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Complex Numbers in Geometry

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Contents

1 Introduction

When we are unable to solve some problem in plane geometry, it is recommended to try to do calculus. There are several techniques for doing calculations instead of geometry. The next text is devoted to one of them – the application of complex numbers.

The plane will be the complex plane and each point has its corresponding complex number. Because of that points will be often denoted by lowercase letters a, b, c, d, . . . , as complex numbers.

The following formulas can be derived easily.

2 Formulas and Theorems

Theorem 1.

- $ab \parallel cd$ *if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = \dfrac{c-d}{\overline{c}-\overline{d}}$.
- a, b, c *are colinear if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = \dfrac{a-c}{\overline{a}-\overline{c}}$.
- $ab \perp cd$ *if and only if* $\dfrac{a-b}{\overline{a}-\overline{b}} = -\dfrac{c-d}{\overline{c}-\overline{d}}$.
- $\varphi = \angle acb$ *(from a to b in positive direction) if and only if* $\dfrac{c-b}{|c-b|} = e^{i\varphi}\dfrac{c-a}{|c-a|}$.

Theorem 2. *Properties of the unit circle:*

- *For a chord* ab *we have* $\dfrac{a-b}{\overline{a}-\overline{b}}=-ab$.
- *If* c *belongs to the chord* ab *then* $\overline{c}=\dfrac{a+b-c}{ab}$.
- *The intersection of the tangents from* a *and* b *is the point* $\dfrac{2ab}{a+b}$.
- *The foot of perpendicular from an arbitrary point* c *to the chord* ab *is the point* $p=\dfrac{1}{2}\Big(a+b+c-ab\overline{c}\Big)$.
- *The intersection of chords* ab *and* cd *is the point* $\dfrac{ab(c+d)-cd(a+b)}{ab-cd}$.

Theorem 3. *The points* a,b,c,d *belong to a circle if and only if*

$$\frac{a-c}{b-c}:\frac{a-d}{b-d}\in\boldsymbol{R}.$$

Theorem 4. *The triangles* abc *and* pqr *are similar and equally oriented if and only if*

$$\frac{a-c}{b-c}=\frac{p-r}{q-r}.$$

Theorem 5. *The area of the triangle* abc *is*

$$p=\frac{i}{4}\begin{vmatrix} a & \overline{a} & 1\\ b & \overline{b} & 1\\ c & \overline{c} & 1\end{vmatrix}=\frac{i}{4}\Big(a\overline{b}+b\overline{c}+c\overline{a}-\overline{a}b-\overline{b}c-\overline{c}a.\Big)$$

Theorem 6.

- *The point* c *divides the segment* ab *in the ratio* $\lambda\neq-1$ *if and only if* $c=\dfrac{a+\lambda b}{1+\lambda}$.
- *The point* t *is the centroid of the triangle* abc *if and only if* $t=\dfrac{a+b+c}{3}$.
- *For the orthocenter* h *and the circumcenter* o *of the triangle* abc *we have* $h+2o=a+b+c$.

Theorem 7. *Suppose that the unit circle is inscribed in a triangle* abc *and that it touches the sides* bc,ca,ab*, respectively at* p,q,r.

- *It holds* $a=\dfrac{2qr}{q+r}, b=\dfrac{2rp}{r+p}$ *and* $c=\dfrac{2pq}{p+q}$;
- *For the orthocenter* h *of the triangle* abc *it holds*

$$h=\frac{2(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r))}{(p+q)(q+r)(r+p)}.$$

- *For the excenter* o *of the triangle* abc *it holds* $o=\dfrac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)}$.

Theorem 8.

- *For each triangle* abc *inscribed in a unit circle there are numbers* u,v,w *such that* $a=u^2,b=v^2,c=w^2$*, and* $-uv,-vw,-wu$ *are the midpoints of the arcs* ab,bc,ca *(respectively) that don't contain* c,a,b.
- *For the above mentioned triangle and its incenter* i *we have* $i=-(uv+vw+wu)$.

Theorem 9. *Consider the triangle $\triangle$ whose one vertex is 0, and the remaining two are x and y.*

- *If h is the orthocenter of $\triangle$ then $h = \dfrac{(\overline{x}\,y + x\overline{y}\,)(x-y)}{x\overline{y}\,-\overline{x}\,y}$.*
- *If o is the circumcenter of $\triangle$, then $o = \dfrac{xy(\overline{x}\,-\overline{y}\,)}{\overline{x}\,y - x\overline{y}}$.*

3 Complex Numbers and Vectors. Rotation

This section contains the problems that use the main properties of the interpretation of complex numbers as vectors (Theorem 6) and consequences of the last part of theorem 1. Namely, if the point b is obtained by rotation of the point a around c for the angle φ (in the positive direction), then $b - c = e^{i\varphi}(a - c)$.

1. (Yug MO 1990, 3-4 grade) Let S be the circumcenter and H the orthocenter of $\triangle ABC$. Let Q be the point such that S bisects HQ and denote by T_1, T_2, and T_3, respectively, the centroids of $\triangle BCQ$, $\triangle CAQ$ and $\triangle ABQ$. Prove that

$$AT_1 = BT_2 = CT_3 = \frac{4}{3}R,$$

where R denotes the circumradius of $\triangle ABC$.

2. (BMO 1984) Let $ABCD$ be an inscribed quadrilateral and let H_A, H_B, H_C and H_D be the orthocenters of the triangles BCD, CDA, DAB, and ABC respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

3. (Yug TST 1992) The squares $BCDE$, $CAFG$, and $ABHI$ are constructed outside the triangle ABC. Let $GCDQ$ and $EBHP$ be parallelograms. Prove that $\triangle APQ$ is isosceles and rectangular.

4. (Yug MO 1993, 3-4 grade) The equilateral triangles BCB_1, CDC_1, and DAD_1 are constructed outside the triangle ABC. If P and Q are respectively the midpoints of B_1C_1 and C_1D_1 and if R is the midpoint of AB, prove that $\triangle PQR$ is isosceles.

5. In the plane of the triangle $A_1A_2A_3$ the point P_0 is given. Denote with $A_s = A_{s-3}$, for every natural number $s > 3$. The sequence of points P_0, P_1, P_2, $\dots$ is constructed in such a way that the point P_{k+1} is obtained by the rotation of the point P_k for an angle 120^o in the clockwise direction around the point A_{k+1}. Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ has to be isosceles.

6. (IMO Shortlist 1992) Let $ABCD$ be a convex quadrilateral for which $AC = BD$. Equilateral triangles are constructed on the sides of the quadrilateral. Let O_1, O_2, O_3, and O_4 be the centers of the triangles constructed on AB, BC, CD, and DA respectively. Prove that the lines O_1O_3 and O_2O_4 are perpendicular.

4 The Distance. Regular Polygons

In this section we will use the following basic relation for complex numbers: $|a|^2 = a\overline{a}$. Similarly, for calculating the sums of distances it is of great advantage if points are colinear or on mutually parallel lines. Hence it is often very useful to use rotations that will move some points in nice positions.

Now we will consider the regular polygons. It is well-known that the equation $x^n = 1$ has exactly n solutions in complex numbers and they are of the form $x_k = e^{i\frac{2k\pi}{n}}$, for $0 \le k \le n-1$. Now we have that $x_0 = 1$ and $x_k = \varepsilon^k$, for $1 \le k \le n-1$, where $x_1 = \varepsilon$.

Let's look at the following example for the illustration:

Problem 1. *Let $A_0A_1A_2A_3A_4A_5A_6$ be a regular 7-gon. Prove that*

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_3}.$$

Solution. As mentioned above let's take $a_k = \varepsilon^k$, for $0 \leq k \leq 6$, where $\varepsilon = e^{i\frac{2\pi}{7}}$. Further, by rotation around $a_0 = 1$ for the angle ε, i.e. $\omega = e^{i\frac{2\pi}{14}}$, the points a_1 and a_2 are mapped to a_1' and a_2' respectively. These two points are collinear with a_3. Now it is enough to prove that $\dfrac{1}{a_1'-1} = \dfrac{1}{a_2'-1} + \dfrac{1}{a_3-1}$. Since $\varepsilon = \omega^2$, $a_1' = \varepsilon(a_1 - 1) + 1$, and $a_2' = \omega(a_2 - 1) + 1$ it is enough to prove that

$$\frac{1}{\omega^2(\omega^2-1)} = \frac{1}{\omega(\omega^4-1)} + \frac{1}{\omega^6-1}.$$

After rearranging we get $\omega^6 + \omega^4 + \omega^2 + 1 = \omega^5 + \omega^3 + \omega$. From $\omega^5 = -\omega^{12}$, $\omega^3 = -\omega^{10}$, and $\omega = -\omega^8$ (which can be easily seen from the unit circle), the equality follows from $0 = \omega^{12} + \omega^{10} + \omega^8 + \omega^6 + \omega^4 + \omega^2 + 1 = \varepsilon^6 + \varepsilon^5 + \varepsilon^4 + \varepsilon^3 + \varepsilon^2 + \varepsilon + 1 = \dfrac{\varepsilon^7 - 1}{\varepsilon - 1} = 0$. $\triangle$

7. Let $A_0A_1 \dots A_{14}$ be a regular 15-gon. Prove that

$$\frac{1}{A_0A_1} = \frac{1}{A_0A_2} + \frac{1}{A_0A_4} + \frac{1}{A_0A_7}.$$

8. Let $A_0A_1 \dots A_{n-1}$ be a regular n-gon inscribed in a circle with radius r. Prove that for every point P of the circle and every natural number $m < n$ we have

$$\sum_{k=0}^{n-1} PA_k^{2m} = \binom{2m}{m} nr^{2m}.$$

9. (SMN TST 2003) Let M and N be two different points in the plane of the triangle ABC such that

$$AM : BM : CM = AN : BN : CN.$$

Prove that the line MN contains the circumcenter of $\triangle ABC$.

10. Let P be an arbitrary point on the shorter arc A_0A_{n-1} of the circle circumscribed about the regular polygon $A_0A_1 \dots A_{n-1}$. Let $h_1, h_2, \dots, h_n$ be the distances of P from the lines that contain the edges A_0A_1, A_1A_2, …, $A_{n-1}A_0$ respectively. Prove that

$$\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_{n-1}} = \frac{1}{h_n}.$$

5 Polygons Inscribed in Circle

In the problems where the polygon is inscribed in the circle, it is often useful to assume that the unit circle is the circumcircle of the polygon. In theorem 2 we can see lot of advantages of the unit circle (especially the first statement) and in practice we will see that lot of the problems can be solved using this method. In particular, we know that each triangle is inscribed in the circle and in many problems from the geometry of triangle we can make use of complex numbers. The only problem in this task is finding the circumcenter. For that you should take a look in the next two sections.

11. The quadrilateral $ABCD$ is inscribed in the circle with diameter AC. The lines AB and CD intersect at M and the tangets to the circle at B and C interset at N. Prove that $MN \perp AC$.

12. (IMO Shorlist 1996) Let H be the orthocenter of the triangle $\triangle ABC$ and P an arbitrary point of its circumcircle. Let E the foot of perpendicular BH and let $PAQB$ and $PARC$ be parallelograms. If AQ and HR intersect in X prove that $EX \| AP$.

13. Given a cyclic quadrilateral $ABCD$, denote by P and Q the points symmetric to C with respect to AB and AD respectively. Prove that the line PQ passes through the orthocenter of $\triangle ABD$.

14. (IMO Shortlist 1998) Let ABC be a triangle, H its orthocenter, O its incenter, and R the circumradius. Let D be the point symmetric to A with respect to BC, E the point symmetric to B with respect to CA, and F the point symmetric to C with respect to AB. Prove that the points D, E, and F are collinear if and only if $OH = 2R$.

15. (Rehearsal Competition in MG 2004) Given a triangle ABC, let the tangent at A to the circumscribed circle intersect the midsegment parallel to BC at the point A_1. Similarly we define the points B_1 and C_1. Prove that the points A_1, B_1, C_1 lie on a line which is parallel to the Euler line of $\triangle ABC$.

16. (MOP 1995) Let AA_1 and BB_1 be the altitudes of $\triangle ABC$ and let $AB \neq AC$. If M is the midpoint of BC, H the orthocenter of $\triangle ABC$, and D the intersection of BC and B_1C_1, prove that $DH \perp AM$.

17. (IMO Shortlist 1996) Let ABC be an acute-angled triangle such that $BC > CA$. Let O be the circumcircle, H the orthocenter, and F the foot of perpendicular CH. If the perpendicular from F to OF intersects CA at P, prove that $\angle FHP = \angle BAC$.

18. (Romania 2005) Let $A_0A_1A_2A_3A_4A_5$ be a convex hexagon inscribed in a circle. Let A_0', A_2', A_4' be the points on that circle such that

$$A_0A_0' \parallel A_2A_4, \quad A_2A_2' \parallel A_4A_0 \quad A_4A_4' \parallel A_2A_0.$$

Suppose that the lines $A_0'A_3$ and A_2A_4 intersect at A_3', the lines $A_2'A_5$ and A_0A_4 intersect at A_5', and the lines $A_4'A_1$ and A_0A_2 intersect at A_1'.
If the lines A_0A_3, A_1A_4, and A_2A_5 are concurrent, prove that the lines A_0A_3', A_4A_1' and A_2A_5' are concurrent as well.

19. (**Simson's line**) If A, B, C are points on a circle, then the feet of perpendiculars from an arbitrary point D of that circle to the sides of ABC are collinear.

20. Let A, B, C, D be four points on a circle. Prove that the intersection of the Simsons line corresponding to A with respect to the triangle BCD and the Simsons line corresponding to B w.r.t. $\triangle ACD$ belongs to the line passing through C and the orthocenter of $\triangle ABD$.

21. Denote by $l(S; PQR)$ the Simsons line corresponding to the point S with respect to the triangle PQR. If the points A, B, C, D belong to a circle, prove that the lines $l(A; BCD)$, $l(B; CDA)$, $l(C, DAB)$, and $l(D, ABC)$ are concurrent.

22. (Taiwan 2002) Let A, B, and C be fixed points in the plane, and D the mobile point of the circumcircle of $\triangle ABC$. Let I_A denote the Simsons line of the point A with respect to $\triangle BCD$. Similarly we define I_B, I_C, and I_D. Find the locus of the points of intersection of the lines I_A, I_B, I_C, and I_D when D moves along the circle.

23. (BMO 2003) Given a triangle ABC, assume that $AB \neq AC$. Let D be the intersection of the tangent to the circumcircle of $\triangle ABC$ at A with the line BC. Let E and F be the points on the bisectors of the segments AB and AC respectively such that BE and CF are perpendicular to BC. Prove that the points D, E, and F lie on a line.

24. (**Pascal's Theorem**) If the hexagon $ABCDEF$ can be inscribed in a circle, prove that the points $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are colinear.

25. (**Brokard's Theorem**) Let $ABCD$ be an inscribed quadrilateral. The lines AB and CD intersect at E, the lines AD and BC intersect in F, and the lines AC and BD intersect in G. Prove that O is the orthocenter of the triangle EFG.

26. (Iran 2005) Let ABC be an equilateral triangle such that $AB = AC$. Let P be the point on the extention of the side BC and let X and Y be the points on AB and AC such that

$$PX \parallel AC, \quad PY \parallel AB.$$

Let T be the midpoint of the arc BC. Prove that $PT \perp XY$.

27. Let $ABCD$ be an inscribed quadrilateral and let K, L, M, and N be the midpoints of AB, BC, CA, and DA respectively. Prove that the orthocenters of $\triangle AKN$, $\triangle BKL$, $\triangle CLM$, $\triangle DMN$ form a parallelogram.

6 Polygons Circumscribed Around Circle

Similarly as in the previous chapter, here we will assume that the unit circle is the one inscribed in the given polygon. Again we will make a use of theorem 2 and especially its third part. In the case of triangle we use also the formulas from the theorem 7. Notice that in this case we know both the incenter and circumcenter which was not the case in the previous section. Also, notice that the formulas from the theorem 7 are quite complicated, so it is highly recommended to have the circumcircle for as the unit circle whenever possible.

28. The circle with the center O is inscribed in the triangle ABC and it touches the sides AB, BC, CA in M, K, E respectively. Denote by P the intersection of MK and AC. Prove that $OP \perp BE$.

29. The circle with center O is inscribed in a quadrilateral $ABCD$ and touches the sides AB, BC, CD, and DA respectively in K, L, M, and N. The lines KL and MN intersect at S. Prove that $OS \perp BD$.

30. (BMO 2005) Let ABC be an acute-angled triangle which incircle touches the sides AB and AC in D and E respectively. Let X and Y be the intersection points of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE. Let Z be the midpoint of BC. Prove that the triangle XYZ is isosceles if and only if $\angle A = 60°$.

31. (**Newtons Theorem**) Given an circumscribed quadrilateral $ABCD$, let M and N be the midpoints of the diagonals AC and BD. If S is the incenter, prove that M, N, and S are colinear.

32. Let $ABCD$ be a quadrilateral whose incircle touches the sides AB, BC, CD, and DA at the points M, N, P, and Q. Prove that the lines AC, BD, MP, and NQ are concurrent.

33. (Iran 1995) The incircle of $\triangle ABC$ touches the sides BC, CA, and AB respectively in D, E, and F. X, Y, and Z are the midpoints of EF, FD, and DE respectively. Prove that the incenter of $\triangle ABC$ belongs to the line connecting the circumcenters of $\triangle XYZ$ and $\triangle ABC$.

34. Assume that the circle with center I touches the sides BC, CA, and AB of $\triangle ABC$ in the points D, E, F, respectively. Assume that the lines AI and EF intersect at K, the lines ED and KC at L, and the lines DF and KB at M. Prove that LM is parallel to BC.

35. (25. Tournament of Towns) Given a triangle ABC, denote by H its orthocenter, I the incenter, O its circumcenter, and K the point of tangency of BC and the incircle. If the lines IO and BC are parallel, prove that AO and HK are parallel.

36. (IMO 2000) Let AH_1, BH_2, and CH_3 be the altitudes of the acute-angled triangle ABC. The incircle of ABC touches the sides BC, CA, AB respectively in T_1, T_2, and T_3. Let l_1, l_2, and l_3 be the lines symmetric to H_2H_3, H_3H_1, H_1H_2 with respect to T_2T_3, T_3T_1, and T_1T_2 respectively. Prove that the lines l_1, l_2, l_3 determine a triagnle whose vertices belong to the incircle of ABC.

7 The Midpoint of Arc

We often encounter problems in which some point is defined to be the midpoint of an arc. One of the difficulties in using complex numbers is distinguishing the arcs of the cirle. Namely, if we define the midpoint of an arc to be the intersection of the bisector of the corresponding chord with the circle, we are getting two solutions. Such problems can be relatively easy solved using the first part of the theorem 8. Moreover the second part of the theorem 8 gives an alternative way for solving the problems with incircles and circumcircles. Notice that the coordinates of the important points are given with the equations that are much simpler than those in the previous section. However we have a problem when calculating the points d, e, f of tangency of the incircle with the sides (calculate them!), so in this case we use the methods of the previous section. In the case of the non-triangular polygon we also prefer the previous section.

37. (Kvant M769) Let L be the incenter of the triangle ABC and let the lines AL, BL, and CL intersect the circumcircle of $\triangle ABC$ at A_1, B_1, and C_1 respectively. Let R be the circumradius and r the inradius. Prove that:

$$\text{(a) } \frac{LA_1 \cdot LC_1}{LB} = R; \quad \text{(b) } \frac{LA \cdot LB}{LC_1} = 2r; \quad \text{(c) } \frac{S(ABC)}{S(A_1B_1C_1)} = \frac{2r}{R}.$$

38. (Kvant M860) Let O and R be respectively the center and radius of the circumcircle of the triangle ABC and let Z and r be respectively the incenter and inradius of $\triangle ABC$. Denote by K the centroid of the triangle formed by the points of tangency of the incircle and the sides. Prove that Z belongs to the segment OK and that $OZ : ZK = 3R/r$.

39. Let P be the intersection of the diagonals AC and BD of the convex quadrilateral $ABCD$ for which $AB = AC = BD$. Let O and I be the circumcenter and incenter of the triangle ABP. Prove that if $O \neq I$ then $OI \perp CD$.

40. Let I be the incenter of the triangle ABC for which $AB \neq AC$. Let O_1 be the point symmetric to the circumcenter of $\triangle ABC$ with respect to BC. Prove that the points A, I, O_1 are colinear if and only if $\angle A = 60°$.

41. Given a triangle ABC, let A_1, B_1, and C_1 be the midpoints of BC, CA, and AB respecctively. Let P, Q, and R be the points of tangency of the incircle k with the sides BC, CA, and AB. Let P_1, Q_1, and R_1 be the midpoints of the arcs QR, RP, and PQ on which the points P, Q, and R divide the circle k, and let P_2, Q_2, and R_2 be the midpoints of arcs QPR, RQP, and PRQ respectively. Prove that the lines A_1P_1, B_1Q_1, and C_1R_1 are concurrent, as well as the lines A_1P_1, B_1Q_2, and C_1R_2.

8 Important Points. Quadrilaterals

In the last three sections the points that we've taken as initial, i.e. those with *known* coordinates have been "equally improtant" i.e. all of them had the same properties (they've been either the points of the same circle, or intersections of the tangents of the same circle, etc.). However, there are numerous problems where it is possible to distinguish one point from the others based on its influence to the other points. That point will be regarded as the origin. This is particularly useful in the case of quadrilaterals (that can't be inscribed or circumscribed around the circle) – in that case the intersection of the diagonals can be a good choice for the origin. We will make use of the formulas from the theorem 9.

42. The squares $ABB'B''$, $ACC'C''$, $BCXY$ are consctructed in the exterior of the triangle ABC. Let P be the center of the square $BCXY$. Prove that the lines CB'', BC'', AP intersect in a point.

43. Let O be the intersection of diagonals of the quadrilateral $ABCD$ and M, N the midpoints of the side AB and CD respectively. Prove that if $OM \perp CD$ and $ON \perp AB$ then the quadrilateral $ABCD$ is cyclic.

44. Let F be the point on the base AB of the trapezoid $ABCD$ such that $DF = CF$. Let E be the intersection of AC and BD and O_1 and O_2 the circumcenters of $\triangle ADF$ and $\triangle FBC$ respectively. Prove that $FE \perp O_1O_2$.

45. (IMO 2005) Let $ABCD$ be a convex quadrilateral whose sides BC and AD are of equal length but not parallel. Let E and F be interior points of the sides BC and AD respectively such that $BE = DF$. The lines AC and BD intersect at P, the lines BD and EF intersect at Q, and the lines EF and AC intersect at R. Consider all such triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

46. Assume that the diagonals of $ABCD$ intersect in O. Let T_1 and T_2 be the centroids of the triangles AOD and BOC, and H_1 and H_2 orthocenters of $\triangle AOB$ and $\triangle COD$. Prove that $T_1T_2 \perp H_1H_2$.

