Tran Phuong

# Diamonds in mathematical inequalities

# Proposal for a book

**Tran Phuong**

*Director of Center for Research and Support Development of Intellectual Products.*

*No 19/89, Thai Ha Street, Dong Da District, Hanoi, Viet nam.*

*Email: tranphuong.math@yahoo.com; tranphuong.math@gmail.com*

*Phone No:*

*(844) 5-376-340; (84)913-220-457*

*Assistants' mobile phone numbers:*

*0061402201501*

*Email:trdieulinh@yahoo.com*

*Fax:*

*(84)(04) 943-5816*

*(84)(04)719-9641*

# I. GENERAL DESCRIPTION

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## I.1 Title

Diamonds in mathematical inequalities.

## I.2. Purpose of the book

The book aims, first, to provide students with a comprehensive and minute system of typical inequality demonstration methods and techniques, ranging from classical to modern ones, which, due to the fact that their importance remains unchanged throughout the flow of time, can be considered as "diamonds in mathematical inequalities". This main purpose is accomplished by a number of valuable mathematic samples with interesting "lead-ins" and, most importantly, a dialectic viewpoint on each solving method.

The major goal of the authors is, therefore, to helping students to acquire multitude of mathematical tools and methods in solving inequalities, then, be able to achieve excellence in this field. Besides, on utilizing a new approach in presenting familiar mathematical facts, the authors also hope to highlight in readers' mind the importance of a good mathematical inequality-based thinking in developing their creativeness as well as their ability to critically evaluate changes in life.

The success of this book will also mean the publication of the authors' other books on other fields of secondary mathematics.

## I.3. Why is this book different from others?

- **The uniqueness of the book can be sensed in the very first lines of its foreword:**

" *Money ?*
*Eventually spent*
*Beauty ?*
*Eventually faded...*
*Only*
*Intellect rooted in mind*
*And feelings rooted in heart*
*Will last*
*forever with time*

***(Tran Phuong, 1990)***

***Comparisons in life***

*The material and spiritual world in the universe is always on the move, ever changing to create different things and phenomena. This diversity can be seen through opposite images: big − small ; wide − narrow ; long − short ; high − low ; plentiful − scarce ; rich − poor ; beautiful − ugly and so on. Extensively, it is that the divergence between Buddhism, Mohammedanism and Catholicism; the differences between the Oriental − Occidental cultures and between the major philosophic thoughts all over the world.*

*Everybody's life is an ongoing search to set his/her own values. Every thing gets its standing in the ever changing world by its own values; however, one does not often realize that everything can only obtain its value in comparison with some other thing. It is this relationship that creates inequalities.*

*In fact, though "all comparisons are odious" as the saying goes, men cannot help resorting to comparison to evaluate things. A kilogram of Korean ginseng is 1000 times smaller than a ton of rice in term of mass, but costs 10 times more. Take the example of a president or a car driver, who is more important? You may say that the head of state is certainly more important than a racer, but only in state affairs. In respect of annual income, former President Clinton's US $200,000 annual salary in office is derisory in comparison with world famous formula 1 car racer Michael Schumacher's US $60,000,000 annual income. Thus, factors in a comparison should be identified, quantified and converted to a same kind of unit - whether you want to or not.*

*In order to develop comparison thinking and evaluation in life, notions of mathematical inequalities are given us as early as school age. Children at kindergarten learn to compare 1 with 2 to have a* $1<2$ *inequality. Fourth formers begin to achieve more difficult comparisons, for instance* $\frac{135}{143}$ *with* $\frac{189}{197}$*, etc. Some of them proceed by reduction to a common denominator, which is rather complicated, i.e.*

$$\boxed{\frac{135}{143}=\frac{135\times197}{143\times197}=\frac{26595}{28171};\frac{189}{197}=\frac{189\times143}{197\times143}=\frac{27027}{28171}\quad \textit{but}\quad \frac{26595}{28171}<\frac{27027}{28171}\Rightarrow\frac{135}{143}<\frac{189}{197}}$$

*Others intelligently operate by fractions complementation as follows:*

$$\boxed{\frac{135}{143}+\frac{8}{143}=1=\frac{8}{197}+\frac{189}{197}\quad;\quad \textit{now}\quad \frac{8}{143}>\frac{8}{197}\quad \textit{hence}\quad \frac{135}{143}<\frac{189}{197}}$$

*Comparison problems grow more and more difficult with more extensive operations. All mathematicians share the common concept "**Basic results of mathematics often are expressed by inequalities in stead of equalities**". The same is true in real life, where one always encounters differences between things and phenomena, and even changes in a single phenomenon by the minute. Indeed, if we are subject to no change after every second, then, according to the induction principle, 50 years later, we would not get any older. On the other* *hand, in social life, inequality-based thinking is always needed to assess business activities, export-import industry, stock exchange market, finances, banking... Therefore, in order to develop thinking and properly assess the changes in life, a good mathematical inequality-based thinking is most necessary. With these introductory words, the author wishes that the readers consider the subject treated in this book as a close and intimate one."*

With such profound philosophical understandings, the writer has always been aware of the necessity to ensure an inspiration and intellectual delight in each technique and method presented. The book, there fore, will encourage students to acquire a multitudes of inequality demonstration methods as effective tools not only in solving mathematics problems but also in critically evaluate changes in life.

- **In respect of penmanship, the major content is presented in two typical ways:**

**1.** The classical "diamonds" were introduced in new angles of view with typical techniques and methods followed by a variety of interesting examples ranging from simple to complex.

To be more specific, the authors insufflate in each inequality demonstration technique a new notion imbued with a forceful philosophic connotation, the notion of **"*point of incidence*"**. The utility of such creative penmanship is rooted from the ideas relating to the point of incidence in the film "Teheran 43". The film, which was produced by the cooperation of Russia, France and Italia, has the setting of the summer 1943, when the World War II was growing fiercer and fiercer. In the context, the U.S., Great Britain and the Soviet Union envisaged opening a Front of Alliance and the Germans plotted to assassinate the three leaders of the Alliances, Roosevelt, Churchill and Stalin. The German spy Schiller trusted professional hit-man Max with the assassination, but bade him to work out the venue and the time of the meeting. From the military developments, Max estimated that the meeting would take place at the end of 1943. Added with diplomatic pretext for the meeting, Max came to the final conclusion that it must take place no sooner or later than Churchill's birthday, which was $30^{\text{th}}$, November and the venue must be in a British colony far away from the front in Europe. The venue, accordingly, must be no other than the British Embassy in Teheran, the Iranian Capital. From such estimation Max flew to Teheran three months before the meeting and hired locals to dig a tunnel ending up in the British Embassy.

Thus, the idea of choosing points of incidence in life springs from considering the evolution and development of things to estimate where they would lead to, with a view to orienting the direction of approach from the outset. As to the techniques of choosing the point of incident in equalities, it is based on the state of variables when the two terms of the inequalities happen to equal. Typical examples illustrating this technique are found precisely presented in the very first introduction on AM-GM, Cauchy-Schwarz, Bernoulli inequalities and throughout this book.

**2.** The modern "diamonds", of which the typical ones are ABC, GTA, GLA, DAC methods, are discussed in details so that readers can easily grasp the spirit of each technique and method and apply them.

After all, this is a profound book on the methods and techniques of mathematical inequality demonstration. The problems given are arranged didactically upwards, i.e. gradually from lower to higher levels. A variety of typical examples are given to illustrate each problem; exercises for readers also range from simple to complex ones with large amplitude. All of them are presented in an interesting way but easy to understand and apply.

Hopefully, readers would feel easy to understand and to perceive this book, and to evaluate the authors' "mathematical creations". It convey new perspectives and view points to many familiar problems and challenges its audience with a number of new ones. In one word, with this book, readers may have chances to develop and to make themselves recognized.

**3.** Mathematics is a language itself and also Inequality should be a public language that can connect everybody throughout the world easily. Therefore the authors used many Mathematics notations to reduce the English language so that even those who don't know English well can also enjoy this book.

## I.4. Table of contents

The book consists of 5 chapters, for a total of about 480 pages. The major content of the book can be summarized as follows:

**Chapter I: Diamonds in classical mathematical inequalities**

§1. AM-GM inequalities

§2. Cauchy-Schwarz inequalities

§3. Holder inequalities

§4. Minkowski inequalities

§5. Chebyshev inequalities

**Chapter II: Diamonds in modern mathematical inequalities**

§6. Schur inequalities

§7. Muirhead inequalities

§8. Permutation inequalities

**Chapter III: Diamonds of analytic method**

§9. Fermat theorem (Derivative method)

§10. Lagrange theorem

§11. Bernoulli inequalities

## I.5. Who is the audience?

The book is intended for a wide range of audience. It is first and particularly meant for capable basic secondary students, candidates to national and international mathematics contests, mathematics teachers at all levels and researchers. On completing this book, the authors, however, did not target the public from only one country. Believing that mathematics language does not change across national boundaries, they hope that the book will be able to reach an international readership and prove to be helpful for everyone who is seeking ways to broaden their mathematical horizons.

## I.6. Final comment

This book is the authors' soul-felt work achieved through years of hard work, based on their vast experience in mathematics and mathematical education. It is a comprehensive work in a field favorable to observation, intuition and creativeness, all of which are outstanding talents of the Vietnamese people. Having devoted themselves entirely to completing this book, the authors are convinced that it will last as long as a close friend of the world's mathematics lovers.

# II. SAMPLE WORK

The initial section of Sample Work is a part of chapter one of the book, which was written with a brand new style for a well – known method: "***Point of incidence***" In this chapter, we will introduce the technique of choosing the "point of incidence" in AM – GM, Cauchy-Schwarz and Bernoulli inequalities. However, it is noted that selecting "point of incidence" is only the typical technique among a total of **30 techniques** of using AM – GM inequalities presented in the manuscript of the book "Collections of Topics, Methods and Techniques in Algebraic Inequality Demonstration", which is **2222-pages** thick and was completed by Tran Phuong due to the impetus raised in his mind on the occasion of President W.J. Clinton visited Vietnam on November, $16^{th}$, 2000. In this part, the ideas relating to the point of incidence in the film "Teheran 43" will be mentioned again.

