The Arnold Hagger Prize

by Dr Oakley aged $39\frac{72}{73}$

30th January 2020

Please read the following before you start

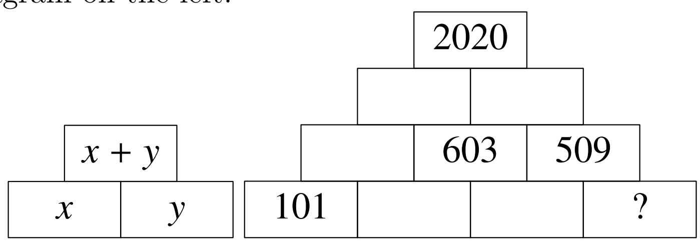
This paper comprises twenty questions. The first fourteen are worth four marks. You are advised to concentrate on these before advancing to the final six questions. Questions 15 to 17 are worth six marks. Questions 18 and 19 are worth eight marks. There are ten marks available for question 20. The answers to the questions 1 to 14 are integers and all that is required is the answer. The last six questions require detailed answers. Please write all answers in the answer booklet. No calculators are allowed. You have 90 minutes to complete as many questions as you can.

Question One

Emily reads every day. Emily reads 20 pages of a book a day for seven days and then 25 pages of this same book each day for six days leaving her 30 pages to read on the fourteenth day. She reads another book of identical length at a constant rate of 40 pages per day. How long does it take her to complete this book?

Question Two

In this number pyramid, the numbers on the second level or above are obtained by summing the two numbers directly beneath it, as shown in the diagram on the left.



What number should replace the question mark?

Question Three

The points A, B, C and D lie on a straight line. AC is of length 20, BD is of length 20 and AD is of length 29. What is the length BC?

Question Four

The value of

$$\left(1+\frac{1}{1+1}\right)\left(1+\frac{1}{1+2}\right)\left(1+\frac{1}{1+4}\right)\left(1+\frac{1}{1+8}\right)\left(1+\frac{1}{1+16}\right)\left(1+\frac{1}{1+32}\right)$$

is $\frac{a}{b}$ $(a > b)$, where a and b are integers with no common factor greater than one.

What is the value of $a + b$?

Question Five

A square-shaped floor is covered with congruent square tiles. If the total number of tiles that lie on the two diagonals is 21, how many tiles cover the floor?

Question Six

This question has been reproduced in honour of Andrew Jobbings, who passed away last year.

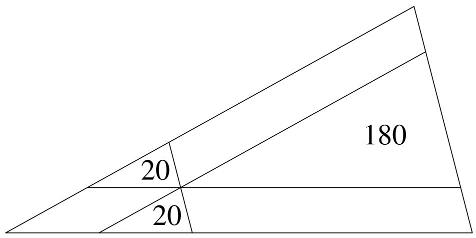
ABC is an isosceles triangle with an apex at B and side lengths 9, 9 and 3. Point D is placed on the line BC, but not at C, such that $AD = 3$. What is the distance BD?

Question Seven

Boris has a selection of blue, red, green and yellow marbles. One third of his marbles are blue, one quarter are red and one sixth are yellow. What is the smallest number of yellow marbles that Boris can have?

Question Eight

A point is chosen in the middle of a given triangle. Three lines through this point, each parallel to one of the sides of the triangle, are drawn. This forms three triangles of areas 20, 20 and 180 as shown.



What is the area of the original triangle?

Question Nine

Mr Lucas drives from his house to work, leaving at the same time every Friday morning. If he averages 40mph then he arrives three minutes late for Third Form Assembly. Had he averaged 60mph, he would be three minutes early. What speed must Mr Lucas average to ensure he get to work exactly on time?

Question Ten

Deborah has some red socks, some purple socks and some orange socks. The ratio of red and purple socks to orange socks is 17:3. The ratio of red socks to orange socks and purple socks is 4:11. She has 35 pairs of purple socks. How many red socks does Deborah possess?

Question Eleven

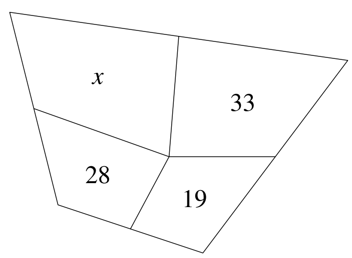
Bill and Ben were working together in house on their Top Schools. Last night, they each solved half of their problems on their own but solved the second half together, putting down exactly the same answers whether right or wrong. Bill had correct answers to only 80% of the problems he solved alone but scored 88% overall. Ben had correct answers to 90% of the problems he solved alone. What was Ben's overall percentage of correct answers?