9 Non-unique Intersections and Viete's formulas

The point of intersection of two lines can be determined from the system of two equations each of which corresponds to the condition that a point correspond to a line. However this method can lead us into some difficulties. As we mentioned before standard methods can lead to non-unique points. For example, if we want to determine the intersection of two circles we will get a quadratic equations. That is not surprising at all since the two circles have, in general, two intersection points. Also, in many of the problems we don't need both of these points, just the direction of the line determined by them. Similarly, we may already know one of the points. In both cases it is more convenient to use Vieta's formulas and get the sums and products of these points. Thus we can avoid "taking the square root of a complex number" which is very suspicious operation by itself, and usually requires some knowledge of complex analysis.

Let us make a remark: If we need explicitly coordinates of one of the intersection points of two circles, and we don't know the other, the only way to solve this problem using complex numbers is to set the given point to be one of the initial points.

47. Suppose that the tangents to the circle Γ at A and B intersect at C. The circle Γ_1 which passes through C and touches AB at B intersects the circle Γ at the point M. Prove that the line AM bisects the segment BC.

48. (Republic Competition 2004, 3rd grade) Given a circle k with the diameter AB, let P be an arbitrary point of the circle different from A and B. The projections of the point P to AB is Q. The circle with the center P and radius PQ intersects k at C and D. Let E be the intersection of CD and PQ. Let F be the midpoint of AQ, and G the foot of perpendicular from F to CD. Prove that $EP = EQ = EG$ and that A, G, and P are colinear.

49. (China 1996) Let H be the orthocenter of the triangle ABC. The tangents from A to the circle with the diameter BC intersect the circle at the points P and Q. Prove that the points P, Q, and H are colinear.

50. Let P be the point on the extension of the diagonal AC of the rectangle $ABCD$ over the point C such that $\angle BPD = \angle CBP$. Determine the ratio $PB : PC$.

51. (IMO 2004) In the convex quadrilateral $ABCD$ the diagonal BD is not the bisector of any of the angles ABC and CDA. Let P be the point in the interior of $ABCD$ such that

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that the quadrilateral $ABCD$ is cyclic if and only if $AP = CP$.

10 Different Problems – Different Methods

In this section you will find the problems that are not closely related to some of the previous chapters, as well as the problems that are related to more than one of the chapters. The useful advice is to carefully think of possible initial points, the origin, and the unit circle. As you will see, the main problem with solving these problems is the time. Thus if you are in competition and you want to use complex numbers it is very important for you to estimate the time you will spend. Having this in mind, it is very important to learn complex numbers as early as possible.

You will see several problems that use theorems 3, 4, and 5.

52. Given four circles k_1, k_2, k_3, k_4, assume that $k_1 \cap k_2 = \{A_1, B_1\}$, $k_2 \cap k_3 = \{A_2, B_2\}$, $k_3 \cap k_4 = \{A_3, B_3\}$, $k_4 \cap k_1 = \{A_4, B_4\}$. If the points A_1, A_2, A_3, A_4 lie on a circle or on a line, prove that the points B_1, B_2, B_3, B_4 lie on a circle or on a line.

53. Suppose that $ABCD$ is a parallelogram. The similar and equally oliented triangles CD and CB are constructed outside this parallelogram. Prove that the triangle FAE is similar and equally oriented with the first two.

54. Three triangles KPQ, QLP, and PQM are constructed on the same side of the segment PQ in such a way that $\angle QPM = \angle PQL = \alpha$, $\angle PQM = \angle QPK = \beta$, and $\angle PQK = \angle QPL = \gamma$. If $\alpha < \beta < \gamma$ and $\alpha + \beta + \gamma = 180°$, prove that the triangle KLM is similar to the first three.

55. *(Iran, 2005) Let n be a prime number and H_1 a convex n-gon. The polygons $H_2, \dots, H_n$ are defined recurrently: the vertices of the polygon H_{k+1} are obtained from the vertices of H_k by symmetry through k-th neighbour (in the positive direction). Prove that H_1 and H_n are similar.

56. Prove that the area of the triangles whose vertices are feet of perpendiculars from an arbitrary vertex of the cyclic pentagon to its edges doesn't depend on the choice of the vertex.

57. The points A_1, B_1, C_1 are chosen inside the triangle ABC to belong to the altitudes from A, B, C respectively. If

$$S(ABC_1) + S(BCA_1) + S(CAB_1) = S(ABC),$$

prove that the quadrilateral $A_1B_1C_1H$ is cyclic.

58. (IMO Shortlist 1997) The feet of perpendiculars from the vertices A, B, and C of the triangle ABC are D, E, end F respectively. The line through D parallel to EF intersects AC and AB respectively in Q and R. The line EF intersects BC in P. Prove that the circumcircle of the triangle PQR contains the midpoint of BC.

59. (BMO 2004) Let O be a point in the interior of the acute-angled triangle ABC. The circles through O whose centers are the midpoints of the edges of $\triangle ABC$ mutually intersect at K, L, and M, (different from O). Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC.

60. Two circles k_1 and k_2 are given in the plane. Let A be their common point. Two mobile points, M_1 and M_2 move along the circles with the constant speeds. They pass through A always at the same time. Prove that there is a fixed point P that is always equidistant from the points M_1 and M_2.

61. (Yug TST 2004) Given the square $ABCD$, let γ be i circle with diameter AB. Let P be an arbitrary point on CD, and let M and N be intersections of the lines AP and BP with γ that are different from A and B. Let Q be the point of intersection of the lines DM and CN. Prove that $Q \in \gamma$ and $AQ : QB = DP : PC$.

62. (IMO Shortlist 1995) Given the triangle ABC, the circle passing through B and C intersect the sides AB and AC again in C' and B' respectively. Prove that the lines BB', CC', and HH' are concurrent, where H and H' orthocenters of the triangles ABC and $A'B'C'$ respectively.

63. (IMO Shortlist 1998) Let M and N be interior points of the triangle ABC such that $\angle MAB = \angle NAC$ and $\angle MBA = \angle NBC$. Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

64. (IMO Shortlist 1998) Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360°$ and $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$. Prove that

$$BC \cdot AE \cdot FD = CA \cdot EF \cdot DB.$$

65. (IMO Shortlist 1998) Let ABC be a triangle such that $\angle A = 90°$ and $\angle B < \angle C$. The tangent at A to its circumcircle ω intersect the line BC at D. Let E be the reflection of A with respect to

BC, X the foot of the perpendicular from A to BE, and Y the midpoint of AX. If the line BY intersects ω in Z, prove that the line BD tangents the circumcircle of $\triangle ADZ$.

Hint: Use some inversion first...

66. (Rehearsal Competition in MG 1997, 3-4 grade) Given a triangle ABC, the points A_1, B_1 and C_1 are located on its edges BC, CA, and AB respectively. Suppose that $\triangle ABC \sim \triangle A_1B_1C_1$. If either the orthocenters or the incenters of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide prove that the triangle ABC is equilateral.

67. (**Ptolomy's inequality**) Prove that for every convex quadrilateral $ABCD$ the following inequality holds

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD.$$

68. (China 1998) Find the locus of all points D such that

$$DA \cdot DB \cdot AB + DB \cdot DC \cdot BC + DC \cdot DA \cdot CA = AB \cdot BC \cdot CA.$$

11 Disadvantages of the Complex Number Method

The bigest difficulties in the use of the method of complex numbers can be encountered when dealing with the intersection of the lines (as we can see from the fifth part of the theorem 2, although it dealt with the chords of the circle). Also, the difficulties may arrise when we have more than one circle in the problem. Hence you should avoid using the comples numbers in problems when there are lot of lines in general position without some special circle, or when there are more then two circles. Also, the things can get very complicated if we have only two circles in general position, and only in the rare cases you are advised to use complex numbers in such situations. The problems when some of the conditions is the equlity with sums of distances between non-colinear points can be very difficult and pretty-much unsolvable with this method.

Of course, these are only the obvious situations when you can't count on help of complex numbers. There are numerous innocent-looking problems where the calculation can give us increadible difficulties.

12 Hints and Solutions

Before the solutions, here are some remarks:

- In all the problems it is assumed that the lower-case letters denote complex numbers corresponding to the points denoted by capital letters (sometimes there is an exception when the unit circle is the incircle of the triangle and its center is denoted by o).
- Some abbreviations are used for addressing the theorems. For example T1.3 denotes the third part of the theorem 1.
- The solutions are quite useless if you don't try to solve the problem by yourself.
- Obvious derivations and algebraic manipulations are skipped. All expressions that are somehow "equally" related to both a and b are probably divisible by $a-b$ or $a+b$.
- To make the things simpler, many conjugations are skipped. However, these are very straightforward, since most of the numbers are on the unit circle and they satisfy $\overline{a} = \dfrac{1}{a}$.
- If you still doesn't believe in the power of complex numbers, you are more than welcome to try these problems with other methods– but don't hope to solve all of them. For example, try the problem 41. Sometimes, complex numbers can give you shorter solution even when comparing to the elementar solution.
- The author has tried to make these solutions available in relatively short time, hence some mistakes are possible. For all mistakes you've noticed and for other solutions (with complex numbers), please write to me to the above e-mail address.

1. Assume that the circumcircle of the triangle abc is the unit circle, i.e. $s=0$ and $|a|=|b|=|c|=1$. According to T6.3 we have $h=a+b+c$, and according to T6.1 we conclude that $h+q=2s=0$, i.e. $q=-a-b-c$. Using T6.2 we get $t_1=\dfrac{b+c+q}{3}=-\dfrac{a}{3}$ and similarly $t_2=-\dfrac{b}{3}$ and $t_3=-\dfrac{c}{3}$. We now have $|a-t_1|=\left|a+\dfrac{a}{3}\right|=\left|\dfrac{4a}{3}\right|=\dfrac{4}{3}$ and similarly $|b-t_2|=|c-t_3|=\dfrac{4}{3}$. The proof is complete. We have assumed that $R=1$, but this is no loss of generality.

2. For the unit circle we will take the circumcircle of the quadrilateral $abcd$. According to T6.3 we have $h_a=b+c+d$, $h_b=c+d+a$, $h_c=d+a+b$, and $h_d=a+b+c$. In order to prove that $abcd$ and $h_ah_bh_ch_d$ are congruent it is enough to establish $|x-y|=|h_x-h_y|$, for all $x,y\in\{a,b,c,d\}$. This is easy to verify.

3. Notice that the point h ca be obtained by the rotation of the point a around b for the angle $\dfrac{\pi}{2}$ in the positive direction. Since $e^{i\frac{\pi}{2}}=i$, using T1.4 we get $(a-b)i=a-h$, i.e. $h=(1-i)a+ib$. Similarly we get $d=(1-i)b+ic$ and $g=(1-i)c+ia$. Since $BCDE$ is a square, it is a parallelogram as well, hence the midpoints of ce and bd coincide, hence by T6.1 we have $d+b=e+c$, or $e=(1+i)b-ic$. Similarly $g=(1+i)c-ia$. The quadrilaterals $beph$ and $cgqd$ are parallelograms implying that $p+b=e+h$ and $c+q=g+d$, or

$$p=ia+b-ic,\quad q=-ia+ib+c.$$

In order to finish the proof it is enough to show that q ca be obtained by the rotation of p around a by an angle $\dfrac{\pi}{2}$, which is by T1.4 equivalent to

$$(p-a)i=p-b.$$

The last identity is easy to verify.

4. The points b_1, c_1, d_1, are obtained by rotation of b, c, d around c, d, and a for the angle $\frac{\pi}{3}$ in the positive direction. If we denote $e^{i\pi/3} = \varepsilon$ using T1.4 we get

$$(b-c)\varepsilon = b_1 - c, \quad (c-d)\varepsilon = c_1 - d, \quad (d-a)\varepsilon = d_1 - a.$$

Since p is the midpoint of b_1c_1 T6.1 gives

$$p = \frac{b_1 + c_1}{2} = \frac{\varepsilon b + c + (1-\varepsilon)d}{2}.$$

Similarly we get $q = \frac{\varepsilon c + d + (1-\varepsilon)a}{2}$. Using T6.1 again we get $r = \frac{a+b}{2}$. It is enough to prove that q can be obtained by the rotation of p around r for the angle $\frac{\pi}{3}$, in the positive direction. The last is (by T1.4) equivalent to

$$(p-r)\varepsilon = q - r,$$

which follows from

$$p - r = \frac{-a + (\varepsilon-1)b + c + (1-\varepsilon)}{2}, \quad q - r = \frac{-\varepsilon a - b + \varepsilon c + d}{2},$$

and $\varepsilon^2 - \varepsilon + 1 = 0$ (since $0 = \varepsilon^3 + 1 = (\varepsilon+1)(\varepsilon^2 - \varepsilon + 1)$).

5. Let $\varepsilon = e^{i\frac{2\pi}{3}}$. According to T1.4 we have $p_{k+1} - a_{k+1} = (p_k - a_{k+1})\varepsilon$. Hence

$$\begin{aligned} p_{k+1} &= \varepsilon p_k + (1-\varepsilon)a_{k+1} = \varepsilon(\varepsilon p_{k-1} + (1-\varepsilon)a_k) + (1-\varepsilon)a_{k+1} = \dots \\ &= \varepsilon^{k+1}p_0 + (1-\varepsilon)\sum_{i=1}^{k+1}\varepsilon^{k+1-i}a_i. \end{aligned}$$

Now we have $p_{1996} = p_0 + 665(1-\varepsilon)(\varepsilon^2 a_1 + \varepsilon a_2 + a_3)$, since $\varepsilon^3 = 1$. That means $p_{1996} = p_0$ if and only if $\varepsilon^2 a_1 + \varepsilon a_2 + a_3 = 0$. Using that $a_1 = 0$ we conclude $a_3 = -\varepsilon a_2$, and it is clear that a_2 can be obtained by the rotation of a_3 around $0 = a_1$ for the angle $\frac{\pi}{3}$ in the positive direction.

6. Since the point a is obtained by the rotation of b around o_1 for the angle $\frac{2\pi}{3} = \varepsilon$ in the positive direction, T1.4 implies $(o_1 - b)\varepsilon = o_1 - a$, i.e. $o_1 = \frac{a - b\varepsilon}{1-\varepsilon}$. Analogously

$$o_2 = \frac{b - c\varepsilon}{1-\varepsilon}, \quad o_3 = \frac{c - d\varepsilon}{1-\varepsilon}, \quad o_4 = \frac{d - a\varepsilon}{1-\varepsilon}.$$

Since $o_1o_3 \perp o_2o_4$ is equivalent to $\frac{o_1 - o_3}{\overline{o_1} - \overline{o_3}} = -\frac{o_2 - o_4}{\overline{o_2} - \overline{o_4}}$, it is enouogh to prove that

$$\frac{a - c - (b-d)\varepsilon}{\overline{a-c} - \overline{(b-d)\varepsilon}} = -\frac{b - d - (c-a)\varepsilon}{\overline{b-d} - \overline{(c-a)\varepsilon}},$$

i.e. that $(a-c)\overline{b-d} - (b-d)\overline{b-d}\,\varepsilon + (a-c)\overline{a-c}\,\overline{\varepsilon} - (b-d)\overline{a-c}\,\varepsilon\overline{\varepsilon} = -\overline{a-c}\,(b-d) + (b-d)\overline{b-d}\,\overline{\varepsilon} - (a-c)\overline{a-c}\,\varepsilon + (a-c)\overline{b-d}\,\varepsilon\overline{\varepsilon}$. The last follows from $\overline{\varepsilon} = \frac{1}{\varepsilon}$ and $|a-c|^2 = (a-c)\overline{a-c} = |b-d|^2 = (b-d)\overline{b-d}$.

7. We can assume that $a_k = \varepsilon^k$ for $0 \leq k \leq 12$, where $\varepsilon = e^{i\frac{2\pi}{15}}$. By rotation of the points a_1, a_2, and a_4 around $a_0 = 1$ for the angles ω^6, ω^5, and ω^3 (here $\omega = e^{i\pi/15}$), we get the points a_1', a_2', and a_4', such that takve da su $a_0, a_7, a_1', a_2', a_4'$ kolinearne. Sada je dovoljno dokazati da je

$$\frac{1}{a_1'-1} = \frac{1}{a_2'-1} + \frac{1}{a_4'-1} + \frac{1}{a_7-1}.$$

From T1.4 we have $a_1' - a_0 = (a_1 - a_0)\omega^6$, $a_2' - a_0 = (a_2 - a_0)\omega^5$ and $a_4' - a_0 = (a_4 - a_0)\omega^3$, as well as $\varepsilon = \omega^2$ and $\omega^{30} = 1$. We get

$$\frac{1}{\omega^6(\omega^2-1)} = \frac{1}{\omega^5(\omega^4-1)} + \frac{1}{\omega^3(\omega^8-1)} - \frac{\omega^{14}}{\omega^{16}-1}.$$

Taking the common denominator and cancelling with $\omega^2 - 1$ we see that it is enough to prove that

$$\omega^8 + \omega^6 + \omega^4 + \omega^2 + 1 = \omega(\omega^{12} + \omega^8 + \omega^4 + 1) + \omega^3(\omega^8 + 1) - \omega^{20}.$$

Since $\omega^{15} = -1 = -\omega^{30}$, we have that $\omega^{15-k} = -\omega^{30-k}$. The required statement follows from $0 = \omega^{28}+\omega^{26}+\omega^{24}+\omega^{22}+\omega^{20}+\omega^{18}+\omega^{16}+\omega^{14}+\omega^{12}+\omega^{10}+\omega^8+\omega^6+\omega^4+\omega^2+1 = \frac{\omega^{30}-1}{\omega^2-1} = 0$.

8. [Obtained from Uroš Rajković] Take the complex plane in which the center of the polygon is the origin and let $z = e^{i\frac{\pi}{k}}$. Now the coordinate of A_k in the complex plane is z^{2k}. Let p ($|p| = 1$) be the coordinate of P. Denote the left-hand side of the equality by S. We need to prove that $S = \binom{2m}{m} \cdot n$. We have that

$$S = \sum_{k=0}^{n-1} PA_k^{2m} = \sum_{k=0}^{n-1} \left|z^{2k} - p\right|^{2m}$$

Notice that the arguments of the complex numbers $(z^{2k} - p) \cdot z^{-k}$ (where $k \in \{0, 1, 2, \dots, n\}$) are equal to the argument of the complex number $(1 - p)$, hence

$$\frac{(z^{2k}-p)\cdot z^{-k}}{1-p}$$

is a positive real number. Since $|z^{-k}| = 1$ we get:

$$S = \sum_{k=0}^{n-1} |z^{2k} - p\,|^{2m} = |1 - p\,|^{2m} \cdot \sum_{k=0}^{n-1} \left(\frac{z^{2k}-p}{1-p}\right)^{2m} = |1-p\,|^{2m} \cdot \frac{\sum_{k=0}^{n-1}(z^{2k}-p)^{2m}}{(1-p)^{2m}}.$$

Since S is a positive real number we have:

$$S = \left|\sum_{k=0}^{n-1}(z^{2k} - p\,)^{2m}\right|.$$

Now from the binomial formula we have:

$$S = \left|\sum_{k=0}^{n-1}\left[\sum_{i=0}^{2m}\binom{2m}{i}\cdot z^{2ki}\cdot(-p)^{2m-i}\right]\cdot z^{-2mk}\right|.$$

After some algebra we get:

$$S = \left|\sum_{k=0}^{n-1}\sum_{i=0}^{2m}\binom{2m}{i}\cdot z^{2k(i-m)}\cdot(-p)^{2m-i}\right|,$$

or, equivalently
$$S=\left|\sum_{i=0}^{2m}\binom{2m}{i}\cdot(-p)^{2m-i}\cdot\sum_{k=0}^{n-1}z^{2k(i-m)}\right|.$$
Since for $i\neq m$ we have:
$$\sum_{k=0}^{n-1}z^{2k(i-m)}=\frac{z^{2n(i-m)}-1}{z^{2(i-m)}-1},$$
for $z^{2n(i-m)}-1=0$ and $z^{2(i-m)}-1\neq 0$, we have
$$\sum_{k=0}^{n-1}z^{2k(i-m)}=0.$$
For $i=m$ we have:
$$\sum_{k=0}^{n-1}z^{2k(i-m)}=\sum_{k=0}^{n-1}1=n.$$
From this we conclude:
$$S=\left|\binom{2m}{m}\cdot(-p)^m\cdot n\right|=\binom{2m}{m}\cdot n\cdot|(-p)^m|.$$
Using $|p|=1$ we get
$$S=\binom{2m}{m}\cdot n$$
and that is what we wanted to prove.

9. Choose the circumcircle of the triangle abc to be the unit circle. Then $o=0$ and $\overline{a}=\dfrac{1}{a}$. The first of the given relations can be written as
$$1=\frac{|a-m||b-n|}{|a-n||b-m|}\Rightarrow 1=\frac{|a-m|^2|b-n|^2}{|a-n|^2|b-m|^2}=\frac{(a-m)(\overline{a}-\overline{m})(a-n)(\overline{a}-\overline{n})}{(a-n)(\overline{a}-\overline{n})(b-m)(\overline{b}-\overline{m})}$$
After some simple algebra we get $(a-m)(\overline{a}-\overline{m})(b-n)(\overline{b}-\overline{n})=(1-\frac{m}{a}-a\overline{m}+m\overline{m})(1-\frac{n}{b}-b\overline{n}+n\overline{n})=1-\frac{m}{a}-a\overline{m}+m\overline{m}-\frac{n}{b}+\frac{mn}{ab}+\frac{a\overline{m}n}{b}-\frac{m\overline{m}n}{b}-b\overline{n}+\frac{bm\overline{n}}{a}+ab\overline{m}\,\overline{n}-bm\overline{m}\,\overline{n}+n\overline{n}-\frac{mn\overline{n}}{a}-a\overline{m}n\overline{n}+m\overline{m}n\overline{n}$. The value of the expression $(a-n)(\overline{a}-\overline{n})(b-m)(\overline{b}-\overline{m})$ we can get from the prevoius one replacing every a with b and vice versa. The initial equality now becomes:
$$\begin{aligned}&1-\frac{m}{a}-a\overline{m}+m\overline{m}-\frac{n}{b}+\frac{mn}{ab}+\frac{a\overline{m}n}{b}-\frac{m\overline{m}n}{b}-b\overline{n}+\\&\frac{bm\overline{n}}{a}+ab\overline{m}\,\overline{n}-bm\overline{m}\,\overline{n}+n\overline{n}-\frac{mn\overline{n}}{a}-a\overline{m}n\overline{n}+m\overline{m}n\overline{n}\\=\;&1-\frac{m}{b}-b\overline{m}+m\overline{m}-\frac{n}{a}+\frac{mn}{ab}+\frac{b\overline{m}n}{a}-\frac{m\overline{m}n}{a}-a\overline{n}+\frac{am\overline{n}}{b}+\\&ab\overline{m}\,\overline{n}-am\overline{m}\,\overline{n}+n\overline{n}-\frac{mn\overline{n}}{b}-b\overline{m}n\overline{n}+m\overline{m}n\overline{n}.\end{aligned}$$
Subtracting and taking $a-b$ out gives
$$\frac{m}{ab}-\overline{m}-\frac{n}{ab}+\frac{(a+b)\overline{m}n}{ab}-\frac{m\overline{m}n}{ab}+\overline{n}-\frac{(a+b)m\overline{n}}{ab}+m\overline{m}\,\overline{n}+\frac{mn\overline{n}}{ab}-\overline{m}n\overline{n}=0.$$

Since $AM/CM = AN/CM$ holds as well we can get the expression analogous to the above when every b is exchanged with c. Subtracting this expression from the previous and taking $b-c$ out we get

$$-\frac{m}{abc}+\frac{n}{abc}-\frac{\overline{m}\,n}{bc}+\frac{m\overline{m}\,n}{abc}+\frac{m\overline{n}}{bc}-\frac{mn\overline{n}}{abc}=0.$$

Writing the same expression with ac instead of bc (this can be obtained from the initial conditions because of the symmetry), subtracting, and simplifying yields $m\overline{n}-n\overline{m}=0$. Now we have $\frac{m-o}{\overline{m}-\overline{o}}=\frac{n-o}{\overline{n}-\overline{o}}$, and by T1.2 the points m,n,o are colinear.