In the remaining section, we will consider the best estimation of ***Nesbit – Shapiro Inequality*** in primary mathematics.

In respect of format, we would like to note here that in the hope of presenting the content of the book in the most convenient way, we choose the paper size of $17\times24^{cm}$ and the page layout of $14\times21^{cm}$, which is very common in Vietnam.

## *INTRODUCTION: FROM THE STORY ABOUT "POINTS OF INCIDENCE" TO POINT OF INCIDENCE IN MATHEMATICAL INEQUALITY*

### IDEAS RELATING TO "POINTS OF INCIDENCE" IN TEHERAN 43

The film, which was produced by the cooperation of Russia, France and Italia, has the setting of the summer 1943, when the World War II was growing fiercer and fiercer. In the context, the U.S., Great Britain and the Soviet Union envisaged opening a Front of Alliance and the Germans plotted to assassinate the three leaders of the Alliances, Roosevelt, Churchill and Stalin. The German spy Schiller trusted professional hit-man Max with the assassination, but bade him to work out the venue and the time of the meeting. From the military developments, Max estimated that the meeting would take place at the end of 1943. Added with diplomatic pretext for the meeting, Max came to the final conclusion that it must take place no sooner or later than Churchill's birthday, which was $30^{th}$, November and the venue must be in a British colony far away from the front in Europe. The venue, accordingly, must be no other than the British Embassy in Teheran, the Iranian Capital. From such estimation Max flew to Teheran three months before the meeting and hired locals to dig a tunnel ending up in the British Embassy.

Thus, the idea of choosing points of incidence in life springs from considering the evolution and development of things to estimate where they would lead to, with a view to orienting the direction of approach from the outset. As to the techniques of choosing the point of incident in equalities, it is based on the state of variables when the two terms of the inequalities happen to equal. Typical examples illustrating this technique will be precisely presented in the very first introduction on AM-GM, Cauchy-Schwarz, Bernoulli inequalities and throughout this book.

**POINT OF INCIDENCE IN EQUALITIES**

***Setting the manner:***

In proving the inequality **A ≥ B** we often follow either the two following patterns:

***Pattern 1:*** Create a sequence of intermediary inequalities

$$\mathbf{A \geq A_1 \geq A_2 \geq \ldots \geq A_{n-1} \geq A_n \geq B}$$

***Pattern 2:*** Create a sequence of component inequalities

$$+\begin{cases} A_1 \geq B_1 \\ A_2 \geq B_2 \\ \ldots\ldots\ldots \\ A_n \geq B_n \end{cases} \quad \text{or} \quad \times\begin{cases} A_1 \geq B_1 \geq 0 \\ A_2 \geq B_2 \geq 0 \\ \ldots\ldots\ldots\ldots \\ A_n \geq B_n \geq 0 \end{cases}$$

$$\Rightarrow A \geq B \qquad\qquad \Rightarrow A \geq B$$

To create intermediary or component inequalities we need to note that if the **'Central inequality A ≥ B'** becomes **'A = B'** at a **standard P** , all the intermediary inequalities in **Pattern 1** or all the component inequalities ***(local inequality)*** in **Pattern 2** also become **equalities**. To find **standard P** we need to pay attention to the symmetry of variables and the conditions for equality to occur in AM – GM Inequality where ***all joining variables are equal***. For estimating at what standard **'A = B'** occurs is to guide algebraic transformations and estimations of intermediary or component inequalities, the work can be called **'Inspecting the condition of equality occurring and point of incidence in the inequality'**.

We will be more familiar with this idea in a variety of examples ranging from single to complex in the detailed discussion of sample exercises later.

# III. ABOUT THE AUTHORS

**III.1. Tran Phuong**, the chief author of the book, is the Director of Center for Research and Development Support of Intellectual Products in Vietnam.

Like most authors of mathematics books, Tran Phuong received systematic and professional education and training specializing in this field. Before entering university, he studied in Mathematics Specialized School at the University of Natural Sciences, Hanoi, the cradle in which up to 60% of Vietnamese IMO medalists were trained and nourished so far. After his school years, he continued pursuing his interest in mathematics in Teacher Training University, Hanoi, the leading center for training mathematics professors in Vietnam.

After graduation, the author, however, pursued his career in his own way which is teaching on the invitations of high schools, universities and mathematics centers in Vietnam. During the years from 1990 to 2000, up to approximately 10 thousands of students attended his classes on mathematics. The majority of them were excellent pupils, including many candidates to both national and international Mathematics contests. Vietnamese IMO 43 gold medalist Pham Gia Vinh Anh is among his best pupils. Since his ultimate goal in teaching mathematics is student’s full development in mathematics-based thinking, not merely the mathematics knowledge itself, many of his students, on their outcome of what they acquired in his classes, contribute actively in various fields of the Vietnamese society.

On realizing his dream of devoting entirely to mathematics science teaching and learning, Tran Phuong has had a great passion for writing books on mathematics, especially on inequalities. His first book, which was published in 1993, was a comprehensive work on mathematical inequalities. From then on, he has completed a total of 10 books, all of which were introduced by five first-ranked publishers in Vietnam (Education Publisher, Youth Publisher, Knowledge Publisher, Ho Chi Minh Publisher and Da Nang Publisher) and soon became the best-sellers among referenced books for secondary pupils in Vietnam. He is also the author of an intellectual game show “Vietnamese Infant Prodigy” on television.

For his great contribution to mathematics teaching and learning in Vietnam, he was awarded the noble “Vietnam’s Genius 2006” by Vietnamese government.

**III.2. Contributors:**

**1. Tran Tuan Anh**

*PhD Candidate, Shool of Mathemmatics Georgia Institute of Technology, Atlanta,GA 30332, USA*

**2. Pham Gia Vinh Anh**

*Bachelor of Information Technology, University of Sydney, Australia, Gold Medal in IMO 43*

**3. Nguyen Anh Cuong**

*Student of School of Computing, Singapore National University, course 2006-2010*

**4. Bui Viet Anh**

*Student of Department of Electronics and Telecommunications , University of Technoly, Ha Noi,course 2005-2009, Vietnam*

**5. Le Trung Kien**

*Mathematics majored students, school period of 2004 - 2007, the Gifted High School of Bac Giang, Vietnam*

**6. Phan Thanh Viet**

*Mathematics majored students, school period of 2004 - 2007, the Gifted High Shool of Luong Van Chanh, Phu yen, Vietnam*

**7. Vo Quoc Ba Can**

*Mathematics majored students, school period of 2003 - 2006, the Gifted High School of Ly Tu Trong, Can Tho, Vietnam*

**8. Hoang Trong Hien**

*Mathematics majored students, school period of 2005 - 2008, the Gifted High School of Le Hong Phong, Nam Dinh, Vietnam*

**9. Nguyen Thuc Vu Hoang**

*Mathematics majored students, school period of 2005 - 2008, the Gifted High School of Le Quy Đon, Quang Tri,Vietnam*

**10. Nguyen Quoc Hung**

*Academic Period of 2005 -2009, Faculty of Mathematics & Informatics, University of Natural Sciences, Ho Chi Minh City, Vietnam*

# CHAPTER ONE: DIAMONDS IN CLASSICAL MATHEMATICAL INEQUALITIES

## *§1.1. DIFFERENT COLORS IN CATCHING POINT OF INCIDENT TECHNIQUE IN AM – GM INEQUALITY*

Main points:

**I. AM – GM inequality**

1. General form
2. Special cases
3. Proof

**II. Different colors in catching point of incident technique in AM – GM inequality**

1. Point of incidence in evaluating from AM to GM
2. Point of incidence in evaluating from GM to AM
3. Point of free incidence or homogeneous principle in AM – GM inequality
4. Specialization in inequality of same degree 4. Specialize in homogeneous inequality
5. Non-symmetric point of incidence in AM – GM inequality
6. Balancing Coefficient Technique (Method of equalizing coefficient )
7. Using AM – GM in homogeneous inequality Applying AM – GM to inequality of different degree
8. Specialize in un-homogeneous inequality Specialization in inequality of different degree
9. The most beautiful solutions for four trigonometry inequalities
10. Selective problems in using point of incidence for AM – GM

### I. AM – GM INEQUALITY

**1. General Form:** Suppose $a_1, a_2, ...a_n$ are $n$ non-negative real numbers, then:

***1.1. Form 1:*** $\dfrac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$

***1.2. Form 2:*** $a_1 + a_2 + ... + a_n \geq n.\sqrt[n]{a_1 a_2 ... a_n}$

***1.3. Form 3:*** $\left(\dfrac{a_1 + a_2 + ... + a_n}{n}\right)^n \geq a_1 a_2 .... a_n$

***1.4. Equality occurs*** $\Leftrightarrow$ $a_1 = a_2 = ... = a_n \geq 0$

***1.5. Corollary***

- If $a_1, a_2, ..., a_n \geq 0$ and $a_1 + a_2 + ... + a_n = S$ is constant, then

$$\text{Max}\left(a_1 a_2 ... a_n\right) = \left(\frac{S}{n}\right)^n \text{ occurs } \Leftrightarrow a_1 = a_2 = ... = a_n = \frac{S}{n}$$

- If $a_1, a_2, ..., a_n \geq 0$ and $a_1 a_2 ... a_n = P$ is constant, then

$$\text{Min}\left(a_1 + a_2 + ... a_n\right) = n.\sqrt[n]{P} \text{ occurs } \Leftrightarrow a_1 = a_2 = ... = a_n = \sqrt[n]{P}$$

**2. Special Cases:**

| Form \ $n$ | $n = 2$ | $n = 3$ | $n = 4$ |
|---|---|---|---|
| Condition | $\forall\ a, b \geq 0$ | $\forall\ a, b, c \geq 0$ | $\forall\ a, b, c, d \geq 0$ |
| Form 1 | $\frac{a+b}{2} \geq \sqrt{ab}$ | $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ | $\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$ |
| Form 2 | $a+b \geq 2.\sqrt{ab}$ | $a+b+c \geq 3.\sqrt[3]{abc}$ | $a+b+c+d \geq 4.\sqrt[4]{abcd}$ |
| Form 3 | $\left(\frac{a+b}{2}\right)^2 \geq ab$ | $\left(\frac{a+b+c}{3}\right)^3 \geq abc$ | $\left(\frac{a+b+c+d}{4}\right)^4 \geq abcd$ |
| Equality | $a = b$ | $a = b = c$ | $a = b = c = d$ |

**3. Proof:** $\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$ , $\forall a_1, a_2, ... a_n \geq 0$ (1)

There are 36 solutions for this inequality; following is the proof by mathematic induction

• For $n = 2$: $\frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} = \frac{\left(\sqrt{a_1} - \sqrt{a_2}\right)^2}{2} \geq 0 \Rightarrow \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$

Equality occurs $\Leftrightarrow a_1 = a_2$. Suppose the inequality is true for $n \geq 2$.

