape which was constructed from five identical circles each of radius "R" arranged as shown below.



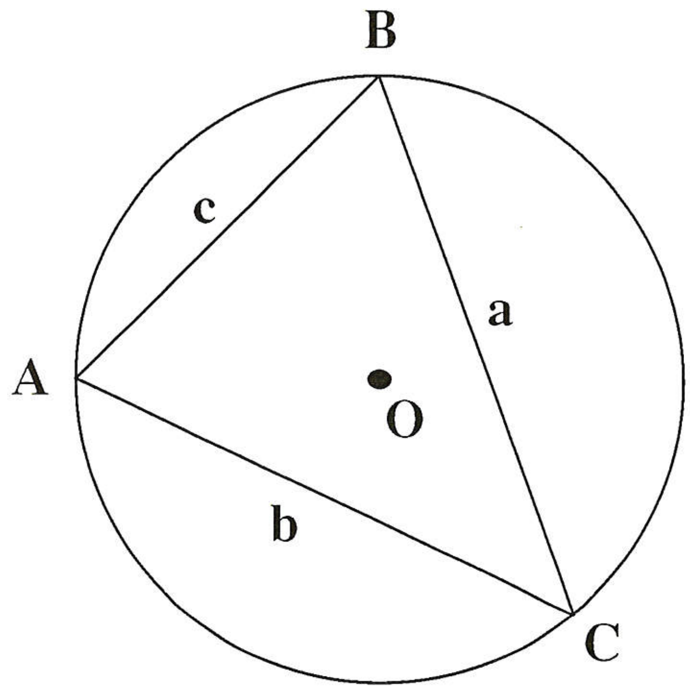
Question Eight : **T·H·E G·R·E·A·T E·I·G·H·T**

[5] State the units digit of;

$$8^{98765432}$$

Question Nine : **G·I·V·E M·E A S·I·N·E**

[10] A, B and C are three points on the circumference of a circle, centre O.
With the usual notation, AB = c, AC = b and BC = a.
The angles and sides are connected by the well known Sine Rule.



i.e.

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Prove that each of these ratios is equal to the diameter of the circle.

Question Ten : **T·H·E R·E·V·E·N·G·E O·F P·Y·T·H·A·G·O·R·A·S·**

[10] Let $x + y = A$ and $x^2 + y^2 = B$, where $x \geqslant y$.
Find, in as simple a form as possible, an expression for $x^3 - y^3$ in terms of A and B.

Question Eleven : **A·N O·D·D R·E·S·U·L·T**

[10] Prove that

$$n^2 \bmod 2n = n$$

where n is any odd number greater than 1

NOTE : The "mod" function is defined as follows: "x mod y" is the remainder when x is divided by y.

EXAMPLES :

"13 mod 5 = 3" means "13 has a remainder upon division by 5 of 3".

"13 mod 7 = 6" means "13 has a remainder upon division by 7 of 6".

Question Twelve : **A·S E·A·S·Y A·S A B C**

[10] a, b, and c are positive real numbers with the property that;

$$ab + bc + ca = 1$$

Show that; $a + b + c \geqslant \sqrt{3}$

Question Thirteen : **A·I·M·I·N·G F·O·R P·E·R·F·E·C·T·I·O·N**

[10] In this question, n is a natural number.

i.e. $n \in \{ 0, 1, 2, 3, 4, 5, ... \}$

Consider the expression $2 + 2\sqrt{1 + 12n^2}$

Prove that if this expression has an integer value then that value is also a perfect square.

END OF PAPER

Martin Hansen, January 2018