10. [Obtained from Uroš Rajković] First we will prove that for the points p, a, and b of the unit circle the distance from p to the line ab is equal to:

$$\frac{1}{2}|(a-p)(b-p)|.$$

Denote by q the foot of perpendicular from p to ab and use T2.4 to get:

$$q=\frac{1}{2}\Big(p+a+b-\frac{ab}{p}\Big).$$

Now the required distance is equal to:

$$|q-p\,|=\frac{1}{2}\left|-p+a+b-\frac{ab}{p}\right|.$$

Since $|p\,|=1$ we can multiply the expression on the right by $-p$ which gives us:

$$\left|\frac{1}{2}(p^2-(a+b)p+ab)\right|.$$

Now it is easy to see that the required distance is indeed equal to:

$$\frac{1}{2}|(a-p)(b-p)|.$$

If we denote $z=e^{i\frac{2\pi}{2n}}$, the coordinate of A_k is z^{2k}. Now we have:

$$2\cdot h_k=|(z^{2k}-p)(z^{2k-2}-p)|.$$

The vector $(z^{2k}-p)\cdot z^{-k}$ is colinear with $1-p$, nece

$$\frac{(z^{2k}-p)\cdot z^{-k}}{1-p}$$

is a positive real number. Hence for $k\in\{1,2,\cdots,n-1\}$ it holds:

$$h_k=\frac{(z^{2k}-p)\cdot(z^{2k-2}-p)\cdot z^{-(2k-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2,$$

since $|z|=1$. We also have:

$$h_n=\frac{(1-p)\cdot(z^{2n-2}-p)\cdot z^{-(n-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2.$$

We need to prove that:

$$\sum_{k=1}^{n-1}\frac{1}{\dfrac{(z^{2k}-p)\cdot(z^{2k-2}-p)\cdot z^{-(2k-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2}=\frac{1}{\dfrac{(1-p)\cdot(z^{2n-2}-p)\cdot z^{-(n-1)}}{2\cdot(1-p)^2}\cdot|1-p|^2}.$$

After cancelling and multiplying by z we get:

$$\sum_{k=1}^{n-1}\frac{z^{2k}}{(z^{2k}-p)\cdot(z^{2k-2}-p)}=\frac{-1}{(1-p)\cdot(z^{2n-2}-p)},$$

since $z^n=-1$. Denote by S the left-hand side of the equality. We have:

$$S-\frac{1}{z^2}S=\sum_{k=1}^{n-1}\frac{(z^{2k}-p)-(z^{2k-2}-p)}{(z^{2k}-p)\cdot(z^{2k-2}-p)}.$$

This implies:

$$(1-\frac{1}{z^2})S=\sum_{k=1}^{n-1}\left(\frac{1}{z^{2k-2}-p}-\frac{1}{z^{2k}-p}\right).$$

After simplifying we get:

$$(1-\frac{1}{z^2})S=\frac{1}{1-p}-\frac{1}{z^{2n-2}-p}=\frac{(z^{2n-2}-p)-(1-p)}{(1-p)\cdot(z^{2n-2}-p)}.$$

Since $z^{2n-2}=\dfrac{1}{z^2}$ (from $z^n=1$) we get:

$$S=\frac{-1}{(1-p)\cdot(z^{2n-2}-p)},$$

q.e.d.

11. Assume that the unit circle is the circumcircle of the quadrilateral $abcd$. Since ac is its diameter we have $c=-a$. Furthermore by T2.5 we have that

$$m=\frac{ab(c+d)-cd(a+b)}{ab-cd}=\frac{2bd+ad-ab}{d+b}.$$

According to T2.3 we have that $n=\dfrac{2bd}{b+d}$, hence $m-n=\dfrac{a(d-b)}{b+d}$ and $\overline{m}-\overline{n}=\dfrac{b-d}{a(b+d)}$. Now we have

$$\frac{m-n}{\overline{m}-\overline{n}}=-\frac{a-c}{\overline{a}-\overline{c}}=a^2,$$

hence according to T1.3 $mn\perp ac$, q.e.d.

12. Assume that the unit circle is the circumcircle of the triangle abc. Using T6.3 we have $h=a+b+c$, and using T2.4 we have $e=\dfrac{1}{2}\Big(a+b+c-\dfrac{ac}{b}\Big)$. Since $paqb$ is a parallelogram the midpoints of pq and ab coincide, and according to T6.1 $q=a+b-p$ and analogously $r=a+c-p$. Since the points x,a,q are colinear, we have (using T1.2)

$$\frac{x-a}{\overline{x}-\overline{a}}=\frac{a-q}{\overline{a}-\overline{q}}=\frac{p-b}{\overline{p}-\overline{b}}=-pb,$$

or, equivalently $\overline{x} = \dfrac{pb + a^2 - ax}{abp}$. Since the points h, r, x are colinear as well, using the same theorem we get

$$\frac{x-h}{\overline{x}-\overline{h}} = \frac{h-r}{\overline{h}-\overline{r}} = \frac{b+p}{\overline{b}+\overline{p}} = bp,$$

i.e.

$$\overline{x} = \frac{x - a - b - c + p + \dfrac{bp}{a} + \dfrac{bp}{c}}{bp}.$$

Equating the expressions obtained for $\overline{x}$ we get

$$x = \frac{1}{2}\Big(2a + b + c - p - \frac{bp}{c}\Big).$$

By T1.1 it is sufficient to prove that

$$\frac{e-x}{\overline{e}-\overline{x}} = \frac{a-p}{\overline{a}-\overline{p}} = -ap.$$

The last follows from

$$e - x = \frac{1}{2}\Big(p + \frac{bp}{c} - a - \frac{ac}{b}\Big) = \frac{bcp + b^2p - abc - ac^2}{2bc} = \frac{(b+c)(bp-ac)}{2bc},$$

by conjugation.

13. We will assume that the circumcircle of the quadrilateral $abcd$ is the unit circle. Using T2.4 and T6.1 we get

$$p = a + b - \frac{ab}{c}, \quad q = a + d + \frac{ad}{c} \quad (1).$$

Let H be the orthocenter of the triangle ABD. By T6.3 we have $h = a + b + d$, hence according to T1.2 it is enough to prove that

$$\frac{p-h}{\overline{p}-\overline{h}} = \frac{q-h}{\overline{q}-\overline{h}}. \quad (2)$$

Chaning for p from (1) we get

$$\frac{p-h}{\overline{p}-\overline{h}} = \frac{a + b - \dfrac{ab}{c} - a - b - d}{\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{c}{ab} - \dfrac{1}{a} - \dfrac{1}{b} - \dfrac{1}{d}} = \frac{abd}{c},$$

and since this expression is symmetric with respect to b and d, (2) is clearly satisfied.

14. Assume that the unit circle is the circumcircle of the triangle abc and assume that a', b', c' are feet of perpendiculars from a, b, c respectively. From T2.4 we have

$$a' = \frac{1}{2}\Big(a + b + c - \frac{bc}{a}\Big), \quad b' = \frac{1}{2}\Big(a + b + c - \frac{ca}{b}\Big), \quad c' = \frac{1}{2}\Big(a + b + c - \frac{ab}{c}\Big).$$

Since a', b', c' are midpoints of ad, be, cf respectively according to T6.1 we have

$$d = b + c - \frac{bc}{a}, \quad e = a + c - \frac{ac}{b}, \quad f = a + b - \frac{ab}{c}.$$

By T1.2 the colinearity of the points d, e, f is equivalent to

$$\frac{d-e}{\overline{d}-\overline{e}} = \frac{f-e}{\overline{f}-\overline{e}}.$$

Since $d-e=b-a+\frac{ac}{b}-\frac{bc}{a}=(b-a)\frac{ab-c(a+b)}{ab}$ and similarly $f-e=(b-c)\frac{bc-a(b+c)}{bc}$, by conjugation and some algebra we get

$$\begin{aligned} 0 &= (a^2b+a^2c-abc)(c-a-b)-(c^2a+c^2b-abc)(a-b-c) \\ &= (c-a)(abc-a^2b-ab^2-a^2c-ac^2-b^2c-bc^2). \quad (1) \end{aligned}$$

Now we want to get the necessary and sufficient condition for $|h|=2$ (the radius of the circle is 1). After the squaring we get

$$\begin{aligned} 4 &= |h|^2=h\overline{h}=(a+b+c)\Big(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\Big) \\ &= \frac{a^2b+ab^2+a^2c+ac^2+b^2c+bc^2+3abc}{abc}. \quad (2) \end{aligned}$$

Now (1) is equivalent to (2), which finishes the proof.

15. Assume that the unit circle is the circumcircle of the triangle abc. Let a',b',c' be the midpoints of bc,ca,ab. Since $aa_1\perp ao$ and since a_1,b',c' are colinear, using T1.3 and T1.2, we get

$$\frac{a-a_1}{\overline{a}-\overline{a_1}}=-\frac{a-o}{\overline{a}-\overline{o}}=-a^2,\quad \frac{b'-c'}{\overline{b'}-\overline{c'}}=\frac{b'-a_1}{\overline{b'}-\overline{a_1}}.$$

From the first equality we have $\overline{a_1}=\frac{2a-a_1}{a^2}$, and since from T6.1 $b'=\frac{a+c}{2}$ and $c'=\frac{a+b}{2}$ we also have $\overline{a_1}=\frac{ab+bc+ca-aa_1}{2abc}$. By equating of the above expressions gives us $a_1=\frac{a^2(a+b+c)-3abc}{a^2-2bc}$. Similarly $b_1=\frac{b^2(a+b+c)-3abc}{2(b^2-ac)}$ and $c_1=\frac{c^2(a+b+c)-3abc}{2(c^2-2ab)}$. Now we have

$$a_1-b_1=\frac{a^2(a+b+c)-3abc}{2(a^2-bc)}-\frac{b^2(a+b+c)-3abc}{2(b^2-ac)}=-\frac{c(a-b)^3(a+b+c)}{2(a^2-bc)(b^2-ac)},$$

and it is easy to verify the condition for $a_1b_1\perp ho$, which is according to T1.3:

$$\frac{a_1-b_1}{\overline{a_1}-\overline{b_1}}=-\frac{h-o}{\overline{h}-\overline{o}}=-\frac{(a+b+c)abc}{ab+bc+ca}.$$

Similarly $a_1c_1\perp ho$, implying that the points a_1, a_2, and a_3 are colinear.

16. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $b_1=\frac{1}{2}\Big(a+b+c-\frac{ac}{b}\Big)$ and $c_1=\frac{1}{2}\Big(a+b+c-\frac{ab}{c}\Big)$, according to T6.1 $m=\frac{b+c}{2}$, and according to T6.3 $h=a+b+c$. Now we will determine the point d. Since d belongs to the chord bc according to T2.2 $\overline{d}=\frac{b+c-d}{bc}$. Furthermore, since the points b_1, c_1, and d are colinear, according to T1.2 we have

$$\frac{d-b_1}{\overline{d}-\overline{b_1}}=\frac{b_1-c_1}{\overline{b_1}-\overline{c_1}}=\frac{a\Big(\frac{b}{c}-\frac{c}{b}\Big)}{\frac{1}{a}\Big(\frac{c}{b}-\frac{b}{c}\Big)}=-a^2.$$

Now we have that $\overline{d}=\frac{a^2\overline{b_1}+b_1-d}{a^2}$, hence

$$d=\frac{a^2b+a^2c+ab^2+ac^2-b^2c-bc^2-2abc}{2(a^2-bc)}.$$

In order to prove that $dh \perp am$ (see T1.3) it is enough to prove that $\dfrac{d-h}{\overline{d}-\overline{h}} = -\dfrac{m-a}{\overline{m}-\overline{a}}$. This however follows from

$$\begin{aligned} d-h &= \frac{b^2c+bc^2+ab^2+ac^2-a^2b-a^2c-2a^3}{2(a^2-bc)} \\ &= \frac{(b+c-2a)(ab+bc+ca+a^2)}{2(a^2-bc)} \end{aligned}$$

and $m-a = \dfrac{b+c-2a}{2}$ by conjugation.

17. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $f = \dfrac{1}{2}\Big(a+b+c-\dfrac{ab}{c}\Big)$. Since a, c, p are colinear and ac is a chord of the unit circle, according to T2.2 we have $\overline{p} = \dfrac{a+c-p}{ac}$. Since $fo \perp pf$ using T1.3 we coclude

$$\frac{f-o}{\overline{f}-\overline{o}} = -\frac{p-f}{\overline{p}-\overline{f}}.$$

From the last two relations we have

$$p = f\frac{2ac\overline{f}-(a+c)}{ac\overline{f}-f} = \frac{\Big(a+b+c-\dfrac{ab}{c}\Big)c^2}{b^2+c^2}.$$

Let $\angle phf = \varphi$, then

$$\frac{f-h}{\overline{f}-\overline{h}} = e^{i2\varphi}\frac{p-h}{\overline{p}-\overline{h}}.$$

Since $p-h = -b\dfrac{ab+bc+ca+c^2}{b^2+c^2}$, and by conjugation

$$\overline{p}-\overline{h} = -\frac{c(ab+bc+ca+b^2)}{ab(b^2+c^2)},$$

$f-h = \dfrac{ab+bc+ca+c^2}{2c}$, $\overline{f}-\overline{h} = \dfrac{ab+bc+ca+c^2}{2abc}$, we see that $e^{i2\varphi} = \dfrac{c}{b}$. On the other hand we have $\dfrac{c-a}{\overline{c}-\overline{a}} = e^{i2\alpha}\dfrac{b-a}{\overline{b}-\overline{a}}$, and using T1.2 $e^{i2\alpha} = \dfrac{c}{b}$. We have proved that $\alpha = \pi+\varphi$ or $\alpha = \varphi$, and since the first is impossible, the proof is complete.

18. First we will prove the following useful lemma.

Lemma 1. *If a, b, c, a', b', c' are the points of the unit circle, then the lines aa', bb', cc' concurrent or colinear if and only if*

$$(a-b')(b-c')(c-a') = (a-c')(b-a')(c-b').$$

Proof. Let x be the intersection of aa' and bb', and let y be the intersection of the lines aa' and cc'. Using T2.5 we have

$$x = \frac{aa'(b+b')-bb'(a+a')}{aa'-bb'}, \quad y = \frac{aa'(c+c')-cc'(a+a')}{aa'-cc'}.$$

Here we assumed that these points exist (i.e. that none of $aa' \parallel bb'$ and $aa' \parallel cc'$ holds). It is obvious that the lines aa', bb', cc' are concurrent if and only if $x=y$, i.e. if and only if

$$(aa'(b+b')-bb'(a+a'))(aa'-cc') = (aa'(c+c')-cc'(a+a'))(aa'-bb').$$

After simplifying we get $aa'b+aa'b'-abb'-a'b'b-bcc'-b'cc'=aa'c+aa'c'-bc'c-bb'c'-acc'-a'cc'$, and since this is equivalent to $(a-b')(b-c')(c-a')=(a-c')(b-a')(c-b')$, the lemma is proven. $\square$

Now assume that the circumcircle of the hexagon is the unit circle. Using T1.1 we get

$$\frac{a_2-a_4}{\overline{a_2}-\overline{a_4}}=\frac{a_0-a_0'}{\overline{a_0}-\overline{a_0'}},\quad \frac{a_4-a_0}{\overline{a_4}-\overline{a_0}}=\frac{a_2-a_2'}{\overline{a_2}-\overline{a_2'}},\quad \frac{a_2-a_0}{\overline{a_2}-\overline{a_0}}=\frac{a_4-a_4'}{\overline{a_4}-\overline{a_4'}},$$

hence $a_0'=\dfrac{a_2a_4}{a_0}, a_2'=\dfrac{a_0a_4}{a_2}$ i $a_4'=\dfrac{a_0a_2}{a_4}$. Similarly, using T2.5 we get

$$a_3'=\frac{a_0'a_3(a_2+a_3)-a_2a_3(a_0'+a_3)}{a_0'a_3-a_2a_4}=\frac{a_4(a_3-a_2)+a_3(a_2-a_0)}{a_3-a_0}.$$

Analogously,

$$a_5'=\frac{a_0(a_5-a_4)+a_5(a_4-a_2)}{a_5-a_2},\quad a_1'=\frac{a_2(a_1-a_0)+a_1(a_0-a_4)}{a_1-a_4}.$$

Assume that the points a_3'', a_1'', a_5'' are the other intersection points of the unit circle with the lines a_0a_3', a_4a_1', a_2a_5' respectively. According to T1.2

$$\frac{a_3'-a_0}{\overline{a_3'}-\overline{a_0}}=\frac{a_3''-a_0}{\overline{a_3''}-\overline{a_0}}=-a_3''a_0,$$

and since $a_0-a_3'=\dfrac{a_3(2a_0-a_2-a_4)+a_2a_4-a_0^2}{a_3-a_0}$, we have

$$a_3''-a_4=\frac{(a_0-a_2)^2(a_3-a_4)}{a_0a_2(a_3-a_0)(\overline{a_0}-\overline{a_3'})},\quad a_3''-a_2=\frac{(a_0-a_4)^2(a_3-a_2)}{a_0a_4(a_3-a_0)(\overline{a_0}-\overline{a_3'})}.$$

Analogously we get

$$a_1''-a_0=a_3''-a_4=\frac{(a_2-a_4)^2(a_1-a_0)}{a_2a_4(a_1-a_4)(\overline{a_4}-\overline{a_1'})},$$

$$a_1''-a_2=a_3''-a_4=\frac{(a_4-a_0)^2(a_1-a_2)}{a_0a_4(a_1-a_4)(\overline{a_4}-\overline{a_1'})},$$

$$a_5''-a_0=a_3''-a_4=\frac{(a_2-a_4)^2(a_5-a_0)}{a_2a_4(a_5-a_0)(\overline{a_2}-\overline{a_5'})},$$

$$a_5''-a_4=a_3''-a_4=\frac{(a_0-a_2)^2(a_5-a_4)}{a_0a_2(a_5-a_4)(\overline{a_2}-\overline{a_5'})}.$$

Using the lemma and the concurrence of the lines a_0a_3, a_1a_4, and a_2a_5 (i.e. $(a_0-a_1)(a_2-a_3)(a_4-a_5)=(a_0-a_5)(a_2-a_1)(a_4-a_3)$) we get the concurrence of the lines a_0a_3'', a_4a_1'', and a_2a_5'', i.e. $(a_0-a_1'')(a_2-a_3'')(a_4-a_5'')=(a_0-a_5'')(a_2-a_1'')(a_4-a_3'')$, since they, obviously, intersect.

19. [Obtained from Uroš Rajković] Assume that the unit circle is the circumcircle of the triangle abc. If A_1, B_1, and C_1 denote the feet of the perpendiculars, we have from T2.4:

$$\begin{aligned} a_1 &= \frac{1}{2}\left(b+c+m-\frac{bc}{m}\right), \\ b_1 &= \frac{1}{2}\left(a+c+m-\frac{ac}{m}\right), \text{ and} \\ c_1 &= \frac{1}{2}\left(a+b+m-\frac{ab}{m}\right). \end{aligned}$$

We further get:

$$\frac{a_1-c_1}{b_1-c_1}=\frac{c-a+\dfrac{ab-bc}{m}}{c-b+\dfrac{ab-ac}{m}}=\frac{(c-a)(m-b)}{(c-b)(m-a)}=\frac{\overline{a}_1-\overline{c}_1}{\overline{b}_1-\overline{c}_1},$$

and, according to T1.2, the points A_1, B_1, and C_1 are colinear.

20. The quadrilateral $ABCD$ is cyclic, and we assume that it's circumcircle is the unti circle. Let a_1, a_2, and a_3 denote the feet of the perpendiculars from a to bc, cd, and db respectively. Denote by b_1, b_2, and b_3 the feet of the perpendiculars from b to ac, cd, and da respectively. According to T2.4 we have that

$$a_1=\frac{1}{2}\Big(a+b+c-\frac{bc}{a}\Big),\ a_2=\frac{1}{2}\Big(a+b+d-\frac{bd}{a}\Big),\ a_3=\frac{1}{2}\Big(a+c+d-\frac{cd}{a}\Big)$$

$$b_1=\frac{1}{2}\Big(b+a+c-\frac{ac}{b}\Big),\ b_2=\frac{1}{2}\Big(b+c+d-\frac{cd}{b}\Big),\ b_3=\frac{1}{2}\Big(b+d+a-\frac{da}{b}\Big)$$

The point x can be obtained from the condition for colinearity. First from the colinearity of x, a_1, a_2 and T1.2 we have that

$$\frac{x-a_1}{\overline{x}-\overline{a_1}}=\frac{a_1-a_2}{\overline{a_1}-\overline{a_2}}=\frac{\frac{1}{2}\Big(c-d+\dfrac{bd}{a}-\dfrac{bc}{a}\Big)}{\frac{1}{2}\Big(\dfrac{1}{c}-\dfrac{1}{d}+\dfrac{a}{bd}-\dfrac{a}{bc}\Big)}=\frac{bcd}{a},$$

and after simplifying

$$\overline{x}=\frac{x-\frac{1}{2}\Big(a+b+c+d-\dfrac{abc+acd+abd+bcd}{a^2}\Big)}{bcd}a.$$

Similarly from the colinearity of the points x, b_1, and b_2 we get

$$\overline{x}=\frac{x-\frac{1}{2}\Big(a+b+c+d-\dfrac{abc+acd+abd+bcd}{b^2}\Big)}{acd}b,$$

and from this we conclude

$$x=\frac{1}{2}\Big(a+b+c+d\Big).$$

Let $h=a+c+d$ (by T6) be the orthocenter of the triangle acd. In order to finish the proof, according to T1.2 it is enough to show that

$$\frac{x-c}{\overline{x}-\overline{c}}=\frac{h-c}{\overline{h}-\overline{c}}=\frac{a+b+d-c}{\overline{a}+\overline{b}+\overline{d}-\overline{c}}.$$

On the other hand $x-c=\frac{1}{2}\Big(a+b+d-c\Big)$, from which the equality is obvious.