• We will prove the inequality is true for $(n + 1)$ numbers $a_1, a_2, ... a_n, a_{n+1} \geq 0$

Using the inductive hypothesis for $n$ numbers $a_1, a_2, ... a_n \geq 0$ we have

$$S_{n+1} = \frac{(a_1 + a_2 + ... + a_n) + a_{n+1}}{n+1} \geq \frac{n.\sqrt[n]{a_1 a_2 ... a_n} + a_{n+1}}{n+1}. \text{ Let } \begin{cases} a_1 a_2 ... a_n = p^{n(n+1)} \\ a_{n+1} = q^{n+1} \text{ for } p, q \geq 0 \end{cases}$$

$\Rightarrow S_{n+1} \geq \frac{np^{n+1} + q^{n+1}}{n+1}$ (1). We will prove $\frac{np^{n+1} + q^{n+1}}{n+1} \geq p^n q = \sqrt[n+1]{a_1 a_2 ... a_n a_{n+1}}$ (2)

We have: $\frac{np^{n+1} + q^{n+1}}{n+1} - p^n q = \frac{1}{n+1}\left[ np^n (p-q) - q\left(p^n - q^n\right) \right]$

$$= \frac{(p-q)^2}{n+1}\left[ p^{n-1} + p^{n-2}(p+q) + ... + p\left(p^{n-2} + p^{n-3}.q + ... + q^{n-2}\right) + \left(p^{n-1} + p^{n-2}q + ... q^{n-1}\right)\right] \geq 0$$

From (1) and (2) $\Rightarrow \frac{a_1 + a_2 + ... + a_n + a_{n+1}}{n+1} \geq \sqrt[n+1]{a_1 a_2 ... a_n a_{n+1}}$

Equality occurs $\Leftrightarrow \begin{cases} a_1 = a_2 = ... = a_n \\ p = q \end{cases} \Leftrightarrow a_1 = a_2 = ... = a_n = a_{n+1}$

According to the induction principle, the inequality is true for $\forall n \geq 2$, $n \in \mathbb{N}$

## II. CATCH POINT OF INCIDENCE TECHNIQUE IN AM – GM INEQUALITY

### 1. POINT OF INCIDENCE IN EVALUATING FROM AM TO GM

**Leading in:**

From a simple problem: Find the minimum value of S = $\frac{a}{b}+\frac{b}{a}$ for $a, b > 0$

We can find out the solution: $S=\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}=2$. When $a = b > 0$, Min $S = 2$ but when we consider the problem in a different domain we will have an interesting problem connecting with fix point of incidence in AM – GM inequality.

**Problem 1.** Given $a \geq 3$. Find the minimum value of **S** = $a+\frac{1}{a}$

***Solution***

- ***Common mistake:*** $S=a+\frac{1}{a}\geq 2\sqrt{a\cdot\frac{1}{a}}=2\Rightarrow \text{Min}\, S=2$
- ***Cause:*** $\text{Min}\, S=2\Leftrightarrow a=\frac{1}{a}=1$ contradicts the assumption $a \geq 3$
- ***Analyzing and finding the solution:***

Consider the variation table of $a, \frac{1}{a}$ and S to estimate Min S

| $a$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ······· | 30 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{1}{a}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ | $\frac{1}{7}$ | $\frac{1}{8}$ | $\frac{1}{9}$ | $\frac{1}{10}$ | $\frac{1}{11}$ | $\frac{1}{12}$ | ······· | $\frac{1}{30}$ |
| S | $3\frac{1}{3}$ | $4\frac{1}{4}$ | $5\frac{1}{5}$ | $6\frac{1}{6}$ | $7\frac{1}{7}$ | $8\frac{1}{8}$ | $9\frac{1}{9}$ | $10\frac{1}{10}$ | $11\frac{1}{11}$ | $12\frac{1}{12}$ | ······· | $30\frac{1}{30}$ |

As we can see from the Variation Table, if $a$ increases, S increases and then it can be estimated that S receives the least value at $a = 3$. We say $\textbf{Min}\, \boldsymbol{S}=\boldsymbol{\frac{10}{3}}$ occurs at the point of incidence $a = 3$.

Since in ***AM – GM Inequality*** the equality occurs on the condition that all joining numbers are equal, at the point of incidence $a = 3$ we cannot use ***AM – GM Inequality*** directly for two numbers $a$ and $\frac{1}{a}$ because $3 \neq \frac{1}{3}$. We assume that ***AM – GM Inequality*** is used for the couple $\left(\frac{a}{\alpha},\frac{1}{a}\right)$ such that at the point of incidence $a = 3$ occur $\frac{a}{\alpha}=\frac{1}{a}$ that means the following 'Point of incidence' Pattern holds:

$$\textbf{\textit{Pattern:}}\ \boxed{a=3}\Rightarrow\begin{cases}\dfrac{a}{\alpha}=\dfrac{3}{\alpha}\\[2mm] \dfrac{1}{a}=\dfrac{1}{3}\end{cases}\Rightarrow\frac{1}{3}=\frac{3}{\alpha}\Rightarrow\boxed{\alpha=9}: \textbf{point of incidence coefficient}$$

We transform S according to the 'Point of incidence' Pattern as above.

• ***Right solution:*** $S = a + \frac{1}{a} = \left(\frac{a}{9} + \frac{1}{a}\right) + \frac{8a}{9} \geq 2.\sqrt{\frac{a}{9} \cdot \frac{1}{a}} + \frac{8.3}{9} = \frac{10}{3}$. For $a = 3$ then $\text{Min } S = \frac{10}{3}$

**Problem 2.** Given $\begin{cases} a, b > 0 \\ a + b \leq 1 \end{cases}$ Find the minimum value of the expression $\boldsymbol{S} = ab + \frac{1}{ab}$

## *Solution*

• ***Common mistake :*** $S = ab + \frac{1}{ab} \geq 2.\sqrt{ab \cdot \frac{1}{ab}} = 2 \Rightarrow \text{Min } S = 2$

• ***Cause:*** $\text{Min } S = 2 \Leftrightarrow ab = \frac{1}{ab} = 1 \Rightarrow 1 = \sqrt{ab} \leq \frac{a+b}{2} \leq \frac{1}{2} \Rightarrow 1 \leq \frac{1}{2}$: illogical

• ***Analyzing and finding the solution:***

The expression S contains two variables $a$, $b$ but if we take $t = ab$ or $t = \frac{1}{ab}$, the expression $S = t + \frac{1}{t}$ will contain only one variable. When changing the variables we must find the defined area of the new variables as follows:

$$t = \frac{1}{ab} \Rightarrow ab = \frac{1}{t} \quad \text{and} \quad t = \frac{1}{ab} \geq \frac{1}{\left(\frac{a+b}{2}\right)^2} \geq \frac{1}{\left(\frac{1}{2}\right)^2} = 4$$

***Problem will be:*** Given $t \geq 4$. Find the minimum value of the expression $S = t + \frac{1}{t}$

• ***'Point of incidence' Pattern:***

$$\boxed{t = 4} \Rightarrow \begin{cases} \frac{t}{\alpha} = \frac{4}{\alpha} \\ \frac{1}{t} = \frac{1}{4} \end{cases} \Rightarrow \frac{1}{4} = \frac{4}{\alpha} \Rightarrow \boxed{\alpha = 16}$$ : **Point of incidence coefficient**

• ***General solution:***

$$S = t + \frac{1}{t} = \left(\frac{t}{16} + \frac{1}{t}\right) + \frac{15t}{16} \geq 2.\sqrt{\frac{t}{16} \cdot \frac{1}{t}} + \frac{15t}{16} = \frac{2}{4} + \frac{15t}{16} \geq \frac{2}{4} + \frac{15.4}{16} = \frac{17}{4}$$

For t = 4 or $a = b = \frac{1}{2}$ then $\text{Min } S = \frac{17}{4}$

• ***Reduced solution:*** Since t = 4 ⇔ $a = b = \frac{1}{2}$ we transform S directly as follows

$$S = ab + \frac{1}{ab} = \left(ab + \frac{1}{16ab}\right) + \frac{15}{16ab} \geq 2.\sqrt{ab \cdot \frac{1}{16ab}} + \frac{15}{16\left(\frac{a+b}{2}\right)^2} \geq \frac{17}{4}$$

For $a = b = \frac{1}{2}$ then $\text{Min } S = \frac{17}{4}$

**Problem 3.** Given $\begin{cases} a,b,c>0 \\ a+b+c\le \dfrac{3}{2} \end{cases}$ Find the minimum value of $\mathbf{S}=a+b+c+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$

***Solution***

• ***Common mistake:*** $S=a+b+c+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\ge 6.\sqrt[6]{abc\cdot\dfrac{1}{a}\cdot\dfrac{1}{b}\cdot\dfrac{1}{c}}=6\Rightarrow \text{Min}\, S=6$

• ***Cause:***

$\text{Min}\, S=6\Leftrightarrow a=b=c=\dfrac{1}{a}=\dfrac{1}{b}=\dfrac{1}{c}=1\ \Rightarrow\ a+b+c=3>\dfrac{3}{2}$ contradicts the assumption.