Question Twelve

Several sets of prime numbers such as $7, 83, 421, 659$ each use digits 1 to 9 exactly once. What is the smallest sum such a set of primes could have? In the example, the sum was $7 + 83 + 421 + 659 = 1170$.

Question Thirteen

The point P is placed in a convex quadrilateral and is connect by line segments to the midpoints of the sides, dividing the quadrilateral into four regions of area 19, 28, 33 and x, as shown in the diagram below. What is the value of x?



Question Fourteen

20! has many factors: 41,040 to be precise. If one of these factors is selected at random, the probability that it is odd can be expressed in the form 1/n, where n is an integer.
What is the value of n?

Recall $20! = 20 \times 19 \times 18 \times 17 \times \ldots \times 3 \times 2 \times 1$.

I hope you are now suitably warmed up. The final six questions require written solutions.

Question Fifteen

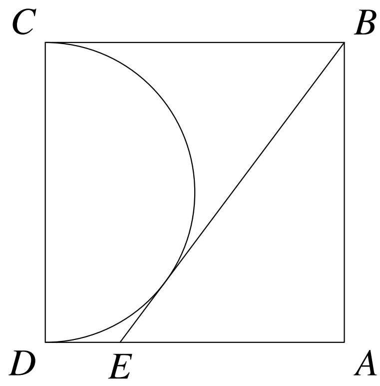
A 40kg mass can be split into four masses such that every integer mass from 1kg to 40kg can be balanced on set of scales using all, or a subset, of the four masses. What are the four masses? You must explain your answer. You do not need to explicitly describe how one would need to balance every mass between 1kg and 40kg.

Question Sixteen

$ABCD$ is a square of side length 20. A semicircle of radius 10, centered at the midpoint of a CD, is drawn in the interior of the square.

A line, tangent to the semicircle, is drawn from B meeting the line AD at point E.

What is the length of BE? You must show full working to obtain full marks.



Question Seventeen

On the island of Knights and Knaves, knights always tell the truth, except when they are mistaken, and knaves always lie. During a meeting 40 islanders sit at a large round table and each declares "I am sitting next to a knight and a knave". Exactly three knights are mistaken – How many knights are there at the meeting?

Question Eighteen

The following question was written by me, aged seventeen. I cannot remember the motivation or inspiration behind it. It was published in the Church Magazine. I received no solutions that month...

Every number has a multiple which is a string of 1s followed by a string of zeroes (the string of zeroes can be of zero length but the string of 1s cannot). I found the lowest such multiples for each number from 1 to 99. For 1, the appropriate multiple is 1. For 2, the multiple is 10; for 3, it is 111; for 4, it is 100; for 26, it is 1111110. As well as calculating these numbers (as if that was not enough), I answered the following questions for each multiple:

a) How many digits in the multiple? b) How many 1s or 0s in the longest string in this multiple?

For 26, the answers are a) 7 and b) 6.

Three consecutive numbers produced these same answers to parts a) and b) i.e. the answer to part (a) is 7 and the answer to part (b) is 6.
What are these three numbers?

Question Nineteen

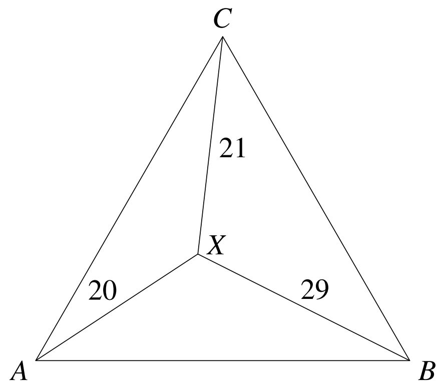
$f(x)$ is a monic quartic polynomial
(i.e. it is of the form $f(x) = x^4 + ax^3 + bx^2 + cx + d$).

$f(1) = 2, f(2) = 9, f(3) = 28$ and $f(4) = 65$.

What is the value of $f(5)$?

Question Twenty

A point X is placed in the interior of an equilateral triangle such that $AX = 20, BX = 29$ and $CX = 21$



What is the exact area of the triangle? Give your answer in the form $a + b\sqrt{p}$, where a and b are rational numbers and p is a prime number.

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