21. Using the last problem we have that the intersection of the lines $l(a;bcd)$ and $l(b;cda)$ is the point $x=\frac{1}{2}\Big(a+b+c+d\Big)$, which is a symmetric expression, hence this point is the intersection of every two of the given lines.

22. Using the last two problems we get the locus of points is the set of all the points of the form $x=\frac{1}{2}\Big(a+b+c+d\Big)$, when d moves along the circle. That is in fact the circle with the radius $\frac{1}{2}$

and center $\dfrac{a+b+c}{2}$, which is the midpoint of the segment connecting the center of the given circle with the orthocenter of the triangle abc.

23. Assume that the unit circle is the circumcircle of the triangle abc. From T1.3 and the condition $ad \perp ao$ we have that
$$\frac{d-a}{\overline{d}-\overline{a}} = -\frac{a-o}{\overline{a}-\overline{o}} = -a^2,$$
and after simplifying $\overline{d} = \dfrac{2a-d}{a^2}$. Since the points b, c, d are colinear and bc is the chord of the unit circle, according to T2.2 $\overline{d} = \dfrac{b+c-d}{bc}$, and solving the given system we get $d = \dfrac{a^2(b+c)-2abc}{a^2-bc}$. Since e belongs to the perpendicular bisector of ab we have $oe \perp ab$. According to T1.3 and $\dfrac{e-o}{\overline{e}-\overline{o}} = -\dfrac{a-b}{\overline{a}-\overline{b}} = ab$, i.e. $\overline{e} = \dfrac{e}{ab}$. From $be \perp bc$, using T1.3 again we get $\dfrac{b-e}{\overline{b}-\overline{e}} = -\dfrac{b-c}{\overline{b}-\overline{c}} = bc$, or equivalently $\overline{e} = \dfrac{c-b+e}{bc} = \dfrac{e}{ab}$. Hence $e = \dfrac{a(c-b)}{c-a}$. Similarly we have $f = \dfrac{a(b-c)}{b-a}$. Using T1.2 we see that it is enough to prove that $\dfrac{d-f}{\overline{d}-\overline{f}} = \dfrac{f-e}{\overline{f}-\overline{e}}$. Notice that
$$\begin{aligned} d-f &= \frac{a^2(b+c)-2abc}{a^2-bc} - \frac{a(b-c)}{b-a} = \frac{a^2b^2+3a^2bc-ab^2c-2a^3b-abc^2}{(a^2-bc)(b-a)} \\ &= \frac{ab(a-c)(b+c-2a)}{(a^2-bc)(b-a)}, \end{aligned}$$
and similarly $d-e = \dfrac{ac(a-b)(b+c-2a)}{(a^2-bc)(c-a)}$. After conjugation we see that the required condition is easy to verify.

24. [Obtained from Uroš Rajković] Assume that the unit circle is the incircle of the hexagon $ABCDEF$. After conjugating and using T2.5 we get:
$$\overline{m} = \frac{a+b-(d+e)}{ab-de},\ \overline{n} = \frac{b+c-(e+f)}{bc-ef},\ \overline{p} = \frac{c+d-(f+a)}{cd-fa},$$
hence:
$$\overline{m} - \overline{n} = \frac{(b-e)(bc-cd+de-ef+fa-ab)}{(ab-de)(bc-ef)},$$
and analogously:
$$\overline{n} - \overline{p} = \frac{(c-f)(cd-de+ef-fa+ab-bc)}{(bc-ef)(cd-fa)}.$$
From here we get:
$$\frac{\overline{m}-\overline{n}}{\overline{n}-\overline{p}} = \frac{(b-e)(cd-fa)}{(f-c)(ab-de)}.$$
Since the numbers $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$, $\overline{e}$, and $\overline{f}$ are equal to $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\frac{1}{e}$, and $\frac{1}{f}$, respectively, we see that it is easy to verify that the complex number on the left-hand side of the last equality equal to its complex conjugate, hence it is real. Now according to T1.2 the points M, N, and P are colinear, q.e.d.

25. Assume that the quadrilateral $abcd$ is inscribed in the unit circle. Using T2.5 we get

$$\begin{aligned} e &= \frac{ab(c+d)-cd(a+b)}{ab-cd}, \\ f &= \frac{ad(b+c)-bc(a+d)}{ad-bc} mbox and \\ g &= \frac{ac(b+d)-bd(a+c)}{ac-bd}. \quad (1) \end{aligned}$$

In order to prove that $o=0$ is the orthocenter of the triangle efg, it is enough to prove that $of \perp eg$ and $og \perp ef$. Because of the symmetry it is enough to prove one of these two relateions. Hence, by T1.3 it is enough to prove that

$$\frac{f-o}{\overline{f}-\overline{o}} = \frac{e-g}{\overline{e}-\overline{g}} \quad (2).$$

From (1) we have that

$$\frac{f-o}{\overline{f}-\overline{o}} = \frac{\dfrac{ad(b+c)-bc(a+d)}{ad-bc}}{\dfrac{(b+c)-(a+d)}{bc-ad}} = \frac{ad(b+c)-bc(a+d)}{a+d-(b+c)}, (3)$$

or equivalently

$$\begin{aligned} e-g &= \frac{(a-d)(ab^2d-ac^2d)+(b-c)(bcd^2-a^2bc)}{(ab-cd)(ac-bd)} \\ &= \frac{(a-d)(b-c)((b+c)ad-(a+d)bc)}{(ab-cd)(ac-bd)} \quad (4) \end{aligned}$$

and by conjugation

$$\overline{e}-\overline{g} = \frac{(a-d)(b-c)(b+c-(a+d))}{(ab-cd)(ac-bd)} \quad (5).$$

Comparing the expressions (3),(4), and (5) we derive the statement.

26. Assume that the unit circle is the circumcircle of the triangle abc and assume that $a=1$. Then $c=\overline{b}$ and $t=-1$. Since p belongs to the chord bc, using T2.2 we get that $\overline{p}=b+\dfrac{1}{b}-p$. Since x belongs to the chord ab, in the similar way we get $\overline{x}=\dfrac{1+b-x}{b}$. Since $px \parallel ac$ by T1.1 we have

$$\frac{p-x}{\overline{p}-\overline{x}} = \frac{a-c}{\overline{a}-\overline{c}} = -\frac{1}{b},$$

i.e. $\overline{x}=pb+\overline{p}-xb$. From this we get $x=\dfrac{b(p+1)}{b+1}$. Similarly we derive $y=\dfrac{p+1}{b+1}$. According to T1.3 it remains to prove that $\dfrac{x-y}{\overline{x}-\overline{y}} = -\dfrac{p-t}{\overline{p}-\overline{t}} = -\dfrac{p+1}{\overline{p}+1}$. This follows from $x-y=\dfrac{(p+1)(b-1)}{b+1}$ and by conjugation

$$\overline{x}-\overline{y} = \frac{(\overline{p}+1)\left(\dfrac{1}{b}-1\right)}{\dfrac{1}{b}+1} = -\frac{(\overline{p}+1)(b-1)}{b+1}.$$

27. Assume that the unit circle is the circumcircle of the quadrilateral $abcd$. Using T6.1 we have $k=\dfrac{a+b}{2}$, $l=\dfrac{b+c}{2}$, $m=\dfrac{c+a}{2}$ and $n=\dfrac{d+a}{2}$. We want to determine the coordinate of the orthocenter of the triangle akn. Let h_1 be that point and denote by h_2, h_3, and h_4 the orthocenters of bkl, clm, and dmn respectively. Then $kh_1 \perp an$ and $nh_1 \perp ak$. By T1.3 we get

$$\frac{k-h_1}{\overline{k}-\overline{h_1}} = -\frac{a-n}{\overline{a}-\overline{n}} \text{ and } \frac{n-h_1}{\overline{n}-\overline{h_1}} = -\frac{a-k}{\overline{a}-\overline{k}}. \quad (1)$$

Since

$$\frac{a-n}{\overline{a}-\overline{n}} = \frac{a-d}{\overline{a}-\overline{d}} = -ad,$$

we have that

$$\overline{h_1} = \frac{\overline{k}\,ad - k + h_1}{ad}.$$

Similarly from the second of the equations in (1) we get

$$\overline{h_1} = \frac{\overline{n}\,ab - n + h_1}{ab}.$$

Solving this system gives us that

$$h_1 = \frac{2a+b+d}{2}.$$

Symmetricaly

$$h_2 = \frac{2b+c+a}{2},\ h_3 = \frac{2c+d+b}{2},\ h_4 = \frac{2d+a+c}{2},$$

and since $h_1 + h_3 = h_2 + h_4$ using T6.1 the midpoints of the segments h_1h_3 and h_2h_4 coincide hence the quadrilateral $h_1h_2h_3h_4$ is a parallelogram.

28. Assume that the unit circle is the circumcircle of the triangle abc. By T2.3 we have that $a = \dfrac{2em}{e+m}$ i $b = \dfrac{2mk}{m+k}$. Let's find the point p. Since the points m, k, and p are colinear and mk is the chord of the unit circle, by T2.2 we have that $\overline{p} = \dfrac{m+k-p}{mk}$. Furthermore the points p, e, and c are colinear. However, in this problem it is more convenient to notice that $pe \perp oe$ and now using T1.3 we have

$$\frac{e-p}{\overline{e}-\overline{p}} = -\frac{e-o}{\overline{e}-\overline{o}} = -e^2$$

and after simplifying $\overline{p} = \dfrac{2e-p}{e^2}$. Equating the two expressions for $\overline{p}$ we get

$$p = e\frac{(m+k)e-2mk}{e^2-mk}.$$

In order to finish the proof using T1.3 it is enough to prove that $\dfrac{p-o}{\overline{p}-\overline{o}} = -\dfrac{e-b}{\overline{e}-\overline{b}}$. This will follow from

$$e - b = \frac{e(m+k)-2mk}{m+k},$$

and after conjugating $\overline{e} - \overline{b} = \dfrac{m+k-2e}{(m+k)e}$ and $\overline{p} = \dfrac{m+k-2e}{mk-e^2}$.

29. Assume that the circle inscribed in $abcd$ is the unit one. From T2.3 we have that

$$a = \frac{2nk}{n+k}, \quad b = \frac{2kl}{k+l}, \quad c = \frac{2lm}{l+m}, \quad d = \frac{2mn}{m+n}. \quad (1)$$

Using T2.5 we get

$$s = \frac{kl(m+n)-mn(k+l)}{kl-mn}. \quad (2)$$

According to T1.1 it is enough to verify that

$$\frac{s-o}{\overline{s}-\overline{o}}=\frac{b-d}{\overline{b}-\overline{d}}.$$

From (1) we have that $b-d=2\dfrac{kl(m+n)-mn(k+l)}{(k+l)(m+n)}$ (3), and after conjugating $\overline{b}-\overline{d}=\dfrac{m+n-(k+l)}{(k+l)(m+n)}$ (4). From (2) we have that

$$\frac{s}{\overline{s}}=\frac{kl(m+n)-mn(k+l)}{kl-mn},\quad (5)$$

and comparing the expressions (3),(4), and (5) we finish the proof.

30. [Obtained from Uroš Rajković] Let P be the point of tangency of the incircle with the line BC. Assume that the incircle is the unit circle. By T2.3 the coordinates of A, B, and C are respectively

$$a=\frac{2qr}{q+r},\quad b=\frac{2pr}{p+r}\text{ i } c=\frac{2pq}{p+q}.$$

Furthermore, using T6.1 we get $x=\dfrac{1}{2}(b+c)=\dfrac{pr}{p+r}+\dfrac{pq}{p+q}$, $y=\alpha b=\alpha\dfrac{2pr}{p+r}$, and $z=\beta c=\beta\dfrac{2pq}{p+q}$ $(\alpha,\beta\in R)$. The values of α and β are easy to compute from the conditions $y\in rq$ and $z\in rq$:

$$\alpha=\frac{(p+r)(q+r)}{2(p+q)r}\text{ i }\beta=\frac{(p+q)(r+q)}{2(p+r)q}.$$

From here we get the coordinates of y and z using p, q, and r:

$$y=\frac{p(q+r)}{(p+q)}\text{ and } z=\frac{p(r+q)}{(p+r)}.$$

We have to prove that:

$$\angle RAQ=60^\circ\iff XYZ\text{ is equilateral.}$$

The first condition is equivalent to $\angle QOR=60^\circ$ i.e. with

$$r=q\cdot e^{i2\pi/3}.$$

The second condition is equivalent to $(z-x)=(y-x)\cdot e^{i\pi/3}$. Notice that:

$$y-x=\frac{p(q+r)}{(p+q)}-\left(\frac{pr}{p+r}+\frac{pq}{p+q}\right)=\frac{pr(r-q)}{(p+q)(p+r)}\text{ and}$$

$$z-x=\frac{p(p+q)}{(p+r)}-\left(\frac{pr}{p+r}+\frac{pq}{p+q}\right)=\frac{pq(q-r)}{(p+q)(p+r)}.$$

Now the second condition is equivalent to:

$$\frac{pq(q-r)}{(p+q)(p+r)}=\frac{pr(r-q)}{(p+q)(p+r)}\,e^{i\pi/3},$$

i.e. with $q=-r\,e^{i\pi/3}$. It remains to prove the equivalence:

$$r=q\,e^{i2\pi/3}\iff q=-r\,e^{i\pi/3},$$

which obviously holds.

31. According to T1.1 it is enough to prove that

$$\frac{m-o}{\overline{m}-\overline{o}}=\frac{n-o}{\overline{n}-\overline{o}}.$$

If p,q,r,s are the points of tangency of the incircle with the sides ab,bc,cd,da respectively using T2.3 we get

$$m=\frac{a+c}{2}=\frac{ps}{p+s}+\frac{qr}{q+r}=\frac{pqs+prs+pqr+qrs}{(p+s)(q+r)},$$

and after conjugating $\overline{m}=\dfrac{p+q+r+s}{(p+s)(q+r)}$ and

$$\frac{m}{\overline{m}}=\frac{pqr+ps+prs+qrs}{p+q+r+s}.$$

Since the last expression is symmetric in p,q,r,s we conclude that $\dfrac{m}{\overline{m}}=\dfrac{n}{\overline{n}}$, as required.

32. Assume that the incircle of the quadrilateral $abcd$ is the unit circle. We will prove that the intersection of the lines mp and nq belongs to bd. Then we can conlude by symmetry that the point also belongs to ac, which will imply that the lines mp, nq, ac, and bd are concurrent. Using T2.3 we have that

$$b=\frac{2mn}{m+n},\quad d=\frac{2pq}{p+q}.$$

If x is the intersection point of mp and nq, using T2.5 we get

$$x=\frac{mp(n+q)-nq(m+p)}{mp-nq}.$$

We have to prove that the points x,b,d are colinear, which is according to T1.2 equivalent to saying that

$$\frac{b-d}{\overline{b}-\overline{d}}=\frac{b-x}{\overline{b}-\overline{x}}.$$

This follows from $b-d=\dfrac{2mn}{m+n}-\dfrac{2pq}{p+q}=2\dfrac{mn(p+q)-pq(m+n)}{(m+n)(p+q)}$ and

$$\begin{aligned} b-x &= \frac{2mn}{m+n}-\frac{mp(n+q)-nq(m+p)}{mp-nq} \\ &= \frac{m^2np-mn^2q-m^2pq+n^2pq+m^2nq-mn^2p}{(mp-nq)(m+n)} \\ &= \frac{(m-n)(mn(p+q)-pq(m+n))}{(m+n)(mp-nq)}, \end{aligned}$$

by conjugation.

33. Assume that the unit circle is the incumcircle of the triangle abc. Using T7.3 we have that the circumecenter has the coordinate

$$o=\frac{2def(d+e+f)}{(d+e)(e+f)(f+d)}.$$

Let's calculate the coordinate of the circumcenter o_1 of the triangle xyz. First, according to T6.1 we have that $x=\dfrac{e+f}{2}$, $y=\dfrac{d+f}{2}$ and $z=\dfrac{d+e}{2}$. Moreover by T1.3 we have that $\dfrac{o_1-\frac{x+y}{2}}{\overline{o_1}-\overline{\frac{x+y}{2}}}=$

$-\dfrac{x-y}{\overline{x}-\overline{y}} = \dfrac{(e-d)/2}{(\overline{e}-\overline{d})/2} = -ed$, and simplifying

$$\overline{o_1} = \frac{-\dfrac{f}{2} + \dfrac{ed}{2f} + o_1}{ed},$$

and similarly $\overline{o_1} = \dfrac{-\dfrac{d}{2} + \dfrac{ef}{2d} + o_1}{ef}$. By equating we get $o_1 = \dfrac{e+f+d}{2}$. Now by T1.2 it is enough to prove that $\dfrac{o_1-i}{\overline{o_1}-\overline{i}} = \dfrac{o-i}{\overline{o}-\overline{i}}$, which can be easily obtained by conjugation of the previous expressions for o and o_1.

34. Assume that the incircle of the triangle abc is the unit circle. Using T7.1 we get $b = \dfrac{2fd}{f+d}$ and $c = \dfrac{2ed}{e+d}$. From some elemetary geometry we conclude that k is the midpoint of segment ef hence by T6.1 we have $k = \dfrac{e+f}{2}$. Let's calculate the coordinate of the point m. Since m belongs to the chord fd by T2.2 we have $\overline{m} = \dfrac{f+d-m}{fd}$. Similarly we have that the points b, m, k are colinear and by T1.2 we get $\dfrac{k-m}{\overline{k}-\overline{m}} = \dfrac{b-k}{\overline{b}-\overline{k}}$, i.e. $\overline{m} = m\dfrac{\overline{b}-\overline{k}}{b-k} + \dfrac{\overline{k}b - k\overline{b}}{b-k}$. Now equating the expressions for $\overline{m}$ one gets

$$m = \frac{(f+d)(b-k) + (k\overline{b} - \overline{k}b)fd}{(\overline{b}-\overline{k})fd + b - k}.$$

Since $b - k = \dfrac{3fd - de - f^2 - ef}{2(f+d)}$ and $k\overline{b} - \overline{k}b = \dfrac{(e+f)(e-d)fd}{e(f+d)}$ we get

$$m = \frac{4ef^2d + efd^2 - e^2d^2 - e^2f^2 - 2f^2d^2 - f^3e}{6efd - e^2d - ed^2 - ef^2 - e^2f - d^2f - df^2}$$

and symmetrically

$$n = \frac{4e^2fd + efd^2 - f^2d^2 - e^2f^2 - 2e^2d^2 - e^3f}{6efd - e^2d - ed^2 - ef^2 - e^2f - d^2f - df^2}.$$

By T1.3 it is enough to prove that $\dfrac{m-n}{\overline{m}-\overline{n}} = -\dfrac{i-d}{\overline{i}-\overline{d}} = -d^2$. This however follows from

$$m - n = \frac{(e-f)(4efd - ed^2 - fd^2 - fe^2 - f^2e)}{6efd - e^2d - ed^2 - ef^2 - e^2f - d^2f - df^2},$$

by conjugation.

35. Assume that the unit circle is the inrcumcircle of the triangle abc. Assume that k, l, and m are the points of tangency of the incircle with the sides bc, ca, and ab, respectively. By T7 we have that

$$o = \frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)}, \quad h = \frac{2(k^2l^2 + l^2m^2 + m^2k^2 + klm(k+l+m))}{(k+l)(l+m)(m+k)}.$$

Since the segments io and bc are parallel we have that $io \perp ik$, which is by T1.3 equivalent to $\dfrac{o-i}{\overline{o}-\overline{i}} = -\dfrac{k-i}{\overline{k}-\overline{i}} = -k^2$. After conjugating the last expression for o becomes

$$klm(k+l+m) + k^2(kl+lm+mk) = 0. \quad (*)$$

Let's prove that under this condition we have $ao \parallel hk$. According to T1.1 it is enough to prove that $\frac{a-o}{\overline{a}-\overline{o}} = \frac{h-k}{\overline{h}-\overline{k}}$. According to T7.1 we have that $a = \frac{2ml}{m+l}$, and

$$a - o = \frac{2ml}{m+l} - \frac{2klm(k+l+m)}{(k+l)(l+m)(m+k)} = \frac{2m^2l^2}{(k+l)(l+m)(m+k)}.$$

Now we get that it is enough to prove that

$$\frac{h-k}{\overline{h}-\overline{k}} = \frac{l^2m^2}{k^2}.$$

Notice that

$$\begin{aligned} h-k &= \frac{2(k^2l^2+l^2m^2+m^2k^2+klm(k+l+m))}{(k+l)(l+m)(m+k)} - k \\ &= \frac{k^2l^2+k^2m^2+2l^2m^2+k^2lm+kl^2m+klm^2-k^2l-k^3m-k^2lm}{(k+l)(l+m)(m+k)} \\ &= \frac{klm(k+l+m)-k^2(k+l+m)+k^2l^2+2l^2m^2+m^2l^2}{(k+l)(l+m)(m+k)} \\ &= \Big(\text{according to (*)}\Big) = \frac{(kl+lm+mk)^2+l^2m^2}{(k+l)(l+m)(m+k)} \\ &= \Big(\text{according to (*)}\Big) = \frac{(kl+lm+mk)^2((k+l+m)^2+k^2)}{(k+l+m)^2(k+l)(l+m)(m+k)}. \end{aligned}$$

After conjugating the last expression for $h-k$ we get

$$\overline{h} - \overline{k} = \frac{(k+l+m)^2+k^2}{(k+l)(l+m)(m+k)},$$

and using the last expression for $h-k$ we get

$$\frac{h-k}{\overline{h}-\overline{k}} = \frac{(kl+lm+mk)^2}{(k+l+m)^2} = \Big(\text{by (*)}\Big) = \frac{l^2m^2}{k^2},$$

which completes the proof.