• ***Analyzing and finding the solution:***

Since S is the symmetrical expression of $a$, $b$, $c$ we estimate $\text{Min}\, S$ occurs at $a=b=c=\dfrac{1}{2}$

• ***'Point of incidence':***

$$\boxed{a=b=c=\frac{1}{2}}\ \Rightarrow \begin{cases} a=b=c=\dfrac{1}{2} \\ \dfrac{1}{\alpha a}=\dfrac{1}{\alpha b}=\dfrac{1}{\alpha c}=\dfrac{2}{\alpha} \end{cases} \Rightarrow \frac{1}{2}=\frac{2}{\alpha}\ \Rightarrow\ \boxed{\alpha=4}$$ : **Point of incidence coefficient**

• ***Method 1:*** $S=a+b+c+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\left(a+b+c+\dfrac{1}{4a}+\dfrac{1}{4b}+\dfrac{1}{4c}\right)+\dfrac{3}{4}\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$

$$\ge 6.\sqrt[6]{abc\cdot\frac{1}{4a}\cdot\frac{1}{4b}\cdot\frac{1}{4c}}+\frac{3}{4}\left(3.\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}}\right)=3+\frac{9}{4}\cdot\frac{1}{\sqrt[3]{abc}}$$

$$\ge 3+\frac{9}{4}\cdot\frac{1}{\frac{a+b+c}{3}}=3+\frac{27}{4}\cdot\frac{1}{a+b+c}\ge 3+\frac{27}{4}\cdot\frac{1}{\frac{3}{2}}=\frac{15}{2}$$. For $a=b=c=\dfrac{1}{2}$ then $\text{Min}\, S=\dfrac{15}{2}$

**Problem 4.** Given $\begin{cases} a,b,c>0 \\ a+b+c\le \dfrac{3}{2} \end{cases}$ Find Min of $\mathbf{S}=\sqrt{a^2+\dfrac{1}{b^2}}+\sqrt{b^2+\dfrac{1}{c^2}}+\sqrt{c^2+\dfrac{1}{a^2}}$

***Solution***

• ***Common mistake:*** $S\ge 3.\sqrt[3]{\sqrt{a^2+\dfrac{1}{b^2}}\cdot\sqrt{b^2+\dfrac{1}{c^2}}\cdot\sqrt{c^2+\dfrac{1}{a^2}}}=3.\sqrt[6]{\left(a^2+\dfrac{1}{b^2}\right)\left(b^2+\dfrac{1}{c^2}\right)\left(c^2+\dfrac{1}{a^2}\right)}$

$$\ge 3.\sqrt[6]{\left(2\cdot\sqrt{a^2\cdot\frac{1}{b^2}}\right)\left(2\cdot\sqrt{b^2\cdot\frac{1}{c^2}}\right)\left(2\cdot\sqrt{c^2\cdot\frac{1}{a^2}}\right)}=3.\sqrt[6]{8}=3\sqrt{2}\Rightarrow \text{Min}\, S=3\sqrt{2}$$

• ***Cause:*** $\text{Min}\, S=3\sqrt{2}\Leftrightarrow a=b=c=\dfrac{1}{a}=\dfrac{1}{b}=\dfrac{1}{c}=1\Rightarrow a+b+c=3>\dfrac{3}{2}$ contradicts the assumption

• ***Analyzing and finding the solution:***

Since S is the symmetrical expression of $a$, $b$, $c$ we estimate Min $S$ occurs at $a=b=c=\dfrac{1}{2}$

- ***'Point of incidence' Pattern:***

$$\boxed{a=b=c=\frac{1}{2}} \Rightarrow \begin{cases} a^2=b^2=c^2=\dfrac{1}{4} \\ \dfrac{1}{\alpha a^2}=\dfrac{1}{\alpha b^2}=\dfrac{1}{\alpha c^2}=\dfrac{4}{\alpha} \end{cases} \Rightarrow \frac{1}{4}=\frac{4}{\alpha} \Rightarrow \boxed{\alpha=16}$$ : **Point of incidence coefficient**

$$S=\sqrt{a^2+\underbrace{\frac{1}{16b^2}+\cdots+\frac{1}{16b^2}}_{\text{16 terms}}}+\sqrt{b^2+\underbrace{\frac{1}{16c^2}+\cdots+\frac{1}{16c^2}}_{\text{16 terms}}}+\sqrt{c^2+\underbrace{\frac{1}{16a^2}+\cdots+\frac{1}{16a^2}}_{\text{16 terms}}}$$

$$\geq \sqrt{17.\sqrt[17]{\frac{a^2}{16^{16}b^{32}}}}+\sqrt{17.\sqrt[17]{\frac{b^2}{16^{16}c^{32}}}}+\sqrt{17.\sqrt[17]{\frac{c^2}{16^{16}a^{32}}}}=\sqrt{17}\left(\sqrt[17]{\frac{a}{16^8b^{16}}}+\sqrt[17]{\frac{b}{16^8c^{16}}}+\sqrt[17]{\frac{c}{16^8a^{16}}}\right)$$

$$\geq\sqrt{17}\left[3\cdot\sqrt[3]{\sqrt[17]{\frac{a}{16^8b^{16}}}\cdot\sqrt[17]{\frac{b}{16^8c^{16}}}\cdot\sqrt[17]{\frac{c}{16^8a^{16}}}}\right]=3\sqrt{17}\cdot\sqrt[17]{\frac{1}{16^8a^5b^5c^5}}$$

$$=\frac{3\sqrt{17}}{2\cdot\sqrt[17]{(2a.2b.2c)^5}}\geq\frac{3\sqrt{17}}{2\cdot\sqrt[17]{\left(\dfrac{2a+2b+2c}{3}\right)^{15}}}\geq\frac{3\sqrt{17}}{2}$$. For $a=b=c=\frac{1}{2}$ then $\text{Min } S=\frac{3\sqrt{17}}{2}$

**Problem 6.** Let $\begin{cases} a,b,c\geq 0 \\ a^2+b^2+c^2=1 \end{cases}$. Find the minimum value of $\text{T}=a+b+c+\dfrac{1}{abc}$

***(Macedonia 1999)***

## *Solution*

- ***Common mistake:*** $a+b+c+\dfrac{1}{abc}\geq 4\cdot\sqrt[4]{a\cdot b\cdot c\cdot\dfrac{1}{abc}}=4 \Rightarrow \text{Min T}=4$

- ***Cause:*** Min T = 4 $\Leftrightarrow$ $a=b=c=\dfrac{1}{abc}\Leftrightarrow a=b=c=1$ contradicts the assumption.

- ***Analyzing and finding the solution:***

Estimate that the point of incidence of Min $T$ is $a=b=c=\dfrac{1}{\sqrt{3}}$, then $\dfrac{1}{abc}=3\sqrt{3}$

- ***'Point of incidence' Pattern:***

$$\boxed{a=b=c=\frac{1}{\sqrt{3}}} \Rightarrow \begin{cases} a=b=c=\dfrac{1}{\sqrt{3}} \\ \dfrac{1}{\alpha abc}=\dfrac{3\sqrt{3}}{\alpha} \end{cases} \Rightarrow \frac{3\sqrt{3}}{\alpha}=\frac{1}{\sqrt{3}} \Rightarrow \boxed{\alpha=9}$$: **Point of incidence coefficient**

- ***Right solution:*** $a+b+c+\dfrac{1}{9abc}+\dfrac{8}{9abc}\geq 4\cdot\sqrt[4]{a\cdot b\cdot c\cdot\dfrac{1}{9abc}}+\dfrac{8}{9\left(\sqrt{\dfrac{a^2+b^2+c^2}{3}}\right)^3}=\dfrac{4}{\sqrt{3}}+\dfrac{8}{\sqrt{3}}=4\sqrt{3}$

**2. POINT OF INCIDENCE IN EVALUATING FROM GM TO AM**

***Remark:*** Consider AM – GM inequality: $\dfrac{a_1+a_2+...+a_n}{n} \geq \sqrt[n]{\underbrace{a_1 a_2 ... a_n}_{n \text{ terms}}}$ , $\forall a_1, a_2, ... a_n \geq 0$

Notices that in RHS expression GM we can see that the root index and the numbers of factors inside the root are equal (both equal to $n$). Therefore, if these two quantities are different from each other, they need adjusting towards being equal. In general, we usually deal with the problems that we need to multiply the expressions with suitable constant so that the numbers of factors inside the root are equal to the root index and suitable with the speculated point of incidence.

**Problem 1.** Given $\begin{cases} a,b,c \geq 0 \\ a+b+c=1 \end{cases}$ Find the minimum value of $S = \sqrt[3]{a+b}+\sqrt[3]{b+c}+\sqrt[3]{c+a}$

***Solution***

• ***Common mistake:***

$$+\begin{cases} \sqrt[3]{a+b}=\sqrt[3]{(a+b).1.1} \leq \dfrac{a+b+1+1}{3} \\ \sqrt[3]{b+c}=\sqrt[3]{(b+c).1.1} \leq \dfrac{b+c+1+1}{3} \\ \sqrt[3]{c+a}=\sqrt[3]{(c+a).1.1} \leq \dfrac{c+a+1+1}{3} \end{cases}$$

$$\Rightarrow S=\sqrt[3]{a+b}+\sqrt[3]{b+c}+\sqrt[3]{c+a} \leq \frac{2(a+b+c)+6}{3}=\frac{8}{3} \quad \Rightarrow \text{Max } S=\frac{8}{3}$$

• ***Causes of the mistake:***

$$Max\, S=\frac{8}{3} \Leftrightarrow +\begin{cases} a+b=1 \\ b+c=1 \\ c+a=1 \end{cases} \quad \Rightarrow 2(a+b+c)=3 \Leftrightarrow 2=3 \Rightarrow \text{illogical}$$

• ***Estimation of the Point of incidence of Max S:***

Since S is the symmetrical expression for a, b, c Max S often occurs when

$$\begin{cases} a=b=c \\ a+b+c=1 \end{cases} \Leftrightarrow a=b=c=\frac{1}{3} \Leftrightarrow a+b=b+c=c+a=\frac{2}{3}$$

