Shrewsbury School

Arnold Hagger Prize

Tuesday 7th February 2023

90 minutes

Answer as many questions as you can.

All answers must be written in this question paper.

There are 100 marks altogether.

You are not necessarily expected to finish the paper. Producing a few complete, elegant solutions is better than doing scraps from each question.

A calculator may **NOT** be used in this examination.

Standard geometrical instruments are allowed.

Good luck! **MPJ**

Name _____ **Year** _____

1. Express as an exact non-decimal value:

$$\frac{\sqrt{18}+\sqrt{27}}{\sqrt{12}+\sqrt{8}}$$

simplifying your answer as much as possible. (3)

(Total 3 marks)

2. In a list of five positive integers, the median is one more than the mean and the mode is one more than the median. If the median is 10, what is the smallest number that could be on the list? (3)

_____ **(Total 3 marks)**

3. A car competes in a time trial, and must complete two laps of the track at an average speed of exactly 60 mph. Unfortunately, the car develops an engine problem, and its average speed for the first lap is only 30 mph. What speed must the car achieve on the second lap to complete the trial successfully? (3)

(Total 3 marks)

4. Make x the subject of the equation

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}$$

(3)

(Total 3 marks)

5. Albert and Bertie take part in the Tucks. Albert runs for half the time and walks for the other half of the time. Bertie runs half the distance and walks the rest. They both walk at 3 mph and run at 6 mph. If Albert takes 40 minutes to complete the course, how long does Bertie take? (3)

(Total 3 marks)

6. For any positive integer n, the factorial function n! is defined to be:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1.$$

For example, $4! = 4 \times 3 \times 2 \times 1 = 24$. If $n = 30$, how many zeros will there be at the end of the answer for 30! when written as a whole number? (5)

(Total 5 marks)

7. Steve the painter takes one hour to paint a wall. Mr Johnson and Mr Johnson would both take 1 hour 20 minutes each to paint a similar wall. If all three start to paint three walls, what is the shortest time in which the job can be finished? In this case, what proportion of the total painting would Steve do? (5)

(Total 5 marks)

8. Hannah has a bag containing n sweets, of which 3 are yellow and the rest are orange. She eats two sweets in succession. If the probability that she eats two yellow sweets is $\frac{1}{15}$, how many sweets were in the bag at the start? (5)

(Total 5 marks)

9. How many real solutions are there to the quartic equation?:

$$x^4 = (x-1)^4$$

Show clearly how you have determined your answer and give the exact values of any solutions you find. (5)

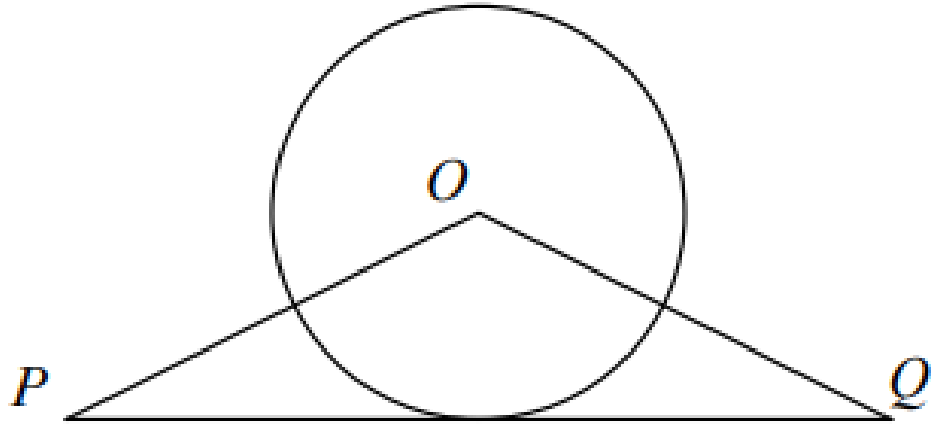
(Total 5 marks)

10. If $25^x + 25^{-x} = 34$, find the value of $5^x + 5^{-x}$, where x is real.

(5)

(Total 5 marks)

11. The diagram shows a circle with centre O and a triangle OPQ. Side PQ is a tangent to the circle. If the area of the circle is equal to the area of the triangle, what is the ratio of the length PQ to the circumference of the circle?



(5)

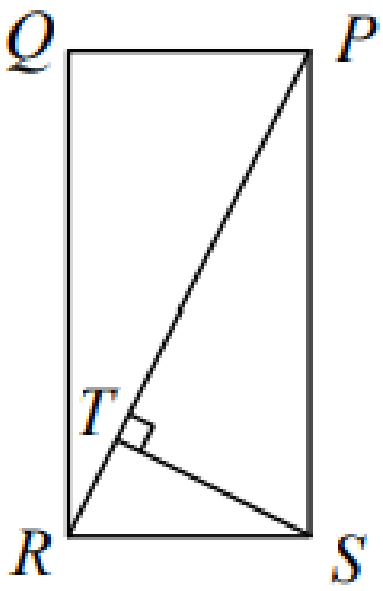
(Total 5 marks)

12. Solve $x^{\sqrt{x}} = \sqrt{x}^{x}$, $x > 0$

(5)

(Total 5 marks)

13. The diagram shows a rectangle PQRS in which the ratio PQ : QR is 1 : 2. The point T on PR is such that TS is perpendicular to PR. What is the ratio of the area of the triangle RST to the area of the rectangle PQRS?



(5)

(Total 5 marks)

14. Algebraic Arabella says that there are infinitely many prime number numbers of the form $4^n - 1$, where n is a positive integer. Numeric Nigel disagrees. Who is correct and why? (5)

(Total 5 marks)

15. For positive integers, a and b, find all ordered pairs (a, b), for $a < b$, that are solutions to the equation

$$\sqrt{a} + \sqrt{b} = \sqrt{2023}$$

(8)

(Total 8 marks)

16. If $a^2 - a - 1 = 0$, find the value of a^6.

(8)

(Total 8 marks)

17. If $4x^2 + x^{-2} = 2$, find the value of $8x^3 + x^{-3}$.

(8)

(Total 8 marks)

18. For real numbers x and y, such that $0 < x < y < 1$, consider the six numbers $x + y, x - y, y - x, xy, \frac{x}{y}, \frac{y}{x}$. Two of these six are found to have equal values, and another pair also have equal values (but different to the first pair). Find the exact values of x and y. (8)

(Total 8 marks)

19. Show that $x^2 + 5y = 243{,}723$ has no integer solutions (x, y).

Hence show that $x^4 + 2y^4 = 26{,}081{,}974$ has no integer solutions (x, y). (8)

(Total 8 marks)

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