36. Assume that the incircle of the triangle abc is the unit circle. Then using T7.1 we have $c = \frac{2t_1t_2}{t_1+t_2}$. Our goal is to first determine the point h_3. From $h_3t_3 \perp it_3$ by T1.3 we have

$$\frac{h_3-t_3}{\overline{h_3}-\overline{t_3}} = -\frac{t_3-i}{\overline{t_3}-\overline{i}} = -t_3^2,$$

i.e. $\overline{h_3} = \frac{2t_3-h_3}{t_3^2}$. Furthermore from $ch_3 \parallel it_3$ and T1.1 we have $\frac{h_3-c}{\overline{h_3}-\overline{c}} = \frac{t_3-i}{\overline{t_3}-\overline{i}} = t_3^2$. Writing the similar expression for $\overline{h_3}$ gives

$$h_3 = \frac{1}{2}\Big(2t_3 + c - \overline{c}\,t_3^2\Big) = t_3 + \frac{t_1t_2-t_3^2}{t_1+t_2}.$$

Similarly we obtain $h_2 = t_2 + \frac{t_1t_3-t_2^2}{t_1+t_3}$. In order to determine the line symmetric to h_2h_3 with repsect to t_2t_3 it is enough to determine the points symmetric to h_2 and h_3 with respect to t_2t_3. Assume that p_2 and p_3 are these two points and let h_2' and h_3' be the feet of perpendiculars from h_2

and h_3 to the line t_2t_3 respectively. According to T2.4 we have $h_2' = \frac{1}{2}\Big(t_2 + t_3 - t_2t_3\overline{h_3}\Big)$ hence by T6.1

$$p_2 = 2h_2' - h_2 = \frac{t_1(t_2^2 + t_3^2)}{t_2(t_1 + t_3)}$$

and symmetrically $p_3 = \frac{t_1(t_2^2 + t_3^2)}{t_3(t_1 + t_2)}$. Furthermore

$$p_2 - p_3 = \frac{t_1^2(t_2^2 + t_3^2)(t_3 - t_2)}{t_1t_3(t_1 + t_2)(t_1 + t_3)},$$

and if the point x belongs to p_2p_3 by T1.2 the following must be satisfied:

$$\frac{x - p_2}{\overline{x} - \overline{p_2}} = \frac{p_2 - p_3}{\overline{p_2} - \overline{p_3}} = -t_1^2.$$

Specifically if x belongs to the unit circle we also have $\overline{x} = \frac{1}{x}$, hence we get the quadratic equation

$$t_2t_3x^2 - t_1(t_2^2 + t_3^2)x + t_1^2t_2t_3 = 0.$$

Its solutions are $x_1 = \frac{t_1t_2}{t_3}$ and $x_2 = \frac{t_1t_3}{t_2}$ and these are the intersection points of the line p_2p_3 with the unit circle. Similarly we get $y_1 = \frac{t_1t_2}{t_3}$, $y_2 = \frac{t_2t_3}{t_1}$, and $z_1 = \frac{t_3t_1}{t_2}$, $z_2 = \frac{t_2t_3}{t_1}$, which finishes the proof.

37. Assume that the circumcircle of the triangle abc is the unit circle. Let u, v, w be the complex numbers described in T8. Using this theorem we get that $l = -(uv + vw + wu)$. By elementary geometry we know that the intersection of the line al and the circumcircle of the triangle abc is the midpoint of the arc bc which doesn't contain the point a. That means $a_1 = -vw$ and similarly $b_1 = -uw$ and $c_1 = -uv$.

(a) The statement follows from the equality

$$1 = \frac{|l - a_1| \cdot |l - c_1|}{|l - b|} = \frac{|u(v + w)| \cdot |w(u + v)|}{|uv + uw + vw + v^2|} = \frac{|v + w| \cdot |u + v|}{|(u + v)(v + w)|} = 1.$$

(b) If x is the point of the tangency of the incircle with the side bc then x is the foot of the perpendicular from the point l l to the side bc and T2.4 implies $x = \frac{1}{2}\Big(b + c + l - bc\overline{l}\Big)$ and consequently $r = |l - x| = \frac{1}{2}\left|\frac{(u + v)(v + w)(w + u)}{u}\right| = \frac{1}{2}|(u + v)(v + w)(w + u)|$. Now the required equality follows from

$$\begin{aligned}\frac{|l - a| \cdot |l - b|}{|l - c_1|} &= \frac{|(u + v)(u + w)| \cdot |(u + v)(v + w)|}{|w(u + v)|} \\ &= |(u + v)(v + w)(w + u)|.\end{aligned}$$

(c) By T5 we have that

$$S(ABC) = \frac{i}{4}\begin{vmatrix} u^2 & 1/u^2 & 1 \\ v^2 & 1/v^2 & 1 \\ w^2 & 1/w^2 & 1 \end{vmatrix} \text{ i } S(A_1B_1C_1) = \frac{i}{4uvw}\begin{vmatrix} vw & u & 1 \\ uw & v & 1 \\ uv & w & 1 \end{vmatrix},$$

hence

$$\begin{aligned}\frac{S(ABC)}{S(A_1B_1C_1)} &= \frac{u^4w^2+w^4v^2+v^4u^2-v^4w^2-u^4v^2-w^4u^2}{uvw(v^2w+uw^2+u^2v-uv^2-u^2w-vw^2)}\\ &= \frac{(u^2-v^2)(uw+vw-uv-w^2)(uw+vw+uv+w^2)}{uvw(u-v)(uv+w^2-uw-vw)}\\ &= -\frac{(u+w)(vw+uw+uv+w^2)}{uvw}\\ &= -\frac{(u+v)(v+w)(w+u)}{uvw}.\end{aligned}$$

Here we consider the oriented surface areas, and substracting the modulus from the last expression gives us the desired equality.

38. First solution. Assume that the circumcircle of the triangle abc is the unit circle and u, v, w are the complex numbers described in T8. Let d, e, f be the points of tangency of the incircle with the sides bc, ca, ab respectively. By T2.4 we have that $f = \frac{1}{2}\Big(a+b+z-ab\overline{z}\Big) = \frac{1}{2}\Big(u^2+v^2+w^2-uv-vw-wu+\frac{uv(u+v)}{2w}\Big)$. By symmetry we get the expressions for e and f and by T6.1 we get

$$k = \frac{1}{3}\Big(u^2+v^2+w^2-uv-vw-wu+\frac{uv(u+v)}{2w}+\frac{vw(v+w)}{2u}-\frac{wu(w+u)}{2v}\Big) =$$
$$= \frac{(uv+vw+wu)(u^2v+uv^2+uw^2+u^2w+v^2w+vw^2-4uvw)}{6uvw}.$$

Now it is easy to verify $\dfrac{z-o}{\overline{z}-\overline{o}} = \dfrac{k-o}{\overline{k}-\overline{o}}$, which is by T1.2 the condition for colinearity of the points z, k, o. Similarly we also have

$$\begin{aligned}\frac{|o-z|}{|z-k|} &= \frac{|uv+vw+wu|}{\left|\dfrac{(uv+vw+wu)(u^2v+uv^2+uw^2+u^2w+v^2w+vw^2+2uvw)}{6uvw}\right|}\\ &= \frac{6}{|(u+v)(v+w)(w+u)|} = \frac{6R}{2r} = \frac{3R}{r},\end{aligned}$$

which completes the proof.

Second solution. Assume that the incircle of the triangle abc is the unit circle and let d, e, f denote its points of tangency with the sides bc, ca, ab respectively. According to T7.3 we have that $o = \dfrac{2def(d+e+f)}{(d+e)(e+f)(f+d)}$ and according to T6.1 $k = \dfrac{d+e+f}{3}$. Now it is easy to verify that $\dfrac{o-z}{\overline{o}-\overline{z}} = \dfrac{k-z}{\overline{k}-\overline{z}}$ which is by T1.2 enough to establish the colinearity of the points o, z, k. We also have that

$$\frac{|o-z|}{|z-k|} = \frac{\left|\dfrac{d+e+f}{(d+e)(e+f)(f+d)}\right|}{\left|\dfrac{d+e+f}{3}\right|} = \frac{3}{|(d+e)(e+f)(f+d)|} = \frac{3R}{r}.$$

39. Assume that the circumcircle of the triangle abc is the unit circle and let u, v, w be the complex numbers described in T8 (here $p = w^2$). According to this theorem we have $i = -uv - vw - wu$. Since $|a-c| = |a-b|$ by T1.4 it holds

$$c - a = e^{i\angle cab}(b-a).$$

By the same theorem we have

$$\frac{-vw-u^2}{\overline{-vw}-\overline{u^2}}=e^{i2\frac{\angle pab}{2}}\frac{v^2-u^2}{\overline{v^2}-\overline{u^2}},$$

hence $e^{i\angle pab}=-\dfrac{w}{v}$. Now we have

$$c=\frac{u^2w+u^2v-v^2w}{v},$$

and symmetrically $d=\dfrac{v^2w+v^2u-u^2w}{u}$. By T1.3 it is enough to prove that

$$\frac{c-d}{\overline{c}-\overline{d}}=-\frac{o-i}{\overline{o}-\overline{i}}=-\frac{uv+vw+wu}{u+v+w}uvw.$$

This follows from $c-d=\dfrac{(u^2-v^2)(uv+vw+wu)}{uv}$ by conjugation.

40. Assume that the circumcircle of the triangle abc is the unit circle. By T8 there are numbers u,v,w such that $a=u^2,b=v^2,c=w^2$ and the incenter is $i=-(uv+vw+wu)$. If o' denotes the foot of the perpendicular from o to bc then by T2.4 we have $o'=\dfrac{1}{2}\Big(b+c\Big)$, and by T6.1 $o_1=2o'=b+c=v^2+w^2$. By T1.2 the points a,i,o_1 are colinear if and only if

$$\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{a-i}{\overline{a}-\overline{i}}.$$

Since

$$\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{o_1-a}{\overline{o_1}-\overline{a}}=\frac{v^2+w^2-u^2}{u^2(v^2+w^2)-v^2w^2}u^2v^2w^2\ mboxand$$

$$\frac{a-i}{\overline{a}-\overline{i}}=\frac{u(u+v+w)+vw}{vw+uw+uv+u^2}u^2vw=u^2vw,$$

we get

$$v^3w+vw^3-u^2vw-(u^2v^2+u^2w^2-v^2w^2)=(vw-u^2)(v^2+w^2+vw)=0.$$

This means that either $vw=u^2$ or $v^2+w^2+vw=0$. If $vw=u^2$ then by T6.1 the points u^2 and $-vw$ belong to the same radius hence abc is isosceles contrary to the assumption. This means that $v^2+w^2+vw=0$. We now want to prove that the triangle with the vertices $o,-vw,w^2$ is equilateral. It is enough to prove that $1=|w^2+vw|=|v+w|$ which is equivalent to $1=(v+w)(\overline{v}+\overline{w})=\dfrac{(v+w)^2}{vw}$ and this to $v^2+w^2+vw=0$. Since $\angle boc=120^\circ$ we have $\alpha=60^\circ$.

41. Assume that the incumcircle of the triangle abc is the unit circle. According to T8 there are complex numbers u,v,w such that $p=u^2,q=v^2,r=w^2$ and $p_1=-vw,q_1=-wu,r_1=-uv$. Then $p_2=vw,q_2=wu,r_2=uv$. By T7.1 we gave

$$a=\frac{2v^2w^2}{v^2+w^2},\ b=\frac{2w^2u^2}{w^2+u^2}\text{ i }c=\frac{2u^2w^2}{u^2+w^2},$$

hence by T6.1

$$a_1=\frac{w^2u^2}{w^2+u^2}+\frac{u^2v^2}{u^2+v^2},\ b_1=\frac{u^2v^2}{u^2+v^2}+\frac{v^2w^2}{v^2+w^2},\ c_2=\frac{v^2w^2}{v^2+w^2}+\frac{w^2u^2}{w^2+u^2}.$$

If the point n is the intersection of the lines a_1p_1 and b_1q_1 then the triplets of points (n, a_1, p_1) and (n, b_1, q_1) are colinear and using T1.2 we get

$$\frac{n-a_1}{\overline{n}-\overline{a_1}}=\frac{a_1-p_1}{\overline{a_1}-\overline{p_1}},\quad \frac{n-b_1}{\overline{n}-\overline{b_1}}=\frac{b_1-q_1}{\overline{b_1}-\overline{q_1}}.$$

Solving this system gives us

$$\begin{aligned} n &= \frac{u^4v^4+v^4w^4+w^4u^4}{(u^2+v^2)(v^2+w^2)(w^2+u^2)}+\\ &\quad \frac{uvw(u^3v^2+u^2v^3+u^3w^2+u^2w^3+v^3w^2+v^2w^3)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)}+\\ &\quad \frac{3u^2v^2w^2(u^2+v^2+w^2)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)}+\\ &\quad \frac{2u^2v^2w^2(uv+vw+wu)}{(u^2+v^2)(v^2+w^2)(w^2+u^2)}. \end{aligned}$$

Since the above expression is symmetric this point belongs to c_1r_1. The second part of the problem can be solved similarly.

42. Assume that a is the origin. According to T1.4 we have $c''-a=e^{i\pi/2}(c-a)$, i.e. $c''=ic$. Similarly we get $b''=-ib$. Using the same theorem we obtain $x-c=e^{i\pi/2}(b-c)$, i.e. $x=(1-i)c+ib$ hence by T6.1 $p=\frac{1+i}{2}b+\frac{1-i}{2}c$. Denote by q the intersection of the lines bc and ap. Then the points a,p,q are colinear as well as the points b,c'',q. Using T1.2 we get

$$\frac{a-p}{\overline{a}-\overline{p}}=\frac{a-q}{\overline{a}-\overline{q}},\quad \frac{b-c''}{\overline{b}-\overline{c''}}=\frac{q-b}{\overline{q}-\overline{b}}.$$

From the first equation we conclude that $\overline{q}=q\frac{(1-i)\overline{b}+(1+i)\overline{c}}{(1+i)b+(1-i)c}$, and from the second $\overline{q}=\frac{q(\overline{b}+i\overline{c})-i(\overline{b}c+b\overline{c})}{b-ic}$. These two imply

$$q=\frac{i(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{2(ib\overline{b}-2b\overline{c}+2\overline{b}c+2ic\overline{c})}=\frac{(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{(b-ic)(\overline{b}+i\overline{c})}.$$

Denote by q' the intersection of ap and cb''. Then the points a,p,q' are colinear as well as the points b'',c,q'. Hence by T1.2

$$\frac{a-p}{\overline{a}-\overline{p}}=\frac{a-q'}{\overline{a}-\overline{q'}},\quad \frac{b''-c}{\overline{b''}-\overline{c}}=\frac{q-c}{\overline{q}-\overline{c}}.$$

The first equation gives $\overline{q'}=q'\frac{(1-i)\overline{b}+(1+i)\overline{c}}{(1+i)b+(1-i)c}$, and the second $\overline{q}=\frac{q(\overline{c}-i\overline{b})+i(\overline{b}c+b\overline{c})}{c+ib}$. By the equating we get

$$q'=\frac{(\overline{b}c+b\overline{c})((1+i)b+(1-i)c)}{(b-ic)(\overline{b}+i\overline{c})},$$

hence $q=q'$, q.e.d.

43. Assume that the origin is the intersection of the diagonals, i.e. $o=0$. From the colinearity of a,o,c and b,o,d using T1.2 we get $a\overline{c}=\overline{a}c$ and $b\overline{d}=\overline{b}d$. By T6.1 we get $m=\frac{a+b}{2}$ and

$n=\dfrac{c+d}{2}$. Since $om\perp cd$ and $on\perp ab$ by T1.3

$$\frac{\frac{c+d}{2}-o}{\frac{\overline{c}+\overline{d}}{2}-\overline{o}}=-\frac{a-b}{\overline{a}-\overline{b}},\quad \frac{\frac{a+b}{2}-o}{\frac{\overline{a}+\overline{b}}{2}-\overline{o}}=-\frac{c-d}{\overline{c}-\overline{d}}.$$

From these two equations we get

$$c=\frac{da(\overline{a}\,b-2b\overline{b}+a\overline{b})}{b(\overline{a}\,b-2a\overline{a}+a\overline{b})}\ \text{ and } c=\frac{da(\overline{a}\,b+2b\overline{b}+a\overline{b})}{b(\overline{a}\,b+2a\overline{a}+a\overline{b})}.$$

The last two expressions give $(\overline{a}\,b+a\overline{b})(a\overline{a}-b\overline{b})=0$. We need to prove that the last condition is sufficient to guarantee that a,b,c,d belong to a circle. According to T3 the last is equivalent to

$$\frac{c-d}{\overline{c}-\overline{d}}\,\frac{b-a}{\overline{b}-\overline{a}}=\frac{b-d}{\overline{b}-\overline{d}}\,\frac{c-a}{\overline{c}-\overline{a}}.$$

Since the points b,d,o are colinear, by T1.2 $\dfrac{b-d}{\overline{b}-\overline{d}}=\dfrac{b-o}{\overline{b}-\overline{o}}=\dfrac{b}{\overline{b}}$ we get $\dfrac{a-c}{\overline{a}-\overline{c}}=\dfrac{a-o}{\overline{a}-\overline{o}}=\dfrac{a}{\overline{a}}$. If $a\overline{b}+\overline{a}\,b=0$ then

$$c-d=d\frac{2ab(\overline{a}-\overline{b})}{b(\overline{a}\,b-2a\overline{a}+a\overline{b})},$$

and the last can be obtained by conjugation. If $a\overline{a}=b\overline{b}$, then

$$c-d=\frac{d(a-b)(\overline{a}\,b+a\overline{b})}{b(\overline{a}\,b-2a\overline{a}+a\overline{b})},$$

and in this case we can get the desired statement by conjugation.

44. Let f be the origin and let $d=\overline{c}$ (this is possible since $FC=FD$). According to T9.2 we have that

$$o_1=\frac{ad(\overline{a}-\overline{d})}{\overline{a}\,d-a\overline{d}},\quad o_2=\frac{bc(\overline{b}-\overline{c})}{\overline{b}\,c-b\overline{c}}.$$

Since $cd\parallel af$ according to T1.1 $\dfrac{a-f}{\overline{a}-\overline{f}}=\dfrac{c-d}{\overline{c}-\overline{d}}=-1$, i.e. $\overline{a}=-a$ and similarly $\overline{b}=-b$. Now we have

$$o_1=\frac{\overline{c}\,(a+c)}{c+\overline{c}},\quad o_2=\frac{c(b+\overline{c})}{c+\overline{c}}.$$

Let's denote the point e. From T1.2 using the colinearity of a,c,e and b,d,e we get the following two equations

$$\frac{a-c}{\overline{a}-\overline{c}}=\frac{e-a}{\overline{e}-\overline{a}},\quad \frac{b-d}{\overline{b}-\overline{d}}=\frac{e-b}{\overline{e}-\overline{b}}.$$

From these equations we get $\overline{e}=\dfrac{a(c+\overline{c})-e(a+\overline{c})}{a-c}$ and $\overline{e}=\dfrac{b(c+\overline{c})-e(b+c)}{b-\overline{c}}$. By equating these two we get

$$e=\frac{a\overline{c}-bc}{a+\overline{c}-b-c}.$$

Using T1.3 the condition $fe\perp o_1o_2$ is equivalent to $\dfrac{o_1-o_2}{\overline{o_1}-\overline{o_2}}=-\dfrac{f-e}{\overline{f}-\overline{e}}$, which trivially follows from $o_1-o_2=\dfrac{a\overline{c}-cb}{c+\overline{c}}$ by conjugation.

45. Assume that the point p is the origin. Let ac be the real axis and let $\angle cpd = \varphi$. Then $a = \alpha, b = \beta e^{i\varphi}, c = \gamma, d = \delta e^{i\varphi}$, where $\alpha, \beta, \gamma, \delta$ are some real numbers. Let $e^{i\varphi} = \Pi$. If $|a - f| = \varepsilon|a - d|$, then $|e - c| = \varepsilon|b - c|$ hence by T6.1 $a - f = \varepsilon(a - d)$ and $e - c = \varepsilon(b - c)$. Thus we have

$$f = \alpha(1 - \varepsilon) + \varepsilon\delta\Pi,\ e = \gamma(1 - \varepsilon) + \varepsilon\beta\Pi.$$

Since q belongs to pd we have that $q = \varrho\Pi$ and since q also belongs to ef by T1.2 we have that $\dfrac{f - q}{\overline{f} - \overline{q}} = \dfrac{e - f}{\overline{e} - \overline{f}}$, hence

$$\frac{\alpha(1 - \varepsilon) + (\varepsilon\delta - \varrho)\Pi}{\alpha(1 - \varepsilon) + (\varepsilon\delta - \varrho)\dfrac{1}{\Pi}} = \frac{(1 - \varepsilon)(\alpha - \gamma) + \varepsilon(\delta - \beta)\Pi}{(1 - \varepsilon)(\alpha - \gamma) + \varepsilon(\delta - \beta)\dfrac{1}{\Pi}}.$$

After some algebra we get $(\Pi - \dfrac{1}{\Pi})(1 - \varepsilon)\Big[(\alpha - \gamma)(\varepsilon\delta - \varrho) - \varepsilon\alpha(\delta - \beta)\Big] = 0$. Since $\Pi \neq \pm 1$ (because $\angle CPD < 180°$) and $\varepsilon \neq 1$ we get $\varrho = \varepsilon\Big[\delta - \dfrac{\alpha(\delta - \beta)}{\alpha - \gamma}\Big]$. Similarly we get $\rho = (1 - \varepsilon)\Big[\alpha - \dfrac{\delta(\alpha - \gamma)}{\delta - \beta}\Big]$, where ρ is the coordinate of the point r. By T9.2 we have

$$\begin{aligned} o_1 &= \frac{rq(\overline{r} - \overline{q})}{\overline{r}q - \overline{q}} = \frac{\rho\varrho\Pi(\rho - \varrho\dfrac{1}{\Pi})}{\rho\varrho\Pi - \rho\varrho\dfrac{1}{\Pi}} = \frac{\rho\Pi - \varrho}{\Pi^2 - 1}\Pi \\ &= \frac{(1 - \varepsilon)\Big[\alpha - \dfrac{\delta(\alpha - \gamma)}{\delta - \beta}\Big]\Pi - \varepsilon\Big[\delta - \dfrac{\alpha(\delta - \beta)}{\alpha - \gamma}\Big]}{\Pi^2 - 1}\Pi. \end{aligned}$$

For any other position of the point e on the line ad such that $ae = \epsilon ad$ the corresponding center of the circle has the coordinate

$$o_2 = \frac{(1 - \epsilon)\Big[\alpha - \dfrac{\delta(\alpha - \gamma)}{\delta - \beta}\Big]\Pi - \epsilon\Big[\delta - \dfrac{\alpha(\delta - \beta)}{\alpha - \gamma}\Big]}{\Pi^2 - 1}\Pi.$$

Notice that the direction of the line o_1o_2 doesn't depend on ε and ϵ. Namely if we denote $A = \alpha - \dfrac{\delta(\alpha - \gamma)}{\delta - \beta}$ and $B = \delta - \dfrac{\alpha(\delta - \beta)}{\alpha - \gamma}$ we have

$$\frac{o_1 - o_2}{\overline{o_1} - \overline{o_2}} = -\frac{A\Pi + B}{A + B\Pi}\Pi.$$

Thus for every three centers o_1, o_2, o_3 it holds $o_1o_2 \| o_2o_3$ hence all the centers are colinear. Since all the circles have a common point, the circles have another common point.
Remark. We have proved more than we've been asked. Namely two conditions $AD = BC$ and $BE = DF$ are substituted by one $BE/BC = DF/AD$.
Another advantage of this solutions is that we didn't have to guess what is the other intersection point.