• ***Right solution:***

$$+\begin{cases} \sqrt[3]{a+b}=\sqrt[3]{\dfrac{9}{4}} \cdot \sqrt[3]{(a+b)\cdot\dfrac{2}{3}\cdot\dfrac{2}{3}} \leq \sqrt[3]{\dfrac{9}{4}} \cdot \dfrac{(a+b)+\dfrac{2}{3}+\dfrac{2}{3}}{3} \\ \sqrt[3]{b+c}=\sqrt[3]{\dfrac{9}{4}} \cdot \sqrt[3]{(b+c)\cdot\dfrac{2}{3}\cdot\dfrac{2}{3}} \leq \sqrt[3]{\dfrac{9}{4}} \cdot \dfrac{(b+c)+\dfrac{2}{3}+\dfrac{2}{3}}{3} \\ \sqrt[3]{c+a}=\sqrt[3]{\dfrac{9}{4}} \cdot \sqrt[3]{(c+a)\cdot\dfrac{2}{3}\cdot\dfrac{2}{3}} \leq \sqrt[3]{\dfrac{9}{4}} \cdot \dfrac{(c+a)+\dfrac{2}{3}+\dfrac{2}{3}}{3} \end{cases}$$

$$\Rightarrow S=\sqrt[3]{a+b}+\sqrt[3]{b+c}+\sqrt[3]{c+a} \leq \sqrt[3]{\frac{9}{4}} \cdot \frac{2(a+b+c)+4}{3}=\sqrt[3]{\frac{9}{4}\cdot\frac{6}{3}}=\sqrt[3]{18}$$

For $a+b=b+c=c+a=\dfrac{2}{3} \Leftrightarrow a=b=c=\dfrac{1}{3}$, Max $S=\sqrt[3]{18}$

**Problem 2.** Given $\begin{cases} a,\, b,\, c > 0 \\ a + b + c = 3 \end{cases}$ Find the greatest value of the expression:

$$\mathbf{S} = \sqrt[3]{a(b+2c)} + \sqrt[3]{b(c+2a)} + \sqrt[3]{c(a+2b)}$$

***Solution***

• ***Estimation of the Point of incidence of Max S:***

Since S is the symmetrical expression for a, b, c Max S often occurs when

$$\begin{cases} a = b = c \\ a + b + c = 3 \end{cases} \Leftrightarrow a = b = c = 1 \Rightarrow \begin{cases} 3a = 3b = 3c = 3 \\ b + 2c = c + 2a = a + 2b = 3 \end{cases}$$

• ***Right solution:***

$$+\begin{cases} \sqrt[3]{a(b+2c)} = \dfrac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3a(b+2c).3} \le \dfrac{1}{\sqrt[3]{9}} \cdot \dfrac{3a + (b+2c) + 3}{3} \\ \sqrt[3]{b(c+2a)} = \dfrac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3b(c+2a).3} \le \dfrac{1}{\sqrt[3]{9}} \cdot \dfrac{3b + (c+2a) + 3}{3} \\ \sqrt[3]{c(a+2b)} = \dfrac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3c(a+2b).3} \le \dfrac{1}{\sqrt[3]{9}} \cdot \dfrac{3c + (a+2b) + 3}{3} \end{cases}$$

$$\Rightarrow S = \sqrt[3]{a(b+2c)} + \sqrt[3]{b(c+2a)} + \sqrt[3]{c(a+2b)} \le \frac{1}{\sqrt[3]{9}} \cdot \frac{6(a+b+c)+9}{3} = \frac{1}{\sqrt[3]{9}} \cdot 9 = 3 \cdot \sqrt[3]{3}$$

For $a = b = c = 1$, Max $S = 3.\sqrt[3]{3}$

**Problem 3.** Given $\begin{cases} a,\, b,\, c > 0 \\ a^2 + b^2 + c^2 = 12 \end{cases}$ Find the greatest value of the expression:

$$\mathbf{S} = a \cdot \sqrt[3]{b^2 + c^2} + b \cdot \sqrt[3]{c^2 + a^2} + c \cdot \sqrt[3]{a^2 + b^2}$$

***Solution***

• ***Estimation of the Point of incidence of Max S:***

Since S is the symmetrical expression for a, b, c Max S often occurs when

$$\begin{cases} a = b = c > 0 \\ a^2 + b^2 + c^2 = 12 \end{cases} \Leftrightarrow a = b = c = 2 \Rightarrow \begin{cases} 2a^2 = 2b^2 = 2c^2 = 8 \\ b^2 + c^2 = c^2 + a^2 = a^2 + b^2 = 8 \end{cases}$$

• ***Right solution:***

$$+\begin{cases} a \cdot \sqrt[3]{b^2 + c^2} = \sqrt[6]{a^6 (b^2 + c^2)^2} = \dfrac{1}{2} \cdot \sqrt[6]{(2a^2)^3 (b^2 + c^2)^2 .8} \le \dfrac{1}{2} \cdot \dfrac{6a^2 + 2(b^2 + c^2) + 8}{6} \\ b \cdot \sqrt[3]{c^2 + a^2} = \sqrt[6]{b^6 (c^2 + a^2)^2} = \dfrac{1}{2} \cdot \sqrt[6]{(2b^2)^3 (c^2 + a^2)^2 .8} \le \dfrac{1}{2} \cdot \dfrac{6b^2 + 2(c^2 + a^2) + 8}{6} \\ c \cdot \sqrt[3]{a^2 + b^2} = \sqrt[6]{c^6 (a^2 + b^2)^2} = \dfrac{1}{2} \cdot \sqrt[6]{(2c^2)^3 (a^2 + b^2)^2 .8} \le \dfrac{1}{2} \cdot \dfrac{6c^2 + 2(a^2 + b^2) + 8}{6} \end{cases}$$

$$\Rightarrow S = a \cdot \sqrt[3]{b^2 + c^2} + b \cdot \sqrt[3]{c^2 + a^2} + c \cdot \sqrt[3]{a^2 + b^2} \le \frac{1}{2} \cdot \frac{10(a^2 + b^2 + c^2) + 24}{6} = 12$$

For $a = b = c = 2$, Max $S = 12$

**Problem 4.** Let be given $a \ge 2$; $b \ge 6$; $c \ge 12$. Find the greatest value of
$$S = \frac{bc\sqrt{a-2} + ca\cdot\sqrt[3]{b-6} + ab\cdot\sqrt[4]{c-12}}{abc}$$

***Solution***

$$+\begin{cases} bc\sqrt{a-2} = \dfrac{bc}{\sqrt{2}}\sqrt{(a-2).2} \le \dfrac{bc}{\sqrt{2}}\cdot\dfrac{(a-2)+2}{2} = \dfrac{abc}{2\sqrt{2}} \\ ca\cdot\sqrt[3]{b-6} = \dfrac{ca}{\sqrt[3]{9}}\cdot\sqrt[3]{(b-6).3.3} \le \dfrac{ca}{\sqrt[3]{9}}\cdot\dfrac{(b-6)+3+3}{2} = \dfrac{abc}{2.\sqrt[3]{9}} \\ ab\cdot\sqrt[4]{c-12} = \dfrac{ab}{\sqrt[4]{64}}\cdot\sqrt[4]{(c-12).4.4.4} \le \dfrac{ab}{\sqrt[4]{64}}\cdot\dfrac{(c-12)+4+4+4}{4} = \dfrac{abc}{8.\sqrt{2}} \end{cases}$$

$$\Rightarrow S \le \frac{1}{abc}\cdot\left(\frac{abc}{2\sqrt{2}} + \frac{abc}{8\sqrt{2}} + \frac{abc}{3.\sqrt[3]{9}}\right) = \frac{5}{8\sqrt{2}} + \frac{1}{2.\sqrt[3]{9}}$$

For $\begin{cases} a-2=2 \\ b-6=3 \\ c-12=4 \end{cases} \Leftrightarrow \begin{cases} a=4 \\ b=9 \\ c=16 \end{cases}$, Max $S = \dfrac{5}{8\sqrt{2}} + \dfrac{1}{3.\sqrt[3]{9}}$

**Problem 5.** Prove that: $S = 1 + \sqrt{\dfrac{2+1}{2}} + \sqrt[3]{\dfrac{3+1}{3}} + \ldots + \sqrt[n]{\dfrac{n+1}{n}} < n+1$

***Solution***

We have $\sqrt[k]{\dfrac{k+1}{k}} = \sqrt[k]{\dfrac{k+1}{k}\cdot\underbrace{1\ldots1}_{k-1\text{ factors}}} < \dfrac{\dfrac{k+1}{k}+(k-1)}{k} = 1 + \dfrac{1}{k^2}$. It follows

$$S < 1 + \left(1+\frac{1}{2^2}\right) + \left(1+\frac{1}{3^2}\right) + \ldots + \left(1+\frac{1}{n^2}\right) = n + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < n + \frac{1}{1\times2} + \frac{1}{2\times3} + \ldots + \frac{1}{(n-1)\times n}$$

$$= n + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = n + \left(1 - \frac{1}{n}\right) < n+1$$

**Problem 6.** Prove that: $\sqrt[n]{1 + \dfrac{\sqrt[n]{n}}{n}} + \sqrt[n]{1 - \dfrac{\sqrt[n]{n}}{n}} < 2$

***Solution***

$$+\begin{cases} \sqrt[n]{1+\dfrac{\sqrt[n]{n}}{n}} = \sqrt[n]{\left(1+\dfrac{\sqrt[n]{n}}{n}\right).\underbrace{1\ldots1}_{n-1\text{ factors}}} < \dfrac{\left(1+\dfrac{\sqrt[n]{n}}{n}\right)+(n-1)}{n} = 1 + \dfrac{\sqrt[n]{n}}{n^2} \\ \sqrt[n]{1-\dfrac{\sqrt[n]{n}}{n}} = \sqrt[n]{\left(1-\dfrac{\sqrt[n]{n}}{n}\right).\underbrace{1\ldots1}_{n-1\text{ factors}}} < \dfrac{\left(1-\dfrac{\sqrt[n]{n}}{n}\right)+(n-1)}{n} = 1 - \dfrac{\sqrt[n]{n}}{n^2} \end{cases}$$

$$\Rightarrow \sqrt[n]{1+\frac{\sqrt[n]{n}}{n}} + \sqrt[n]{1-\frac{\sqrt[n]{n}}{n}} < \left(1+\frac{\sqrt[n]{n}}{n^2}\right) + \left(1-\frac{\sqrt[n]{n}}{n^2}\right) = 2$$

### 3. POINT OF FREE INCIDENCE OR HOMOGENEOUS PRINCIPLE IN AM – GM INEQUALITY

Inequalities mentioned in this section are illustrated by polynomial functions for three variables $a, b, c > 0$, which does not lose either the essence or the generality of the issue. We will approach this technique from the simplest problems.

• ***Degree of the monomial:*** The monomial $a^{\alpha}b^{\beta}c^{\gamma}$ has its degree as $(\alpha+\beta+\gamma)$

For example: 3-degree monomials: $a^3, b^3, c^3, a^2b, b^2c, c^2a, \frac{a^4}{b}, \frac{b^4}{c}, \frac{c^4}{a}, \frac{a^5}{b^2}, \frac{b^5}{c^2}, \frac{c^5}{a^2},$

$\frac{a^5}{bc}, \frac{b^5}{ca}, \frac{c^5}{ab}, \frac{a^6}{b^3}, \frac{a^6}{b^2c}, \frac{a^7}{b^4}, \frac{a^7}{bc^3}, \frac{a^7}{b^2c^2}, \frac{a^8}{b^5}, \frac{a^8}{b^3c^2}, \frac{a^8}{bc^4}, \frac{a^9}{b^4c^2}\ldots\ldots\ldots$

Thus there are countless 3-degree monomials written by variables $a$, $b$, $c$ and generally we have countless given $k$-degree monomials written by three variables $a$, $b$, $c$.

• ***Setting the matter:*** First of all, we will compare $F(a, b, c)$ with $G(a, b, c)$ as polynomials of different degrees, in the following representative problem:

***Problem:*** Prove that: $\forall a, b, c > 0$ we have $a^{2000} + b^{2000} + c^{2000} \geq a + b + c$ (*)

***Analysis:*** Suppose (*) is true, then let $a = b = c > 0$ we have

$$(*) \Leftrightarrow 3a^{2000} \geq 3a \Leftrightarrow a^{2000} \geq a$$

Since $a^{2000} \geq a$ is true for $a \geq 1$ and $a^{2000} \geq a$ is false for $a \in (0, 1)$, it follows

• If $a, b, c > 0$, $a^{2000} + b^{2000} + c^{2000} \geq a + b + c$ is false

• If reducing the definite area $a, b, c \geq 1$ then $a^{2000} \geq a$, $b^{2000} \geq b$, $c^{2000} \geq c$ and we will have a mediocre inequality $a^{2000} + b^{2000} + c^{2000} \geq a + b + c$, $\forall a, b, c \geq 1$

• If reducing the definite area $a, b, c \in (0,1)$ then $a^{2000} < a$, $b^{2000} < b$, $c^{2000} < c$ and we will also have a mediocre inequality $a^{2000} + b^{2000} + c^{2000} < a + b + c$ $\forall a, b, c \in (0, 1)$

• ***Conclusion:*** We should not compare polynomials of different degrees on the definite area $\mathbb{R}^+$. Since there are infinite ways of writing given $k$-degree monomials, the following problems set the matter that compares polynomial functions of same degree with degrees of monomials written in different forms.

• ***General principle:*** When applying this technique we need to note that the monomials added to use ***AM – GM Inequality*** must have the same degrees with the monomials taken from the polynomials in the inequality to be proved. This technique will be illustrated more clearly in the following problems.

All inequalities following are generally in the form of fraction. Therefore, the most common technique to solve them is eliminating denominators to turn them into the form of polynomials. If we choose to eliminate denominators by equivalent transformation, i.e. equalizing denominators, the solution will be very lengthy. Here, we will eliminate denominators by adding reasonable expressions to use AM – GM inequality. All inequalities in the following sections can be generated by the same ideas. However the generating idea is not so special so it will be remained for the readers.

**Problem 1.** Prove that $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}\ge a+b+c \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 1-degree monomials

Using ***AM − GM Inequality*** we have

$$+\begin{cases} \dfrac{a^2}{b}+b\ge 2\sqrt{\dfrac{a^2}{b}\cdot b}=2\sqrt{a^2}=2a \\ \dfrac{b^2}{c}+c\ge 2\sqrt{\dfrac{b^2}{c}\cdot c}=2\sqrt{b^2}=2b \\ \dfrac{c^2}{a}+a\ge 2\sqrt{\dfrac{c^2}{a}\cdot a}=2\sqrt{c^2}=2c \end{cases}$$

$$\Rightarrow\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)\ge 2(a+b+c) \Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 2.** Prove that $\dfrac{a^3}{b^2}+\dfrac{b^3}{c^2}+\dfrac{c^3}{a^2}\ge a+b+c, \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 1-degree monomials

Using ***AM − GM Inequality*** we have

$$+\begin{cases} \dfrac{a^3}{b^2}+b+b\ge 3.\sqrt[3]{\dfrac{a^3}{b^2}\cdot b\cdot b}=3a \\ \dfrac{b^3}{c^2}+c+c\ge 3.\sqrt[3]{\dfrac{b^3}{c^2}\cdot c\cdot c}=3b \\ \dfrac{c^3}{a^2}+a+a\ge 3.\sqrt[3]{\dfrac{c^3}{a^2}\cdot a\cdot a}=3c \end{cases}$$

$$\Rightarrow\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)+2(a+b+c)\ge 3(a+b+c) \Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 3.** Prove that $\dfrac{a^3}{b^2}+\dfrac{b^3}{c^2}+\dfrac{c^3}{a^2}\ge\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}\quad\forall a,b,c>0$

## *Proof*

• ***1st solution:*** •**Lemma:** $\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}\ge a+b+c\quad\forall a,b,c>0$

***Application:***

$$+\begin{cases}\dfrac{a^3}{b^2}+a\ge 2\sqrt{\dfrac{a^3}{b^2}\cdot a}=2\dfrac{a^2}{b}\\ \dfrac{b^3}{c^2}+b\ge 2\sqrt{\dfrac{b^3}{c^2}\cdot b}=2\dfrac{b^2}{c}\\ \dfrac{c^3}{a^2}+c\ge 2\sqrt{\dfrac{c^3}{a^2}\cdot c}=2\dfrac{c^2}{a}\end{cases}$$

$$\Rightarrow\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)+(a+b+c)\ge 2\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\ge\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)$$

$\Rightarrow\dfrac{a^3}{b^2}+\dfrac{b^3}{c^2}+\dfrac{c^3}{a^2}\ge\dfrac{a^2}{b}+\dfrac{b^2}{c}+\dfrac{c^2}{a}$. Equality occurs $\Leftrightarrow a=b=c>0$

• ***2nd solution:*** • **Lemma:** $x^3+y^3\ge xy(x+y)\quad\forall x,y>0$

***Proof:***

$$+\begin{cases}x^3+x^3+y^3\ge 3.\sqrt[3]{x^3x^3y^3}=3x^2y\\ y^3+y^3+x^3\ge 3.\sqrt[3]{y^3y^3x^3}=3y^2x\end{cases}$$

$$\Rightarrow\ 3(x^3+y^3)\ge 3xy(x+y)\Rightarrow x^3+y^3\ge xy(x+y)$$

***Or proof :*** $x^3+y^3=(x+y)(x^2+y^2-xy)\ge(x+y)(2xy-xy)=xy(x+y)$

***Application:***

$$+\begin{cases}\dfrac{a^3}{b^2}+b=\dfrac{a^3+b^3}{b^2}\ge\dfrac{ab(a+b)}{b^2}=\dfrac{a^2}{b}+a\\ \dfrac{b^3}{c^2}+c=\dfrac{b^3+c^3}{c^2}\ge\dfrac{bc(b+c)}{c^2}=\dfrac{b^2}{c}+b\\ \dfrac{c^3}{a^2}+a=\dfrac{c^3+a^3}{a^2}\ge\dfrac{ca(c+a)}{a^2}=\dfrac{c^2}{a}+c\end{cases}$$

$$\Rightarrow\left(\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\right)+(a+b+c)\ge\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)$$

$$\Rightarrow\frac{a^3}{b^2}+\frac{b^3}{c^2}+\frac{c^3}{a^2}\ge\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 4.** Prove that $\dfrac{a^3}{bc}+\dfrac{b^3}{ca}+\dfrac{c^3}{ab} \geq a+b+c, \quad \forall a,b,c>0$ ***(Canada MO 2002)***

***Proof***

**Remark:** Both sides are polynomials containing 1-degree monomials

Using ***AM − GM Inequality*** we have

$$+\left\{\begin{array}{l} \dfrac{a^3}{bc}+b+c \geq 3\cdot\sqrt[3]{\dfrac{a^3}{bc}\cdot b\cdot c} = 3a \\ \dfrac{b^3}{ca}+c+a \geq 3\cdot\sqrt[3]{\dfrac{b^3}{c}\cdot c\cdot a} = 3b \\ \dfrac{c^3}{a}+a+b \geq 3\cdot\sqrt[3]{\dfrac{c^3}{a}\cdot a\cdot b} = 3c \end{array}\right.$$

$$\Rightarrow \frac{a^3}{bc}+\frac{b^3}{ca}+\frac{c^3}{ab}+2(a+b+c)\geq 3(a+b+c) \Rightarrow \text{(q.e.d)}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 5.** Prove that $\dfrac{a^3}{b}+\dfrac{b^3}{c}+\dfrac{c^3}{a} \geq ab+bc+ca, \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 2-degree monomials