46. Let o be the origin. According to the property T9.1 we have that $h_1 = \dfrac{(a - b)(\overline{a}b + a\overline{b})}{a\overline{b} - \overline{a}b}$, $h_2 = \dfrac{(c - d)(\overline{c}d + c\overline{d})}{c\overline{d} - \overline{c}d}$, and according to the theorem 6 $t_1 = \dfrac{a + c}{3}$, $t_2 = \frac{b+d}{3}$. Since the points a, c, and o are colinear as well as the points b, d, and o by T1.2 we have $\overline{c} = \dfrac{c\overline{a}}{a}$, $\overline{d} = \dfrac{d\overline{b}}{b}$, hence

$h_2 = \dfrac{(c-d)(a\overline{b}+\overline{a}\,b)}{a\overline{b}-\overline{a}\,b}$. In order to prove that $t_1t_2 \perp h_1h_2$, by T1.3, it is enough to verify

$$\frac{t_1-t_2}{\overline{t_1}-\overline{t_2}} = -\frac{h_1-h_2}{\overline{h_1}-\overline{h_2}}.$$

This follows from

$$h_1-h_2 = \frac{a\overline{b}+\overline{a}\,b}{a\overline{b}-\overline{a}\,b}\Big(a+c-b-d\Big),$$

by conjugation.

47. Let Γ be the unit circle. Using T2.3 we get $c = \dfrac{2ab}{a+b}$. Let o_1 be the center of Γ_1. Then $o_1b \perp ab$ (because ab is a tangent) hence by T1.3 $\dfrac{o_1-b}{\overline{o_1}-\overline{b}} = -\dfrac{a-b}{\overline{a}-\overline{b}} = ab$. After simplifying $\overline{o_1} = \dfrac{o_1+a-b}{ab}$. We have also $|o_1-b| = |o_1-c|$, and after squaring $(o_1-b)(\overline{o_1}-\overline{b}) = (o_1-c)(\overline{o_1}-\overline{c})$, i.e. $\overline{o_1} = \dfrac{o_1}{b^2} - \dfrac{a-b}{b(a+b)}$. Now we have

$$o_1 = \frac{ab}{a+b}+b.$$

Since the point m belongs to the unit circle it satisfies $\overline{m} = \dfrac{1}{m}$ and since it belongs to the circle with the center o_1 it satisfies $|o_1-m| = |o_1-b|$. Now we have

$$\overline{o_1}\,m^2 - \Big(\frac{o_1}{b}+\overline{o_1}\,b\Big)m + o_1 = 0.$$

This quadratic equation defines both m and b, and by Vieta's formulas we have $b+m = \dfrac{o_1}{\overline{o_1}\,b}+b$, i.e.

$$m = b\,\frac{2a+b}{a+2b}.$$

It remains to prove that the points a, m, and the midpoint of the segment bc colinear. The midpoint of bc is equal to $(b+c)/2$ by T6.1. According to T1.2 it is enough to prove that

$$\frac{a-\dfrac{b+c}{2}}{\overline{a}-\overline{\dfrac{b+c}{2}}} = \frac{a-m}{\overline{a}-\overline{m}} = -am,$$

which is easy to verify.

48. Assume that the circle k is unit and assume that $b=1$. The $a=-1$ and since $p \in k$ we have $\overline{p} = \dfrac{1}{p}$. According to T2.4 we have that $q = \dfrac{1}{2}\Big(p+\dfrac{1}{p}\Big)$, and according to T6.1 we have that $f = \dfrac{\Big(p+\dfrac{1}{p}\Big)-1}{2} = \dfrac{(p-1)^2}{4p}$. Furthermore since c belongs to the circle with the center p and radius $|p-q|$ we have $|p-q| = |p-c|$ and after squaring

$$(p-q)(\overline{p}-\overline{q}) = (p-c)(\overline{p}-\overline{c}).$$

Since $c \in k$ we have $\overline{c} = \dfrac{1}{c}$. The relation $p-q = \dfrac{1}{2}\Big(p-\dfrac{1}{p}\Big)$ implies

$$4pc^2-(p^4+6p^2+1)c+4p^3 = 0.$$

Notice that what we obtained is the quadratic equation for c. Since d satisfies the same conditions we used for c, then the point d is the second solution of this quadratic equation. Now from Vieta's formulas we get

$$c+d=\frac{p^4+6p^2+1}{4p^3},\quad cd=p^2.$$

Since the point g belongs to the chord cd by T2.2 we get

$$\overline{g}=\frac{c+d-g}{cd}=\frac{p^4+6p^2+1-4pg}{4p^3}.$$

From $gf\perp cd$ T1.3 gives $\dfrac{g-f}{\overline{g}-\overline{f}}=-\dfrac{c-d}{\overline{c}-\overline{d}}=cd=p^2$. Solving this system gives us

$$g=\frac{p^3+3p^2-p+1}{4p}.$$

The necessair and sufficient condition for colinearity of the points a,p,g is (according to T1.2) $\dfrac{a-g}{\overline{a}-\overline{g}}=\dfrac{a-p}{\overline{a}-\overline{p}}=p$. This easily follows from $a-g=\dfrac{p^3+3p^3+3p+1}{4p}$ and by conjugating $\overline{a}-\overline{g}=\dfrac{1+3p+3p^2+p^3}{4p^2}$. Since e belongs to the chord cd we have by T2.2 $\overline{e}=\dfrac{c+d-g}{cd}=\dfrac{p^4+6p^2+1-4pe}{4p^3}$, and since $pe\perp ab$ T1.3 implies $\dfrac{e-p}{\overline{e}-\overline{p}}=-\dfrac{a-b}{\overline{a}-\overline{b}}=-1$, or equivalently $\overline{e}=p+\frac{1}{p}-e$. It follows that $e=\dfrac{3p^2+1}{4p}$. Since $p-q=\dfrac{p^2-1}{2p}=2\dfrac{p^2-1}{4p}=2\Big(e-q\Big)$, we get $|e-p|=|e-q|$. Furthermore since $g-e=\dfrac{p^2-1}{4}$ from $|p|=1$, we also have $|e-q|=|g-e|$, which finishes the proof.

49. Assume that the circle with the diameter bc is unit and that $b=-1$. Now by T6.1 we have that $b+c=0$, i.e. $c=1$, and the origin is the midpoint of the segment bc. Since p belongs to the unit circle we have $\overline{p}=\dfrac{1}{p}$, and since $pa\perp p0$, we have according to T1.3 $\dfrac{a-p}{\overline{a}-\overline{p}}=-\dfrac{p-0}{\overline{p}-\overline{0}}=-p^2$. Simplification yields

$$\overline{a}p^2-2p+a=0.$$

Since this quadratic equation defines both p and q, according to Vieta's formulas we have

$$p+q=\frac{2}{\overline{a}},\quad pq=\frac{a}{\overline{a}}.$$

Let h' be the intersection of the perpendicular from a to bc with the line pq. Since $h'\in pq$ T2.2 gives $\overline{h'}=\dfrac{p+q-h'}{pq}=\dfrac{2-\overline{a}h}{a}$. Since $ah\perp bc$ according to T1.3 we have $\dfrac{a-h}{\overline{a}-\overline{h}}=-\dfrac{b-c}{\overline{b}-\overline{c}}=-1$, i.e. $\overline{h}=a+\overline{a}-h$. Now we get

$$h=\frac{a\overline{a}+a^2-2}{a-\overline{a}}.$$

It is enough to prove that $h'=h$, or $ch\perp ab$ which is by T1.3 equivalent to $\dfrac{h-c}{\overline{h}-\overline{c}}=-\dfrac{a-b}{\overline{a}-\overline{b}}$. The last easily follows from

$$h-1=\frac{a\overline{a}+a^2-2-a+\overline{a}}{a-\overline{a}}=\frac{(a+1)(a+\overline{a}-2)}{a-\overline{a}}$$

and $a-b=a+1$ by conjugation.

50. Assume that the origin of our coordinate system is the intersection of the diagonals of the rectangle and that the line ab is parallel to the real axis. We have by T6.1 $c + a = 0$, $d + b = 0$, $c = \overline{b}$, and $d = \overline{a}$. Since the points $p, a, 0$ are colinear T1.2 implies $\frac{p}{\overline{p}} = \frac{a}{\overline{a}}$, i.e. $\overline{p} = -\frac{b}{a}p$. Let $\varphi = \angle dpb = \angle pbc$. By T1.4 we have

$$\frac{c-p}{\overline{c}-\overline{p}} = e^{i2\varphi}\frac{b-p}{\overline{b}-\overline{p}}, \quad \frac{p-b}{\overline{p}-\overline{b}} = e^{i2\varphi}\frac{c-b}{\overline{c}-\overline{b}},$$

and after multiplying these equalities and expressing in terms of a and b

$$\frac{p+b}{bp+a^2} = \frac{a(p-b)^2}{(bp-a^2)^2}.$$

In the polynomial form this writes as

$$(b^2-ab)p^3 + p^2(b^3-2a^2b-a^3+2ab^2) + p(a^4-2a^2b^2-ab^3+2a^3b) + a^4b - a^3b^2$$
$$= (b-a)(bp^3 + (a^2+3ab+b^2)p^2 - ap(a^2+3ab+b^2) - a^3b) = 0.$$

Notice that a is one of those points p which satisfy the angle condition. Hence a is one of the zeroes of the polynomial. That means that p is the root of the polynomial which is obtained from the previous one after division by $p-a$ i.e. $bp^2 + (a^2+3ab+b^2)p + a^2b = 0$. Let's now determine the ratio $|p-b| : |p-c|$. From the previous equation we have $bp^2 + a^2b = -(a^2+3ab+b^2)$, hence

$$\frac{PB^2}{PC^2} = \frac{(p-b)(\overline{p}-\overline{b})}{(p-c)(\overline{p}-\overline{c})} = \frac{bp^2-(a^2+b^2)p+a^2b}{bp^2+2abp+a^b} = \frac{-2(a^2+b^2+2ab)}{-(a^2+b^2+2ab)} = 2,$$

and the required ratio is $\sqrt{2} : 1$.

51. Assume first that the quadrilateral $abcd$ is cyclic and that its cicrumcircle is the unit circle. If $\angle abd = \varphi$ and $\angle bda = \theta$ by T1.4 after squaring we have

$$\frac{d-b}{\overline{d}-\overline{b}} = e^{i2\varphi}\frac{a-b}{\overline{a}-\overline{b}}, \quad \frac{c-b}{\overline{c}-\overline{b}} = e^{i2\varphi}\frac{p-b}{\overline{p}-\overline{b}},$$

$$\frac{c-d}{\overline{c}-\overline{d}} = e^{i2\theta}\frac{p-d}{\overline{p}-\overline{d}}, \quad \frac{b-d}{\overline{b}-\overline{d}} = e^{i2\theta}\frac{a-d}{\overline{a}-\overline{d}}.$$

From the first of these equalities we get $e^{i2\varphi}\frac{a}{d}$, and from the fourth $e^{i2\theta} = \frac{b}{a}$. From the second equality we get $\overline{p} = \frac{ac+bd-pd}{abc}$, and from the third $\overline{p} = \frac{ac+bd-pb}{acd}$. Now it follows that

$$p = \frac{ac+bd}{b+d}.$$

We have to prove that $|a-p|^2 = (a-p)(\overline{a}-\overline{p}) = |c-p|^2 = (c-p)(\overline{c}-\overline{p})$, which follows from

$$a-p = \frac{ab+ad-ac-bd}{b+d}, \quad \overline{a}-\overline{p} = \frac{cd+bc-bd-ac}{ac(b+d)},$$

$$c-p = \frac{bc+cd-ac-bd}{b+d}, \quad \overline{c}-\overline{p} = \frac{ad+ab-bd-ac}{ac(b+d)}.$$

Assume that $|a-p| = |c-p|$. Assume that the circumcircle of the triangle abc is unit. Squaring the last equality gives us that $a\overline{p} + \frac{p}{a} = c\overline{p} + \frac{p}{c}$, i.e. $(a-c)(\overline{p} - \frac{p}{ac}) = 0$. This means that $\overline{p} = \frac{p}{ac}$. Let

d belong to the chord $d'c$. Then according to T2.2 $\overline{d} = \dfrac{c+d'-d}{cd'}$. By the condition of the problem we have $\angle dba = \angle cbp = \varphi$ and $\angle adb = \angle pdc = \theta$, and squaring in T1.4 yields

$$\frac{a-b}{\overline{a}-\overline{b}} = e^{i2\varphi}\frac{d-b}{\overline{d}-\overline{b}}, \quad \frac{p-b}{\overline{p}-\overline{b}} = e^{i2\varphi}\frac{c-b}{\overline{c}-\overline{b}},$$

$$\frac{b-d}{\overline{b}-\overline{d}} = e^{i2\theta}\frac{a-d}{\overline{a}-\overline{d}}, \quad \frac{c-d}{\overline{c}-\overline{d}} = e^{i2\theta}\frac{p-d}{\overline{p}-\overline{d}}.$$

Multiplying the first two equalities gives us

$$\frac{a-b}{\overline{a}-\overline{b}}\frac{c-b}{\overline{c}-\overline{b}} = ab^2c = \frac{p-b}{\overline{p}-\overline{b}}\frac{d-b}{\overline{d}-\overline{b}}.$$

After some algebra we conclude

$$p = \frac{ac+bd-b(ac\overline{d}+b)}{d-b^2\overline{d}} = \frac{bdd'+acd'-abd'-abc+abd-b^2d'}{cd'd-b^2d'+b^2d-b^2c}.$$

Since the pionts d, c, d' are colinear, according to T1.2 we get $\dfrac{d-c}{\overline{d}-\overline{c}} = \dfrac{c-d'}{\overline{c}-\overline{d'}} = -cd'$, and myltiplying the third and fourth equality gives

$$(-cd')(d-a)(\overline{d}-\overline{b})(\overline{d}-\overline{p}) - (\overline{d}-\overline{a})(d-b)(d-p) = 0.$$

Substituting values for p gives us a polynomial f in d. It is of the most fourth degree and observing the coefficient next to d^4 of the left and right summand we get that the polynomial is of the degree at most 3. It is obvious that a and b are two of its roots. We will now prove that its third root is d' and that would imply $d = d'$. For $d = d'$ we get

$$p = \frac{bd'd+acd'-abc-b^2d'}{c(d'^2-b^2)} = \frac{ac+bd'}{b+d'}, \quad d-p = \frac{d'^2-ac}{b+d'}$$

$$\overline{d}-\overline{p} = -bd'\frac{d'^2-ac}{ac(b+d')} \quad \frac{d-a}{\overline{d}-\overline{a}} = -d'a, \quad \frac{d-b}{\overline{d}-\overline{b}} = -d'b$$

and the statement is proved. Thus $d = d'$ hence the quadrilateral $abcd'$ is cyclic.

52. Since the rectangles $a_1b_2a_2b_1$, $a_2b_3a_3b_2$, $a_3b_4a_4b_3$, and a_4, b_1, a_1, b_4 are cyclic T3 implies that the numbers

$$\frac{a_1-a_2}{b_2-a_2} : \frac{a_1-b_1}{b_2-b_1}, \quad \frac{a_2-a_3}{b_3-a_3} : \frac{a_2-b_2}{b_3-b_2},$$

$$\frac{a_3-a_4}{b_4-a_4} : \frac{a_3-b_3}{b_4-b_3}, \quad \frac{a_4-a_1}{b_1-a_1} : \frac{a_4-b_4}{b_1-b_4},$$

are real. The product of the first and the third divided by the product of the second and the fourth is equal to

$$\frac{a_1-a_2}{a_2-a_3}\cdot\frac{a_3-a_4}{a_4-a_1}\cdot\frac{b_2-b_1}{b_3-b_2}\cdot\frac{b_4-b_3}{b_1-b_4},$$

and since the points a_1, a_2, a_3, a_4 lie on a circle according to the theorem 4 the number $\dfrac{a_1-a_2}{a_2-a_3}\cdot\dfrac{a_3-a_4}{a_4-a_1}$ is real, hence the number $\dfrac{b_2-b_1}{b_3-b_2}\cdot\dfrac{b_4-b_3}{b_1-b_4}$ is real as well. According to T3 the points b_1, b_2, b_3, b_4 are cyclic or colinear.

53. Assume that the origin is the intersection of the diagonals of the parallelogram. Then $c = -a$ and $d = -b$. Since the triangles cde and fbc are similar and equally orientged by T4

$$\frac{c-b}{b-f} = \frac{e-d}{d-c},$$

hence $f = \dfrac{be + c^2 - bc - cd}{e-d} = \dfrac{be + a^2}{e+b}$. In order for triangles cde and fae to be similar and equally oriented (as well as for fbc and fae), according to T4 it is necessairy and sufficient that the following relation holds:

$$\frac{c-d}{d-e} = \frac{f-a}{a-e}.$$

The last equaliy follows from

$$f - a = \frac{be + a^2 - ea - ab}{e+b} = \frac{(e-a)(b-a)}{e+b},$$

and $c - d = c + b$, $d - e = -(b+e)$, $c + b = b - a$.

54. Let $p = 0$ and $q = 1$. Since $\angle mpq = \alpha$, according to T1.4 we have that $\dfrac{q-p}{\overline{q}-\overline{p}} = e^{i2\alpha}\dfrac{m-p}{\overline{m}-\overline{p}}$, i.e. $\dfrac{m}{\overline{m}} = e^{i2\alpha}$. Since $\angle pqm = \beta$, the same theorem implies $\dfrac{m-q}{\overline{m}-\overline{q}} = e^{i2\beta}\dfrac{p-q}{\overline{p}-\overline{q}}$, i.e. $1 = e^{i2\beta}\dfrac{m-1}{\overline{m}-1}$. Solving this system (with the aid of $e^{i2(\alpha+\beta+\gamma)} = 1$) we get

$$m = \frac{e^{i2(\alpha+\gamma)} - 1}{e^{i2\gamma} - 1},$$

and symmetrically

$$l = \frac{e^{i2(\beta+\gamma)} - 1}{e^{i2\beta} - 1}, \quad k = \frac{e^{i2(\alpha+\beta)} - 1}{e^{i2\alpha} - 1}.$$

According to T4 in order to prove that the triangles klm and kpq are similar and equally oriented it is enough to prove that

$$\frac{k-l}{l-m} = \frac{k-p}{p-q} = -k.$$

The last follows from

$$\begin{aligned}
\frac{k-l}{l-m} &= \frac{\dfrac{e^{i(2\alpha+4\beta)} - e^{i2\beta} - e^{i(2\alpha+2\beta)} + e^{i(2\beta+2\gamma)} + e^{i2\alpha} - 1}{(e^{i2\alpha}-1)(e^{i2\beta}-1)}}{\dfrac{e^{i(2\beta+4\gamma)} - e^{i2\gamma} - e^{i(2\beta+2\gamma)} + e^{i(2\alpha+2\gamma)} + e^{i2\beta} - 1}{(e^{i2\beta}-1)(e^{i2\gamma}-1)}} \\
&= \frac{e^{i2(\alpha+\beta)}\left(e^{i(2\beta+4\gamma)} - e^{i2\gamma} - e^{i(2\beta+2\gamma)} + e^{i(2\alpha+2\gamma)} + e^{i2\beta} - 1\right)}{e^{i(2\beta+4\gamma)} - e^{i2\gamma} - e^{i(2\beta+2\gamma)} + e^{i(2\alpha+2\gamma)} + e^{i2\beta} - 1} \cdot \\
&\quad \frac{e^{i2\gamma}-1}{e^{i2\alpha}-1} \\
&= \frac{1 - e^{i2(\alpha+\beta)}}{e^{i2\alpha}-1} = -k.
\end{aligned}$$

Since the triangles kpq, qlp, pqm are mutually similar and equally oriented the same holds for all four of the triangles.

55. Assume that the coordinates of the vertices of the i-th polygon are denoted by $a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}$, respectively in positive direction. smeru. According to T6.1 and the given recurrent relation we have that for each i and k:

$$a_i^{(k+1)} = 2a_{i+k}^{(k)} - a_i^{(k)},$$

where the indices are modulo n. Our goal is to determine the value of $a_i^{(n)}$, using the values of $a_1^{(1)}, a_2^{(1)}, \dots, a_n^{(1)}$. The following

$$\begin{aligned} a_i^{(k+1)} &= 2a_{i+k}^{(k)} - a_k^{(i)} = 4a_{i+k+k-1}^{(k-1)} - 2a_{i+k}^{(k-1)} - 2a_{i+k-1}^{(k-1)} + a_i^{(k-1)} \\ &= 4(2a_{i+k+k-1+k-2}^{(k-2)} - -a_{i+k+k-1}^{(k-2)}) - 2(2a_{i+k+k-2}^{(k-2)} - a_{i+k}^{(k-2)}) - \\ &\quad 2(2a_{i+k-1+k-2}^{(k-2)} - a_{i+k-1}^{(k-2)}) + 2a_{i+k-2}^{(k-2)} - a_i^{(k-2)} \\ &= 8a_{i+k+k-1+k-2}^{(k-2)} - 4(a_{i+k+k-1}^{(k-2)} + a_{i+k+k-2}^{(k-2)} + a_{i+k-1+k-2}^{(k-2)}) + \\ &\quad 2(a_{i+k}^{(k-2)} + a_{i+k-1}^{(k-2)} + a_{i+k-2}^{(k-2)}) - a_i^{(k-2)}, \end{aligned}$$

yields that

$$a_i^{(k)} = 2^{k-1}s_k^{(k)}(i) - 2^{k-2}s_{k-1}^{(k)}(i) + \ldots + (-1)^k s_0^{(k)}(i),$$

where $s_j^{(k)}(i)$ denotes the sum of all the numbers of the form $a_{i+s_k(j)}$ and $s_k(j)$ is one of the numbers obtained as the sum of exactly j different natural numbers not greater than n. Here we assume that $s_0^{(k)}(i) = a_i$. The last formula is easy to prove by induction. Particularly, the formula holds for $k = n$ hence

$$a_i^{(n)} = 2^{n-1}s_n^{(n)}(i) - 2^{n-2}s_{n-1}^{(n)}(i) + \ldots + (-1)^n s_0^{(n)}(i).$$

Now it is possible to prove that $s_l^{(n)}(i) = s_l^{(n)}(j)$, for each $1 \le l \le n-1$ which is not very difficult problem in the number theory. Since n is prime we have that $n + n - 1 + \ldots + 1$ is divisible by n hence

$$\begin{aligned} a_i^{(n)} - a_j^{(n)} &= 2^{n-1}a_{i+n+n-1+\ldots+1}^{(1)} - 2^{n-1}a_{j+n+n-1+\ldots+1}^{(1)} + \\ &\quad (-1)^n a_i^{(1)} - (-1)^n a_j^{(1)} \\ &= (2^{n-1} + (-1)^n)(a_i^{(1)} - a_j^{(1)}), \end{aligned}$$

which by T4 finishes the proof.