Using ***AM − GM Inequality*** we have

$$+\left\{\begin{array}{l} \dfrac{a^3}{b}+\dfrac{b^3}{c}+bc \geq 3.\sqrt[3]{\dfrac{a^3}{b}\cdot\dfrac{b^3}{c}\cdot bc} = 3ab \\ \dfrac{b^3}{c}+\dfrac{c^3}{a}+ca \geq 3.\sqrt[3]{\dfrac{b^3}{c}\cdot\dfrac{c^3}{a}\cdot ca} = 3bc \\ \dfrac{c^3}{a}+\dfrac{a^3}{b}+ab \geq 3.\sqrt[3]{\dfrac{c^3}{a}\cdot\dfrac{a^3}{b}\cdot ab} = 3ca \end{array}\right.$$

$$\Rightarrow 2\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+(ab+bc+ca)\geq 3(ab+bc+ca)$$

$$\Rightarrow \frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a} \geq ab+bc+ca\,.$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 6.** Prove that $\frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\ge a^2+b^2+c^2 \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 2-degree monomials

Using ***AM − GM Inequality*** we have

$$+\begin{cases}\frac{a^5}{b^3}+\frac{a^5}{b^3}+b^2+b^2+b^2\ge 5.\sqrt[5]{\frac{a^5}{b^3}\cdot\frac{a^5}{b^3}\cdot b^2\cdot b^2\cdot b^2}=5a^2\\ \frac{b^5}{c^3}+\frac{b^5}{c^3}+c^2+c^2+c^2\ge 5.\sqrt[5]{\frac{b^5}{c^3}\cdot\frac{b^5}{c^3}\cdot c^2\cdot c^2\cdot c^2}=5b^2\\ \frac{c^5}{a^3}+\frac{c^5}{a^3}+a^2+a^2+a^2\ge 5.\sqrt[5]{\frac{c^5}{a^3}\cdot\frac{c^5}{a^3}\cdot a^2\cdot a^2\cdot a^2}=5c^2\end{cases}$$

$$\Rightarrow\ 2\left(\frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\right)+3\left(a^2+b^2+c^2\right)\ge 5\left(a^2+b^2+c^2\right)$$

$\Rightarrow\ \frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\ge a^2+b^2+c^2.$ Equality occurs $\Leftrightarrow\ a=b=c>0$

**Problem 7.** Prove that $\frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\ge \frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}, \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 2-degree monomials

Using ***AM − GM Inequality*** we have

$$+\begin{cases}\frac{a^5}{b^3}+\frac{a^5}{b^3}+\frac{a^5}{b^3}+\frac{a^5}{b^3}+b^2\ge 5.\sqrt[5]{\left(\frac{a^5}{b^3}\right)^4\cdot b^2}=5\frac{a^4}{b^2}\\ \frac{b^5}{c^3}+\frac{b^5}{c^3}+\frac{b^5}{c^3}+\frac{b^5}{c^3}+c^2\ge 5.\sqrt[5]{\left(\frac{b^5}{c^3}\right)^4\cdot c^2}=5\frac{b^4}{c^2}\\ \frac{c^5}{a^3}+\frac{c^5}{a^3}+\frac{c^5}{a^3}+\frac{c^5}{a^3}+a^2\ge 5.\sqrt[5]{\left(\frac{c^5}{a^3}\right)^4\cdot a^2}=5\frac{c^4}{a^2}\end{cases}$$

$$\Rightarrow 4\left(\frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\right)+(a^2+b^2+c^2)\ge 5\left(\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}\right) \quad (1)$$

On the other hand

$$+\begin{cases} \dfrac{a^4}{b^2}+b^2 \geq 2\sqrt{\dfrac{a^4}{b^2}\cdot b^2} = 2\sqrt{a^4} = 2a^2 \\ \dfrac{b^4}{c^2}+c^2 \geq 2\sqrt{\dfrac{b^4}{c^2}\cdot c^2} = 2\sqrt{b^4} = 2b^2 \\ \dfrac{c^4}{a^2}+a^2 \geq 2\sqrt{\dfrac{c^4}{a^2}\cdot a^2} = 2\sqrt{c^4} = 2c^2 \end{cases}$$

$$\Rightarrow \left(\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}\right)+(a^2+b^2+c^2) \geq 2(a^2+b^2+c^2)$$

$\Rightarrow \dfrac{a^4}{b^2}+\dfrac{b^4}{c^2}+\dfrac{c^4}{a^2} \geq a^2+b^2+c^2$ (2). From (1) and (2) follows

$$4\left(\frac{a^5}{b^3}+\frac{b^5}{c^3}+\frac{c^5}{a^3}\right)+(a^2+b^2+c^2) \geq 4\left(\frac{a^4}{b^2}+\frac{b^4}{c^2}+\frac{c^4}{a^2}\right)+(a^2+b^2+c^2)$$

$\Leftrightarrow \dfrac{a^5}{b^3}+\dfrac{b^5}{c^3}+\dfrac{c^5}{a^3} \geq \dfrac{a^4}{b^2}+\dfrac{b^4}{c^2}+\dfrac{c^4}{a^2}$ (q.e.d.).

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 8.** Prove that $\dfrac{a^3}{b^3}+\dfrac{b^3}{c^3}+\dfrac{c^3}{a^3} \geq \dfrac{a^2}{b^2}+\dfrac{b^2}{c^2}+\dfrac{c^2}{a^2} \quad \forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 0-degree monomials

Using ***AM − GM Inequality*** we have

$$+\begin{cases} \dfrac{a^3}{b^3}+\dfrac{a^3}{b^3}+1 \geq 3.\sqrt[3]{\dfrac{a^3}{b^3}\cdot\dfrac{a^3}{b^3}\cdot 1} = 3\cdot\dfrac{a^2}{b^2} \\ \dfrac{b^3}{c^3}+\dfrac{b^3}{c^3}+1 \geq 3.\sqrt[3]{\dfrac{b^3}{c^3}\cdot\dfrac{b^3}{c^3}\cdot 1} = 3\cdot\dfrac{b^2}{c^2} \\ \dfrac{c^3}{a^3}+\dfrac{c^3}{a^3}+1 \geq 3.\sqrt[3]{\dfrac{c^3}{a^3}\cdot\dfrac{c^3}{a^3}\cdot 1} = 3\cdot\dfrac{c^2}{a^2} \end{cases}$$

$$\Rightarrow 2\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)+3 \geq 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)$$

$$\geq 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+3.\sqrt[3]{\frac{a^2}{b^2}\cdot\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}} = 2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+3$$

$\Rightarrow \dfrac{a^3}{b^3}+\dfrac{b^3}{c^3}+\dfrac{c^3}{a^3} \geq \dfrac{a^2}{b^2}+\dfrac{b^2}{c^2}+\dfrac{c^2}{a^2}$ Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 9.** Prove that $\dfrac{a^2}{b^5}+\dfrac{b^2}{c^5}+\dfrac{c^2}{a^5}\ge\dfrac{1}{b^3}+\dfrac{1}{c^3}+\dfrac{1}{a^3}\quad\forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing (–3)-degree monomials

$$+\begin{cases}\dfrac{a^2}{b^5}+\dfrac{a^2}{b^5}+\dfrac{a^2}{b^5}+\dfrac{1}{a^3}+\dfrac{1}{a^3}\ge 5.\sqrt[5]{\dfrac{a^2}{b^5}\cdot\dfrac{a^2}{b^5}\cdot\dfrac{a^2}{b^5}\cdot\dfrac{1}{a^3}\cdot\dfrac{1}{a^3}}=5\cdot\dfrac{1}{b^3}\\ \dfrac{b^2}{c^5}+\dfrac{b^2}{c^5}+\dfrac{b^2}{c^5}+\dfrac{1}{b^3}+\dfrac{1}{b^3}\ge 5.\sqrt[5]{\dfrac{b^2}{c^5}\cdot\dfrac{b^2}{c^5}\cdot\dfrac{b^2}{c^5}\cdot\dfrac{1}{b^3}\cdot\dfrac{1}{b^3}}=5\cdot\dfrac{1}{c^3}\\ \dfrac{c^2}{a^5}+\dfrac{c^2}{a^5}+\dfrac{c^2}{a^5}+\dfrac{1}{c^3}+\dfrac{1}{c^3}\ge 5.\sqrt[3]{\dfrac{c^2}{a^5}\cdot\dfrac{c^2}{a^5}\cdot\dfrac{c^2}{a^5}\cdot\dfrac{1}{c^3}\cdot\dfrac{1}{c^3}}=5\cdot\dfrac{1}{a^3}\end{cases}$$

$$\Rightarrow 3\left(\frac{a^2}{b^5}+\frac{b^2}{c^5}+\frac{c^2}{a^5}\right)+2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)\ge 5\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

$\Rightarrow\dfrac{a^2}{b^5}+\dfrac{b^2}{c^5}+\dfrac{c^2}{a^5}\ge\dfrac{1}{a^3}+\dfrac{1}{b^3}+\dfrac{1}{c^3}$. Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 10.** Prove that $\dfrac{1}{a^3+b^3+abc}+\dfrac{1}{b^3+c^3+abc}+\dfrac{1}{c^3+a^3+abc}\le\dfrac{1}{abc}$, $\forall a,b,c>0$

***(USAMO 1998)***

***Proof***

***Remark:*** $x^3+y^3=(x+y)(x^2+y^2-xy)\ge(x+y)(2xy-xy)=(x+y)xy$, $\forall x,y\ge 0$

$$+\begin{cases}\dfrac{1}{a^3+b^3+abc}\le\dfrac{1}{(a+b)ab+abc}=\dfrac{1}{ab(a+b+c)}=\dfrac{c}{abc(a+b+c)}\\ \dfrac{1}{b^3+c^3+abc}\le\dfrac{1}{(b+c)bc+abc}=\dfrac{1}{bc(a+b+c)}=\dfrac{a}{abc(a+b+c)}\\ \dfrac{1}{c^3+a^3+abc}\le\dfrac{1}{(c+a)ca+abc}=\dfrac{1}{ca(a+b+c)}=\dfrac{b}{abc(a+b+c)}\end{cases}$$

$$\Rightarrow\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{a+b+c}{abc(a+b+c)}=\frac{1}{abc}$$

**Problem 11.** Prove that $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\ge\dfrac{a}{a+b}+\dfrac{b}{b+c}+\dfrac{c}{c+a}$, $\forall a,b,c>0$

***Proof***

**Remark:** Both sides are polynomials containing 0-degree monomials

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)$$

$$=\frac{b+c}{a+b}+\frac{c+a}{b+c}+\frac{a+b}{c+a}\ge 3\cdot\sqrt[3]{\frac{b+c}{a+b}\cdot\frac{c+a}{b+c}\cdot\frac{a+b}{c+a}}=3\quad(1)$$

$$\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)+\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)=\frac{a+b}{a+b}+\frac{b+c}{b+c}+\frac{c+a}{c+a}=3 \quad (2)$$

From (1) and (2) it follows: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$ (q.e.d.)