56. Assume that the pentagon $abcde$ is inscribed in the unit circle and that x, y, and z are feet of perpendiculars from a to bc, cd, and de respectively. According to T2.4 we have that

$$x = \frac{1}{2}\Big(a + b + c - \frac{bc}{a}\Big), \quad y = \frac{1}{2}\Big(a + c + d - \frac{cd}{a}\Big), \quad z = \frac{1}{2}\Big(a + d + e - \frac{de}{a}\Big),$$

and according to T5 we have

$$S(xyz) = \frac{i}{4}\begin{vmatrix} x & \overline{x} & 1 \\ y & \overline{y} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = \frac{i}{8}\begin{vmatrix} a + b + c - \frac{bc}{a} & \overline{a} + \overline{b} + \overline{c} - \frac{\overline{b}\,\overline{c}}{\overline{a}} & 1 \\ a + c + d - \frac{cd}{a} & \overline{a} + \overline{c} + \overline{d} - \frac{\overline{c}\,\overline{d}}{\overline{a}} & 1 \\ a + d + e - \frac{de}{a} & \overline{a} + \overline{d} + \overline{e} - \frac{\overline{d}\,\overline{e}}{\overline{a}} & 1 \end{vmatrix}.$$

Since the determinant is unchanged after substracting some columns from the others, we can substract the second column from the third, and the first from the second. After that we get

$$\begin{aligned} S(xyz) &= \frac{i}{8}\begin{vmatrix} a+b+c-\frac{bc}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{b}\,\overline{c}}{\overline{a}} & 1 \\ \frac{(d-b)(a-c)}{a} & \frac{(d-b)(a-c)}{bcd} & 0 \\ \frac{(e-c)(a-d)}{a} & \frac{(e-c)(a-d)}{a} & 0 \end{vmatrix} \\ &= \frac{i(a-c)(d-b)(a-d)(e-c)}{8}\cdot \\ &\quad \begin{vmatrix} a+b+c-\frac{bc}{a} & \overline{a}+\overline{b}+\overline{c}-\frac{\overline{b}\,\overline{c}}{\overline{a}} & 1 \\ \frac{1}{a} & \frac{1}{bcd} & 0 \\ \frac{1}{a} & \frac{1}{a} & 0 \end{vmatrix}, \end{aligned}$$

and finally

$$\begin{aligned} S(xyz) &= \frac{i(a-c)(d-b)(a-d)(e-c)}{8}\Big(\frac{1}{acde}-\frac{1}{abcd}\Big) \\ &= \frac{i(a-c)(d-b)(a-d)(e-c)(b-e)}{8abcde}. \end{aligned}$$

Since the last expression is symmetric with respet to a, b, c, d, and e the given area doesn't depend on the choice of the vertex (in this case a).

57. Assume that the unit circle is the circumcircle of the triangle abc. Since $\frac{S(bca_1)}{S(abc)} = 1 - \frac{|a-a_1|}{|a-a'|} = 1 - \frac{a-a_1}{a-a'}$ (where a' is the foot of the perpendicular from a to bc), the given equality becomes

$$2 = \frac{a-a_1}{a-a'} + \frac{b-b_1}{b-b'} + \frac{c-c_1}{c-c'}.$$

According to T2.4 we have $a' = \frac{1}{2}\Big(a+b+c-\frac{bc}{a}\Big)$, hence $a-a' = \frac{1}{2}\Big(a+\frac{bc}{a}-b-c\Big) = \frac{(a-b)(a-c)}{2a}$ and after writing the symmetric expressions we get

$$\begin{aligned} 2 &= \frac{2a(a-a_1)}{(a-b)(a-c)} + \frac{2b(b-b_1)}{(b-a)(b-c)} + \frac{2c(c-c_1)}{(c-a)(c-b)} \\ &= -2\frac{a(a-a_1)(b-c)+b(b-b_1)(c-a)+c(c-c_1)(a-b)}{(a-b)(b-c)(c-a)}, \end{aligned}$$

and after simplying

$$aa_1(b-c)+bb_1(c-a)+cc_1(a-b)=0.$$

By T4 points a_1, b_1, c_1, h lie on a circle if and only if

$$\frac{a_1-c_1}{\overline{a_1}-\overline{c_1}}\frac{b_1-h}{\overline{b_1}-\overline{h}} = \frac{a_1-h}{\overline{a_1}-\overline{h}}\frac{b_1-c_1}{\overline{b_1}-\overline{c_1}}.$$

Since h is the orthocenter by T6.3 we have $h = a+b+c$, and since $aa_1 \perp bc$ T1.3 implies $\frac{a_1-a}{\overline{a_1}-\overline{a}} = -\frac{b-c}{\overline{b}-\overline{c}}$, i.e. $\overline{a_1} = \frac{bc+aa_1-a^2}{abc}$, and symmetrically $\overline{b_1} = \frac{ac+bb_1-b^2}{abc}$ and $\overline{c_1} =$

$\dfrac{ab+cc_1-c^2}{abc}$. Similarly from $a_1h \perp bc$ and $b_1h \perp ac$

$$\frac{a_1-h}{\overline{a_1}-\overline{h}} = -\frac{b-c}{\overline{b}-\overline{c}} = bc, \quad \frac{b_1-h}{\overline{b_1}-\overline{h}} = -\frac{a-c}{\overline{a}-\overline{c}} = ac.$$

It is enough to prove that

$$\frac{a(a_1-c_1)}{aa_1-cc_1+(c-a)(a+b+c)} = \frac{b(b_1-c_1)}{bb_1-cc_1+(c-b)(a+b+c)}.$$

Notice that

$$a(b-c)a_1-a(b-c)c_1 = -b_1b(c-a)a-cc_1(a-b)a-a(b-c)c_1 = ab(c-a)(c_1-b_1),$$

and the result follows by the conjugation.

58. Assume that the unit circle is the circumcircle of the triangle abc. By T2.4 we have that $d = \frac{1}{2}\Big(a+b+c-\frac{ab}{c}\Big)$, $e = \frac{1}{2}\Big(a+b+c-\frac{ac}{b}\Big)$, and $f = \frac{1}{2}\Big(a+b+c-\frac{bc}{a}\Big)$. According to T6.1 we get $a_1 = \frac{b+c}{2}$ (where a_1 is the midpoint of the sidebc). Since q belongs to the chord ac T2.2 implies $\overline{q} = \frac{a+c-q}{ac}$, and since $qd \parallel ef$ T1.1 implies $\frac{q-d}{\overline{q}-\overline{d}} = \frac{e-f}{\overline{e}-\overline{f}} = -a^2$. Solving this system gives us

$$q = \frac{a^3+a^2b+abc-b^2c}{2ab}.$$

Symmetrically we get $r = \frac{a^3+a^2c+abc-bc^2}{2ac}$. Since p belongs to the chord bc T2.2 implies $\overline{p} = \frac{b+c-p}{bc}$, and from the colinearity of the points e, f, and p from T1.2 we conclude $\frac{p-e}{\overline{p}-\overline{e}} = \frac{e-f}{\overline{e}-\overline{f}} = -a^2$. After solving this system we get

$$p = \frac{a^2b+a^2c+ab^2+ac^2-b^2c-bc^2-2abc}{2(a^2-bc)} = \frac{b+c}{2} + \frac{a(b-c)^2}{2(a^2-bc)}.$$

By T4 it is sufficient to prove that

$$\frac{p-a_1}{p-r}\frac{q-r}{q-a_1} = \frac{\overline{p}-\overline{a_1}}{\overline{p}-\overline{r}}\frac{\overline{q}-\overline{r}}{\overline{q}-\overline{a_1}}.$$

Since

$$q-r = \frac{a(c-b)(a^2+bc)}{2abc}, \quad p-a_1 = \frac{a(b-c)^2}{2(a^2-bc)},$$

$$p-r = \frac{(a^2-c^2)(b^2c+abc-a^3-a^2c)}{2ac(a^2-bc)}, \quad q-a_1 = \frac{a^3+a^2b-b^2c-ab^2}{2ab}$$

the required statement follows by conjugation.

59. Let O be the circumcenter of the triangle abc. We will prove that O is the incenter as well. Assume that the circumcircle of the triangle abc is unit. According to T6.1 we have that $c_1 = \frac{a+b}{2}$, $b_1 = \frac{a+c}{2}$, and $a_1 = \frac{b+c}{2}$. Assume that k_1, k_2, k_3 are the given circles with the centers a_1, b_1, and c_1. Let $k_1 \cap k_2 = \{k, o\}$, $k_2 \cap k_3 = \{m, o\}$, and $k_3 \cap k_1 = \{l, o\}$. Then we have $|a_1-k| = |a_1-o|$,

$|b_1-k|=|b_1-o|$. After squaring $(a_1-k)(\overline{a_1}-\overline{k})=a_1\overline{a_1}$ and $(b_1-k)(\overline{b_1}-\overline{k})=b_1\overline{b_1}$. After solving this system we obtain

$$k=\frac{(a+c)(b+c)}{2c}.$$

Symmetrically we get $l=\dfrac{(b+c)(a+b)}{2b}$ and $m=\dfrac{(a+c)(a+b)}{2a}$. Let $\angle mko=\varphi$. According to T1.4 we have that $\dfrac{o-k}{\overline{o}-\overline{k}}=e^{i2\varphi}\dfrac{m-k}{\overline{m}-\overline{k}}$, and since $k-m=\dfrac{b(a^2-c^2)}{2ac}$, after conjugation $e^{i2\varphi}=-\dfrac{a}{b}$. If $\angle okl=\psi$, we have by T1.4 $\dfrac{o-k}{\overline{o}-\overline{k}}=e^{i2\psi}\dfrac{l-k}{\overline{l}-\overline{k}}$, hence $e^{i\psi}=-\dfrac{a}{b}$. Now we have $\varphi=\psi$ or $\varphi=\psi\pm\pi$, and since the second condition is impossible (why?), we have $\varphi=\psi$. Now it is clear that o is the incenter of the triangle klm.

For the second part of the problem assume that the circle is inscribed in the triangle klm is the unit circle and assume it touches the sides kl,km,lm at u,v,w respectively. According to T7.1 we have that

$$k=\frac{2uv}{u+v},\quad l=\frac{2uw}{w+u},\quad m=\frac{2vw}{v+w}.$$

Let a_1 be the circumcenter of the triangle kol. Then according to T9.2 we have

$$a_1=\frac{kl(\overline{k}-\overline{l})}{\overline{k}l-k\overline{l}}=\frac{2uvw}{k(u+v)(u+w)}$$

and symmetrically $b_1=\dfrac{2uvw}{(u+v)(v+w)}$ and $c_1=\dfrac{2uvw}{(w+u)(w+v)}$ (b_1 and c_1 are circumcenters of the triangles kom and mol respectively). Now T6.1 implies

$$a+b=2c_1,\quad b+c=2a_1,\quad a+c=2b_1,$$

and after solving this system we get $a=b_1+c_1-a_1$, $b=a_1+c_1-b_1$, and $c=a-1+b_1-c_1$. In order to finish the proof it is enough to establish $ab\perp oc_1$ (the other can be proved symmetrically), i.e. by T1.3 that $\dfrac{c_1-o}{\overline{c_1}-\overline{o}}=-\dfrac{a-b}{\overline{a}-\overline{b}}=-\dfrac{b_1-a_1}{\overline{b_1}-\overline{a_1}}$. The last easily follows from

$$b_1-a_1=\frac{2uvw(u-v)}{(u+v)(v+w)(w+u)},$$

by conjugation.

60. Let b and c be the centers of the circles k_1 and k_2 respectively and assume that bc is the real axis. If the points m_1 and m_2 move in the same direction using T1.4 we get that m_1 and m_2 satisfy

$$m_1-b=(a-b)e^{i\varphi},\quad m_2-c=(a-c)e^{i\varphi}.$$

If ω is the requested point, we must have $|\omega-m_1|=|\omega-m_2|$, and after squaring $(\omega-m_1)(\overline{\omega}-\overline{m_1})=(\omega-m_2)(\overline{\omega}-\overline{m_2})$. From the last equation we get

$$\overline{\omega}=\frac{m_1\overline{m_1}-m_2\overline{m_2}-\omega(\overline{m_1}-\overline{m_2})}{m_1-m_2}.$$

After simplification (with the usage of $\overline{b}=b$ and $\overline{c}=c$ where $e^{i\varphi}=z$)

$$\overline{w}(1-z)=2(b+c)-a-\overline{a}+az+\overline{a}\,\overline{z}-(b+c)(z+\overline{z})-(1-\overline{z})\omega.$$

Since $\overline{z}=\dfrac{1}{z}$, we have

$$(b+c-a-\overline{w})z^2-(2(b+c)-a-\overline{a}-\omega-\overline{\omega})z+b+c-\overline{a}-\omega\equiv 0.$$

The last polynomial has to be identical to 0 hence each of its coefficients is 0, i.e. $\omega = b + c - \overline{a}$. From the previous relations we conclude that this point satisfies the conditions of the problem.
The problem is almost identical in the case of the oposite orientition.

61. Let γ be the unit circle and let $a = -1$. Then $b = 1$, $c = 1 + 2i$, and $d = -1 + 2i$. Since the points n, b, p are colinear we can use T1.2 to get

$$\frac{a - p}{\overline{a} - \overline{p}} = \frac{a - m}{\overline{a} - \overline{m}} = -am = m,$$

and after some algebra $\overline{p} = \dfrac{p + 1 - m}{m}$ (1). Since the points c, d, p are colinear using the same argument we get that

$$\frac{c - n}{\overline{c} - \overline{n}} = \frac{c - d}{\overline{c} - \overline{d}} = 1,$$

hence $\overline{p} = p - 4i$. Comparing this with (1) one gets $p = 4i \cdot \dfrac{m}{m - 1} - 1$. Furthermore, since the points b, n, p are colinear we have

$$\frac{p - 1}{\overline{p} - \overline{1}} = \frac{1 - n}{\overline{1} - \overline{n}} = n,$$

i.e.

$$n = \frac{m(1 - 2i) - 1}{2i + 1 - m}.$$

Let q' be the intersection point of the circle γ and the line dm. If we show that the points q', n, c are colinear we would have $q = q'$ and $q \in \gamma$, which will finish the first part of the problem. Thus our goal is to find the coordinate of the point q'. Since q' belongs to the unit circle we have $q'\overline{q'} = 1$, and since d, m, q' are colinear, we have using T1.2 that

$$\frac{d - m}{\overline{d} - \overline{m}} = \frac{q' - m}{\overline{q'} - \overline{m}} = -q'm,$$

and after simplification

$$q' = -\frac{m + 1 - 2i}{m(1 + 2i) + 1}.$$

In order to prove that the points q', n, c are colinear it suffices to show that $\dfrac{q - c}{\overline{q} - \overline{c}} = \dfrac{n - q}{\overline{n} - \overline{q}} = -nq$, i.e. $n = \dfrac{q - 1 - 2i}{(\overline{q} - 1 + 2i)q}$, which is easy to verify. This proves the first part of the problem.

Now we are proving the second part. Notice that the required inequality is equivalent to $|q - a| \cdot |p - c| = |d - p| \cdot |b - q|$. From the previously computed values for p and q, we easily obtain

$$|q - a| = 2\left|\frac{m + 1}{m(1 + 2i) + 1}\right|,\ |p - c| = 2\left|\frac{m(1 + i) + 1 - i}{m(1 + 2i) + 1}\right|,$$

$$|d - p| = 2\left|\frac{m + 1}{m + 1}\right|,\ |b - q| = 2\left|\frac{m(i - 1) + 1 + 1}{m - 1}\right|,$$

and since $-i((i - 1)m + 1 + i) = m(1 + i) + 1 - i$ the required equality obviously holds.

62. In this problem we have plenty of possibilities for choosing the unit circle. The most convenient choice is the circumcircle of $bcb'c'$ (try if you don't believe). According T2.5 we have that the intersection point x of bb' and cc' satisfy

$$x = \frac{bb'(c + c') - cc'(b + b')}{bb' - cc'}.$$

Since $bh \perp cb'$ and $ch \perp bc'$ T1.3 implies the following two equalities

$$\frac{b-h}{\overline{b}-\overline{h}} = -\frac{b'-c}{\overline{b'}-\overline{c}} = b'c, \quad \frac{c-h}{\overline{c}-\overline{h}} = -\frac{b-c'}{\overline{b}-\overline{c'}} = bc'.$$

From the first we get $\overline{h} = \dfrac{bh-b^2+b'c}{bb'c}$, and from the second $\overline{h} = \dfrac{ch-c^2+bc'}{bcc'}$. After equating the two relations we get

$$h = \frac{b'c'(b-c)+b^2c'-b'c^2}{bc'-b'c}.$$

Symmetrically we obtain $h' = \dfrac{bc(b'-c')+b'^2c-bc'^2}{b'c-bc'}$. It suffices to prove that the points h, h' and x are colinear, or after applying T1.2 we have to verify

$$\frac{h-h'}{\overline{h}-\overline{h'}} = \frac{h-x}{\overline{h}-\overline{x}}.$$

The last follwos from

$$\begin{aligned}
h-h' &= \frac{bc(b'-c')+b'c'(b-c)+bc'(b-c')+b'c(b'-c)}{bc'-b'c}\\
&= \frac{(b+b'-c-c')(bc'+b'c)}{bc'-b'c},\\
h-x &= \frac{b^2b'^2c'+b^3b'c'+b'c^2c'^2+b'c^3c'}{(bc'-b'c)(bb'-cc')} -\\
&\quad \frac{b^2b'cc'+b^2b'c'^2+bb'c^2c'+b'^2c^2c'}{(bc'-b'c)(bb'-cc')}\\
&= \frac{b'c'(b^2-c^2)(b'+b-c-c')}{(bc'-b'c)(bb'-cc')}
\end{aligned}$$

by conjugation.

63. From elementary geometry we know that $\angle nca = \angle mcb$ (such points m and n are called harmonic conjugates). Let $\angle mab = \alpha$, $\angle abm = \beta$, and $\angle mca = \gamma$. By T1.4 we have that

$$\begin{aligned}
\frac{a-b}{|a-b|} &= e^{i\alpha}\frac{a-m}{|a-m|}, \ \frac{a-n}{|a-n|} = e^{i\alpha}\frac{a-c}{|a-c|},\\
\frac{b-c}{|b-c|} &= e^{i\beta}\frac{b-n}{|b-n|}, \ \frac{b-m}{|b-m|} = e^{i\beta}\frac{b-a}{|b-a|},\\
\frac{c-a}{|c-a|} &= e^{i\gamma}\frac{c-n}{|c-n|}, \quad \frac{c-m}{|c-m|} = e^{i\gamma}\frac{c-b}{|c-b|},
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{AM\cdot AN}{AB\cdot AC}+\frac{BM\cdot BN}{BA\cdot BC}+\frac{CM\cdot CN}{CA\cdot CB}\\
= &\frac{(m-a)(n-a)}{(a-b)(a-c)}+\frac{(m-b)(n-b)}{(b-a)(b-c)}+\frac{(m-c)(n-c)}{(c-a)(c-b)}.
\end{aligned}$$

The last expression is always equal to 1 which finishes our proof.

64. Let $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$, $\angle D = \delta$, $\angle E = \varepsilon$, and $\angle F = \varphi$. Applying T1.4 gives us

$$\frac{b-c}{|b-c|} = e^{i\beta}\frac{b-a}{|b-a|}, \quad \frac{d-e}{|d-e|} = e^{i\delta}\frac{d-c}{|d-c|}, \quad \frac{f-a}{|f-a|} = e^{i\varphi}\frac{f-e}{|f-e|}.$$

Multiplying these equalities and using the given conditions (from the conditions of the problem we read $e^{i(\beta+\delta+\varphi)}=1$) we get

$$(b-c)(d-e)(f-a)=(b-a)(d-c)(f-e).$$

From here we can immediately conclude that

$$(b-c)(a-e)(f-d)=(c-a)(e-f)(d-b),$$

and the result follows by placing the modulus in the last expression.

65. We first apply the inversion with repsect to the circle ω. The points a,b,c,e,z are fixed, and the point d is mapped to the intersection of the lines ae and bc. Denote that intersection by s. The circumcircle of the triangle azd is mapped to the circumcircle of the triangle azs, the line bd is mapped to the line bd, hence it is sufficient to prove that bd is the tangent to the circle circumscribed about azs. The last is equivalent to $az\perp sz$.

Let ω be the unit circle and let $b=1$. According to T6.1 we have $c=-1$ and $e=\overline{a}=\dfrac{1}{a}$. We also have $s=\dfrac{a+\overline{a}}{2}=\dfrac{a^2+1}{2a}$. Since $eb\perp ax$ using T1.3 we get

$$\frac{a-x}{\overline{a}-\overline{x}}=-\frac{e-b}{\overline{e}-\overline{b}}=-\frac{1}{a},$$

and since the point x belongs to the chord eb by T2.2 it satisfies $\overline{x}=\dfrac{1+\overline{a}-x}{\overline{a}}$. Solving this system gives sistema dobijamo $x=\dfrac{a^3+a^2+a-1}{2a^2}$. Since y is the midpoint of ax by T6.1

$$y=\frac{a+x}{2}=\frac{3a^3+a^2+a-1}{4a^2}.$$

Since the points b,y,z are colinear and z belongs to the unit circle according to T1.2 and T2.1 we get

$$\frac{b-y}{\overline{b}-\overline{y}}=\frac{b-z}{\overline{b}-\overline{z}}=-z.$$

After simplifying we get $z=\dfrac{1+3a^2}{(3+a^2)a}$. In order to prove that $az\perp zs$ by T1.3 it is sufficient to prove that

$$\frac{a-z}{\overline{a}-\overline{z}}=-\frac{s-z}{\overline{s}-\overline{z}}.$$

The last follows from

$$a-z=\frac{a^4-1}{a(3+a^2)},\quad s-z=\frac{a^4-2a^2+1}{2a(3+a^2)},$$

by conjugation.

66. Assume first that the orthocenters of the given triangles coincide. Assume that the circumcircle of abc is unit. According to T6.3 we have $h=a+b+c$. Consider the rotation with respect to h for the angle ω in the negative direction. The point a_1 goes to the point a_1' such that a_1, a_1', and h are colinear. Assume that the same rotation maps b_1 to b_1' and c_1 to c_1'. Since the triangles abc and $a_1b_1c_1$ are similar and equally oriented we get that the points b,b_1',h are clinear as well as c,c_1',h. Moreover $a_1'b_1'\parallel ab$ (and similarly for $b_1'c_1'$ and $c_1'a_1'$). Now according to T1.4 $e^{i\omega}(a_1'-h)=(a_1-h)$

(since the rotation is in the negative direction), and since the points a, a_1', h are colinear, according to T1.2 we have $\dfrac{a_1'-h}{a-h}=\lambda\in\mathbf{R}$. This means that $a_1=h+\lambda e^{i\omega}(a-h)$ and analogously

$$b_1=h+\lambda e^{i\omega}(b-h),\quad c_1=h+\lambda e^{i\omega}(c-h).$$

Since the point a_1 belongs to the chord bc of the unit circle, by T2.2 we get $\overline{a_1}=\dfrac{b+c-a_1}{bc}$. On the other hand by conjugation of the previous expression for a_1 we get $\overline{a_1}=\overline{h}+\lambda\dfrac{\overline{a}-\overline{h}}{e^{i\omega}}$. Solving for λ gives

$$\lambda=\frac{e^{i\omega}(a(a+b+c)+bc)}{a(b+c)(e^{ie\omega}+1)}.\quad (1)$$

Since λ has the same role in the formulas for b_1 also, we must also have

$$\lambda=\frac{e^{i\omega}(b(a+b+c)+ac)}{b(a+c)(e^{ie\omega}+1)}.\quad (2)$$

By equating (1) and (2) we get

$$\begin{aligned} & ab(a+c)(a+b+c)+b^2c(a+c)-ab(b+c)(a+b+c)-a^2c(b+c)\\ = \; & (a-b)(ab(a+b+c)-abc-ac^2-bc^2)=(a^2-b^2)(ab-c^2).\end{aligned}$$

Since $a^2\neq b^2$ we conclude $ab=c^2$. Now we will prove that this is necessair condition for triangle abc to be equilateral, i.e. $|a-b|=|a-c|$. After squaring the last expression we get that the triangle is equilateral if and only if $0=\dfrac{(a-c)^2}{ac}-\dfrac{(a-b)^2}{ab}=\dfrac{(b-c)(a^2-bc)}{abc}$, and since $b\neq c$, this part of the problem is solved.

Assume now that the incenters of the given triangles coincide. Assume that the incircle of the triangle abc is unit and let d, e, f be the points of tangency of the incircle with the sides ab, bc, ca respectively. Similarly to the previous part of the problem we prove

$$a_1=i+\lambda e^{i\omega}(a-i),\quad b_1=i+\lambda e^{i\omega}(b-i),\quad c_1=i+\lambda e^{i\omega}(c-i).$$

Together with the condition $i=0$ T2.3 and conjugation imply $\overline{a_1}=\dfrac{2\lambda}{e^{i\omega}(e+f)}$. Also, since the points a_1, b, c are colinear we have $a_1d\perp di$ hence according to T1.3 $\dfrac{a_1-d}{\overline{a_1}-\overline{d}}=-\dfrac{d-i}{\overline{d}-\overline{i}}=-d^2$. Solving this system gives

$$\lambda=\frac{d(e+f)}{d^2+efe^{i\omega}}.$$

Since λ has the same roles in the formulas for a_1 and b_1 we must have

$$\lambda=\frac{e(d+f)}{e^2+dfe^{i\omega}},$$

and equating gives us

$$e^{i2\omega}=\frac{ed(e+d+f)}{f(de+ef+fd)}.$$

Symmetry implies $e^{i2\omega}=\dfrac{ef(e+d+f)}{d(de+ef+fd)}$ and since $f^2\neq d^2$ we must have $e+d+f=0$. It is easy to prove that the triangle def is equilateral in this case as well as abc.

67. Since $(a-b)(c-d)+(b-c)(a-d)=(a-c)(b-d)$ the triangle inequality implies $|(a-b)(c-d)|+|(b-c)(a-d)|\geq|(a-c)(b-d)|$, which is exactly an expression of the required inequality. The equality holds if and only if the vectors $(a-b)(c-d)$, $(b-c)(a-d)$, and $(a-c)(c-d)$ are colinear. The first two of them are colinear if and only if

$$\frac{(a-b)(c-d)}{(b-c)(a-d)} \in \mathbf{R},$$

which is according to T3 precisely the condition that a, c, b, d belong to a circle. Similarly we prove that the other two vectors are colinear.

68. Since $(d-a)(d-b)(a-b)+(d-b)(d-c)(b-c)+(d-c)(d-a)(c-a) = (a-b)(b-c)(c-a)$, we have $|(d-a)(d-b)(a-b)|+|(d-b)(d-c)(b-c)|+|(d-c)(d-a)(c-a)| \geq |(a-b)(b-c)(c-a)|$ where the equality holds if and only if $(d-a)(d-b)(a-b)$, $(d-b)(d-c)(b-c)$, $(d-c)(d-a)(c-a)$ and $(a-b)(b-c)(c-a)$ are colinear. The condition for colinearity of the first two vectors can be expressed as

$$\frac{(d-a)(a-b)}{(d-c)(b-c)} = \frac{(\overline{d}-\overline{a})(\overline{a}-\overline{b})}{(\overline{d}-\overline{c})(\overline{b}-\overline{c})}.$$

Assume that the circumcircle of abc is unit. Now the given expression can be written as

$$d\overline{d}a - a^2\overline{d} - \frac{da}{c} + \frac{a^2}{c} = d\overline{d}c - c^2\overline{d} - \frac{dc}{a} + \frac{c^2}{a}$$

and after some algebra $d\overline{d}(a-c) = (a-c)\Big((a+c)\Big(\overline{d} + \frac{d}{ac} - \frac{a+c}{ac}\Big) + 1\Big)$ or

$$d\overline{d} = (a+c)\Big(\overline{d} + \frac{d}{ac} - \frac{a+c}{ac}\Big) + 1.$$

Similarly, from the colinearity of the first and the third vector we get $d\overline{d} = (b+c)\Big(\overline{d} + \frac{d}{bc} - \frac{b+c}{bc}\Big) + 1$. Substracting the last two expressions yields $(a-b)\Big(\overline{d} - \frac{d}{ab} + \frac{c^2-ab}{abc}\Big) = 0$, i.e.

$$\overline{d} - \frac{d}{ab} + \frac{c^2-ab}{abc} = 0.$$

Similarly $\overline{d} - \frac{d}{ac} + \frac{b^2-ac}{abc} = 0$ and after substracting and simplifying we get $d = a+b+c$. It is easy to verify that for $d = a+b+c$, i.e. the orthocenter of the triangle abc, all four of the above mentioned vectors colinear.

13 Problems for Indepent Study

For those who want more, here is the more. Many of the following problems are similar to the problems that are solved above. There are several quite difficult problems (towards the end of the list) which require more attention in choosing the known points, and more time. As in the case with solved problems, I tried to put lot of problems from math competitions from all over the world.

1. (Regional competition 2002, 2nd grade) In the acute-angled triangle ABC, B' and C' are feet of perpendiculars from the vertices B and C respectively. The circle with the diameter AB intersects the line CC' at the points M and N, and the circle with the diameter AC intersects the line BB' at P and Q. Prove that the quadrilateral $MPNQ$ is cyclic.

2. (Yug TST 2002) Let $ABCD$ be a quadrilateral such that $\angle A = \angle B = \angle C$. Prove that the point D, the circumcenter, and the orthocenter of $\triangle ABC$ are colinear.

3. (Republic competition 2005, 4th grade) The haxagon $ABCDEF$ is inscribed in the circle k. If the lengths of the segments AB, CD, and EF are equal to the radius of the circle k prove that the midpoints of the remaining three edges form an equilateral trinagle.

4. (USA 1997) Three isosceles triangles BCD, CAE, and ABF with the bases BC, CA, and AB respectively are constructed in the exterior of the triangle ABC. Prove that the perpendiculars from A, B, and C to the lines EF, FD, and DE repsectively are concurrent.

5. Prove that the side length of the regular 9-gon is equal to the difference of the largest and the smallest diagonal.

6. If $h_1, h_2, \ldots, h_{2n}$ denote respectively the distances of an arbitrary point P of the circle k circumscribed about the polygon $A_1A_2 \ldots A_{2n}$ from the lines that contain the edges A_1A_2, $A_2A_3, \ldots$, $A_{2n}A_1$, prove that $h_1h_3 \cdots h_{2n-1} = h_2h_4 \cdots h_{2n}$.

7. Let $d_1, d_2, \ldots, d_n$ denote the distances of the vertices $A_1, A_2, \ldots, A_n$ of the regular n-gon $A_1A_2 \ldots A_n$ from an arbitrary point P of the smaller arc A_1A_n of the circumcircle. Prove that

$$\frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \ldots + \frac{1}{d_{n-1}d_n} = \frac{1}{d_1d_n}.$$

8. Let $A_0A_1 \ldots A_{2n}$ be a regular polygon, P a point of the smaller arc A_0A_{2n} of the circumcircle and m an integer such that $0 \le m < n$. Prove that

$$\sum_{k=0}^{n} PA_{2k}^{2m+1} = \sum_{k=1}^{n} PA_{2k-1}^{2m+1}.$$

9. (USA 2000) Let $ABCD$ be a cyclic quadrilateral and let E and F be feet of perpendiculars from the intersection of the diagonals to the lines AB and CD respectively. Prove that EF if perpendicular to the line passing through the midpoints of AD and BC.

10. Prove that the midpoints of the altitudes of the traingle are colinear if and only if the triangle is rectangular.

11. (BMO 1990) The feet of preprendiculars of the acute angled triangle ABC are A_1, B_1, and C_1. If A_2, B_2, and C_2 denote the points of tangency of the incircle of $\triangle A_1B_1C_1$ prove that the Euler lines of the triangles ABC and $A_2B_2C_2$ coincide.

12. (USA 1993) Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD are perpendicular. Assume that $AC \cup BD = E$. Prove that the points symmetric to E with respect to the lines AB, BC, CD, and DA form a cyclic quadrilateral.

13. (India 1998) Let AK, BL, CM be the altitudes of the triangle ABC, and let H be its orthocenter. Let P be the midpoint of the segment AH. If BH and MK intersect at the point S, and LP and AM in the point T, prove that TS is perpendicular to BC.

14. (Vietnam 1995) Let AD, BE, and CF be the altitudes of the triangle $\triangle ABC$. For each $k \in R$, $k \neq 0$, let A_1, B_1, and C_1 be such that $AA_1 = kAD$, $BB_1 = kBE$, and $CC_1 = kCF$. Find all k such that for every non-isosceles triangle ABC the triangles ABC and $A_1B_1C_1$ are similar.

15. (Iran 2005) Let ABC be a triangle and D, E, F the points on its edges BC, CA, AB respectively such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = \frac{1-\lambda}{\lambda}$$

where λ is a real number. Find the locus of circumcenters of the triangles DEF as $\lambda \in \mathbf{R}$.

16. Let H_1 and H_2 be feet of perpendiculars from the orthocenter H of the triangle ABC to the bisectors of external and internal angles at the vertex C. Prove that the line H_1H_2 contains the midpoint of the side AB.

17. Given an acute-angled triangle ABC and the point D in its interior, such that $\angle ADB = \angle ACB + 90^\circ$ and $AB \cdot CD = AD \cdot BC$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD}.$$

18. The lines AM and AN are tangent to the circle k, and an arbitrary line through A intersects k at K and L. Let l be an arbitrary line parallel to AM. Assume that KM and LM intersect the line l at P and Q, respectively. Prove that the line MN bisects the segment PQ.

19. The points D, E, and F are chosen on the edges BC, CA, and AB of the triangle ABC in such a way that $BD = CE = AF$. Prove that the triangles ABC and DEF have the common incenter if and only if ABC is equilateral.

20. Given a cyclic quadrilateral $ABCD$, prove that the incircles of the triangles ABC, BCD, CDA, DAB form an rectangle.

21. (India 1997) Let I be the incenter of the triangle ABC and let D and E be the midpoints of the segments AC and AB respectively. Assume that the lines AB and DI intersect at the point P, and the lines AC and EI at the point Q. Prove that $AP \cdot AQ = AB \cdot AC$ if and only if $\angle A = 60^\circ$.

22. Let M be an interior point of the square $ABCD$. Let A_1, B_1, C_1, D_1 be the intersection of the lines AM, BM, CM, DM with the circle circumscribed about the square $ABCD$ respectively. Prove that

$$A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1.$$

23. Let $ABCD$ be a cyclic quadrilateral, $F = AC \cap BD$ and $E = AD \cap BC$. If M and N are the midpoints of the segments AB and CD prove that

$$\frac{MN}{EF} = \frac{1}{2} \cdot \left| \frac{AB}{CD} - \frac{CD}{AB} \right|.$$

24. (Vietnam 1994) The points A', B', and C' are symmetric to the points A, B, and C with respect to the lines BC, CA, and AB respectively. What are the conditions that $\triangle ABC$ has to satisfy in order for $\triangle A'B'C'$ to be equilateral?

25. Let O be the circumcenter of the triangle ABC and let R be its circumradius. The incircle of the triangle ABC touches the sides BC, CA, AB, at A_1, B_1, C_1 and its radius is r. Assume that the lines determined by the midpoints of AB_1 and AC_1, BA_1 and BC_1, CA_1 and CB_1 intersect at the points C_2, A_2, and B_2. Prove that the circumcenter of the triangle $A_2B_2C_2$ coincides with O, and that its circumradius is $R + \frac{r}{2}$.

26. (India 1994) Let $ABCD$ be a nonisosceles trapezoid such that $AB \parallel CD$ and $AB > CD$. Assume that $ABCD$ is circumscribed about the circle with the center I which tangets CD in E. Let M be the midpoint of the segment AB and assume that MI and CD intersect at F. Prove that $DE = FC$ if and only if $AB = 2CD$.

27. (USA 1994) Assume that the hexagon $ABCDEF$ is inscribed in the circle, $AB = CD = EF$, and that the diagonals AD, BE, and CF are concurrent. If P is the intersection of the lines AD and CE, prove that $\frac{CP}{PE} = \left(\frac{AC}{CE}\right)^2$.

28. (Vietnam 1999) Let ABC be a troiangle. The points A', B', and C' are the midpoints of the arcs BC, CA, and AB, which don't contain A, B, and C, respectively. The lines $A'B'$, $B'C'$, and $C'A'$ partition the sides of the triangle into six parts. Prove that the "middle" parts are equal if and only if the triangle ABC is equilateral.

29. (IMO 1991 shortlist) Assume that in $\triangle ABC$ we have $\angle A = 60^\circ$ and that IF is parallel to AC, where I is the incenter and F belongs to the line AB. The point P of the segment BC is such that $3BP = BC$. Prove that $\angle BFP = \angle B/2$.

30. (IMO 1997 shortlist) The angle A is the smallest in the triangle ABC. The points B and C divide the circumcircle into two arcs. Let U be the interior point of the arc between B and C which doesn't contain A. The medians of the segments AB and AC intersect the line AU respectively at the points V and W. The lines BV and CW intersect at T. Prove that $AU = TB + TC$.

31. (Vietnam 1993) Let $ABCD$ be a convex quadrilateral such that AB is not parallel to CD and AD is not parallel to BC. The points P, Q, R, and S are chosen on the edges AB, BC, CD, and DA, respectively such that $PQRS$ is a parallelogram. Find the locus of centroids of all such quadrilaterals $PQRS$.

32. The incircle of the triangle ABC touches BC, CA, AB at E, F, G respectively. Let AA_1, BB_1, CC_1 the angular bisectors of the triangle ABC (A_1, B_1, C_1 belong to the corresponding edges). Let K_A, K_B, K_C respectively be the points of tangency of the other tangents to the incircle from A_1, B_1, C_1. Let P, Q, R be the midpoints of the segments BC, CA, AB. Prove that the lines PK_A, QK_B, RK_C intersect on the incircle of the triangle ABC.

33. Assume that I and I_a are the incenter and the excenter corresponding to the edge BC of the triangle ABC. Let II_a intersect the segment BC and the circumcircle of $\triangle ABC$ at A_1 and M respectively (M belongs to I_a and I) and let N be the midpoint of the arc MBA which contains C. Assume that S and T are intersections of the lines NI and NI_a with the circumcircle of $\triangle ABC$. Prove that the points S, T, and A_1 are colinear.

34. (Vietnam 1995) Let AD, BE, CF be the altitudes of the triangle ABC, and let A', B', C' be the points on the altitudes such that

$$\frac{AA'}{AD} = \frac{BB'}{BE} = \frac{CC'}{CF} = k.$$

Find all values for k such that $\triangle A'B'C' \sim \triangle ABC$.

35. Given the triangle ABC and the point T, let P and Q be the feet of perpendiculars from T to the lines AB and AC, respectively and let R and S be the feet of perpendiculars from A to the lines TC and TB, respectively. Prove that the intersection point of the lines PR and QS belongs to the line BC.

36. (APMO 1995) Let $PQRS$ be a cyclic quadrilateral such that the lines PQ and RS are not parallel. Consider the set of all the circles passing through P and Q and all the circles passing through R and S. Determine the set of all points A of tangency of the circles from these two sets.

37. (YugMO 2003, 3-4 grade) Given a circle k and the point P outside of it. The variable line s which contains point P intersects the circle k at the points A and B. Let M and N be the midpoints of the arcs determined by the points A and B. If C is the point of the segment AB such that

$$PC^2 = PA \cdot PB,$$

prove that the measure of the angle $\angle MCN$ doesn't depend on the choice of s.

38. (YugMO 2002, 2nd grade) Let A_0,A_1,...,A_{2k}, respectevly be the points which divide the circle into $2k+1$ congruent arcs. The point A_0 is connected by the chords to all other points. Those $2k$ chords divide the circle into $2k+1$ parts. Those parts are colored alternatively in white and black in such a way that the number of white parts is by 1 bigger than the number of black parts. Prove that the surface area of teh black part is greater than the surface area of the white part.

39. (Vietnam 2003) The circles k_1 and k_2 touch each other at the point M. The radius of the circle k_1 is bigger than the radius of the circle k_2. Let A be an arbitrary point of k_2 which doesn't belong to the line connecting the centers of the circles. Let B and C be the points of k_1 such that AB and AC are its tangents. The lines BM and CM intersect k_2 again at E and F respectively. The point D is the intersection of the tangent at A with the line EF. Prove that the locus of points D (as A moves along the circle) is a line.

40. (Vietnam 2004) The circles k_1 and k_2 are given in the plane and they intersect at the points A and B. The tangents to k_1 at those points intersect at K. Let M be an arbitrary point of the circle k_1. Assume that $MA \cup k_2 = \{A, P\}$, $MK \cup k_1 = \{M, C\}$, and $CA \cup k_1 = \{A, Q\}$. Prove that the midpoint of the segment PQ belongs to the line MC and that PQ passes through a fixed point as M moves along k_1.

41. (IMO 2004 shortlist) Let $A_1A_2 \dots A_n$ be a regular n-gon. Assume that the points B_1, B_2, ..., B_{n-1} are determined in the following way:

- for $i = 1$ or $i = n-1$, B_i is the midpoint of the segment A_iA_{i+1};
- for $i \neq 1$, $i \neq n-1$, and S intersection of A_1A_{i+1} and A_nA_i, B_i is the intersection of the bisectors of the angle A_iS_{i+1} with A_iA_{i+1}.

Prove that $\angle A_1B_1A_n + \angle A_1B_2A_n + \ldots + \angle A_1B_{n-1}A_n = 180°$.

69. (**Dezargue's Theorem**) The triangles are perspective with respect to a point if and only if they are perspective w.r.t to a line.

42. (IMO 1998 shortlist) Let ABC be a triangle such that $\angle ACB = 2\angle ABC$. Let D be the point of the segment BC such that $CD = 2BD$. The segment AD is extended over the point D to the point E for which $AD = DE$. Prove that

$$\angle ECB + 180° = 2\angle EBC.$$

43. Given a triangle $A_1A_2A_3$ the line p passes through the point P and intersects the segments A_2A_3, A_3A_1, A_1A_2 at the points X_1, X_2, X_3, respectively. Let A_iP intersect the circumcircle of $A_1A_2A_3$ at R_i, for $i = 1, 2, 3$. Prove that X_1R_1, X_2R_2, X_3R_3 intersect at the point that belongs to the circumcircle of the triangle $A_1A_2A_3$.

44. The points O_1 and O_2 are the centers of the circles k_1 and k_2 that intersect. Let A be one of the intersection points of these circles. Two common tangents are constructed to these circles. BC are EF the chords of these circles with endpoints at the points of tangency of the common chords with the circles (C and F are further from A). If M and N are the midpoints of the segments BC and EF, prove that $\angle O_1AO_2 = \angle MAN = 2\angle CAF$.

45. (BMO 2002) Two circles of different radii intersect at points A and B. The common chords of these circles are MN and ST respectively. Prove that the orthocenters of $\triangle AMN$, $\triangle AST$, $\triangle BMN$, and $\triangle BST$ form a rectangle.

46. (IMO 2004 shortlist) Given a cyclic quadrilateral $ABCD$, the lines AD and BC intersect at E where C is between B and E. The diagonals AC and BD intersect at F. Let M be the midpoint of CD and let $N \neq M$ be the point of the circumcircle of the triangle ABM such that $AN/BN = AM/BM$. Prove that the points E, F, N are colinear.

47. (IMO 1994 shortlist) The diameter of the semicircle Γ belongs to the line l. Let C and D be the points on Γ. The tangents to Γ at C and D intersect the line l respectively at B and A such that the center of the semi-circle is between A and B. Let E be the intersection of the lines AC and BD, and F the foot of perpendicular from E to l. Prove that EF is the bisector of the angle $\angle CFD$.