Equality occurs $\Leftrightarrow$ $a=b=c>0$

**Problem 12.** Prove that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2},\ \forall a,b,c>0$

***Proof***

$$S=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b};A=\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b};B=\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}$$

$$A+S=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\geq 3\cdot\sqrt[3]{\frac{a+b}{b+c}\cdot\frac{b+c}{c+a}\cdot\frac{c+a}{a+b}}=3$$

$$B+S=\frac{c+a}{b+c}+\frac{a+b}{c+a}+\frac{b+c}{a+b}\geq 3\cdot\sqrt[3]{\frac{c+a}{b+c}\cdot\frac{a+b}{c+a}\cdot\frac{b+c}{a+b}}=3$$

$$A+B=\frac{b+c}{b+c}+\frac{c+a}{c+a}+\frac{a+b}{a+b}=3$$

$$\Rightarrow 6\leq(A+S)+(B+S)=(A+B)+2S=3+2S\Rightarrow 3\leq 2S\Rightarrow S\geq\frac{3}{2}$$

**Problem 13.** Prove that $\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\geq 2+\frac{2(a+b+c)}{\sqrt[3]{abc}}$ (1) ***(APMO 1998)***

***Proof***

$$(1)\Leftrightarrow 2+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq 2+\frac{2(a+b+c)}{\sqrt[3]{abc}}\Leftrightarrow\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq\frac{2(a+b+c)}{\sqrt[3]{abc}} \quad (2)$$

Using ***AM − GM Inequality*** we have:

$$+\begin{cases}\frac{a}{b}+\frac{a}{b}+\frac{b}{c}\geq 3\cdot\sqrt[3]{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{b}{c}}=\frac{3a}{\sqrt[3]{abc}}\\ \frac{b}{c}+\frac{b}{c}+\frac{c}{a}\geq 3\cdot\sqrt[3]{\frac{b}{c}\cdot\frac{b}{c}\cdot\frac{c}{a}}=\frac{3b}{\sqrt[3]{abc}}\\ \frac{c}{a}+\frac{c}{a}+\frac{a}{b}\geq 3\cdot\sqrt[3]{\frac{c}{a}\cdot\frac{c}{a}\cdot\frac{a}{b}}=\frac{3c}{\sqrt[3]{abc}}\end{cases}$$

$$\Rightarrow\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b+c}{\sqrt[3]{abc}} \quad (3)$$

$$+\begin{cases}\frac{b}{a}+\frac{b}{a}+\frac{c}{b}\geq 3\cdot\sqrt[3]{\frac{b}{a}\cdot\frac{b}{a}\cdot\frac{c}{b}}=\frac{3b}{\sqrt[3]{abc}}\\ \frac{c}{b}+\frac{c}{b}+\frac{a}{c}\geq 3\cdot\sqrt[3]{\frac{c}{b}\cdot\frac{c}{b}\cdot\frac{a}{c}}=\frac{3c}{\sqrt[3]{abc}}\\ \frac{a}{c}+\frac{a}{c}+\frac{b}{a}\geq 3\cdot\sqrt[3]{\frac{a}{c}\cdot\frac{a}{c}\cdot\frac{b}{a}}=\frac{3a}{\sqrt[3]{abc}}\end{cases}$$

$$\Rightarrow\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\geq\frac{a+b+c}{\sqrt[3]{abc}} \quad (4)$$

From (3) and (4) it follows q.e.d.

**Problem 14.** Prove that $a^7+b^7+c^7 \ge a^5b^2+b^5c^2+c^5a^2$ , $\forall a,b,c>0$

***Proof***

Using ***AM − GM Inequality*** we have:

$$+\begin{cases} a^7+a^7+a^7+a^7+a^7+b^7+b^7 \ge 7\cdot\sqrt[7]{a^7\cdot a^7\cdot a^7\cdot a^7\cdot a^7\cdot b^7\cdot b^7}=7a^5b^2 \\ b^7+b^7+b^7+b^7+b^7+c^7+c^7 \ge 7\cdot\sqrt[7]{b^7\cdot b^7\cdot b^7\cdot b^7\cdot b^7\cdot c^7\cdot c^7}=7b^5c^2 \\ c^7+c^7+c^7+c^7+c^7+a^7+a^7 \ge 7\cdot\sqrt[7]{c^7\cdot c^7\cdot c^7\cdot c^7\cdot c^7\cdot a^7\cdot a^7}=7c^5a^2 \end{cases}$$

$\Rightarrow 7\left(a^7+b^7+c^7\right)\ge 7\left(a^5b^2+b^5c^2+c^5a^2\right) \Rightarrow$ (q.e.d)

**Problem 15.** Prove that $\dfrac{a^n+b^n+c^n}{3}\ge\left(\dfrac{a+b+c}{3}\right)^n$ , $\forall a,b,c>0; n\in\mathbb{N}$

***Proof***

Using ***AM − GM Inequality*** we have:

$$+\begin{cases} a^n+(n-1)\left(\dfrac{a+b+c}{3}\right)^n \ge n\cdot\sqrt[n]{a^n\left(\dfrac{a+b+c}{3}\right)^{n(n-1)}}=n\left(\dfrac{a+b+c}{3}\right)^{n-1}a \\ b^n+(n-1)\left(\dfrac{a+b+c}{3}\right)^n \ge n\cdot\sqrt[n]{b^n\left(\dfrac{a+b+c}{3}\right)^{n(n-1)}}=n\left(\dfrac{a+b+c}{3}\right)^{n-1}b \\ c^n+(n-1)\left(\dfrac{a+b+c}{3}\right)^n \ge n\cdot\sqrt[n]{c^n\left(\dfrac{a+b+c}{3}\right)^{n(n-1)}}=n\left(\dfrac{a+b+c}{3}\right)^{n-1}c \end{cases}$$

$\Rightarrow\left(a^n+b^n+c^n\right)\ge n\left(\dfrac{a+b+c}{3}\right)^{n-1}(a+b+c)-3(n-1)\left(\dfrac{a+b+c}{3}\right)^n=3\left(\dfrac{a+b+c}{3}\right)^n \Rightarrow$ (q.e.d)

**Problem 16.** Prove that $a^n+b^n+c^n\ge\left(\dfrac{a+2b}{3}\right)^n+\left(\dfrac{b+2c}{3}\right)^n+\left(\dfrac{c+2a}{3}\right)^n$ , $\forall a,b,c>0; n\in\mathbb{N}$

***Proof***

• ***Lemma:*** $\dfrac{x^n+y^n+z^n}{3}\ge\left(\dfrac{x+y+z}{3}\right)^n$ , $\forall x,y,z>0; n\in\mathbb{N}$

• ***Application:*** Using lemma we have:

$$+\begin{cases} \dfrac{a^n+b^n+b^n}{3}\ge\left(\dfrac{a+b+b}{3}\right)^n=\left(\dfrac{a+2b}{3}\right)^n \\ \dfrac{b^n+c^n+c^n}{3}\ge\left(\dfrac{b+c+c}{3}\right)^n=\left(\dfrac{b+2c}{3}\right)^n \\ \dfrac{c^n+a^n+a^n}{3}\ge\left(\dfrac{c+a+a}{3}\right)^n=\left(\dfrac{c+2a}{3}\right)^n \end{cases}$$

$\Rightarrow a^n+b^n+c^n\ge\left(\dfrac{a+2b}{3}\right)^n+\left(\dfrac{b+2c}{3}\right)^n+\left(\dfrac{c+2a}{3}\right)^n$ (q.e.d)

**Problem 17.** Let be given $a$, $b$, $c$, $d > 0$. Find the minimum value of the expression:

$$S = \frac{a}{b+c+d}+\frac{b}{c+d+a}+\frac{c}{a+b+b}+\frac{d}{a+b+c}+$$
$$+\ \frac{b+c+d}{a}+\frac{c+a+d}{b}+\frac{d+a+b}{c}+\frac{a+b+c}{d}$$

***Solution***

• ***Common mistake:***

Using ***AM − GM Inequality*** directly for eight factors

$$S \geq 8\cdot\sqrt[8]{\frac{a}{b+c+d}.\frac{b}{c+d+a}.\frac{c}{d+a+b}.\frac{d}{a+b+c}.\frac{b+c+d}{a}.\frac{c+d+a}{b}.\frac{d+a+b}{c}.\frac{a+b+c}{d}} = 8 \ \Rightarrow \text{Min S} = 8$$

• ***Causes of the mistakes:***

$$\text{Min S} = 8 \Leftrightarrow \begin{cases} a = b+c+d \\ b = c+d+a \\ c = d+a+b \\ d = a+b+c \end{cases} \Rightarrow\ a+b+c+d = 3(a+b+c+d) \Rightarrow 1 = 3 : \text{illogical}$$

• ***Analysis and solutions:***

To find Min S we need to note that $S$ is a symmetric expression for $a$, $b$, $c$, $d$; thus, Min $S$ (or Max $S$) will occur at the "free point of incidence": $a=b=c=d>0$.

Thus let be given $a = b = c = d > 0$ and estimate Min S = $\frac{4}{3}+12=13\frac{1}{3}$

It follows that the estimated condition for equality to occur in all the component inequalities is the sub-set of the estimated condition $a=b=c=d>0$

• ***Point of incidence Pattern:*** Let be given $a = b = c = d > 0$ we have

$$\begin{cases} \dfrac{a}{b+c+d}=\dfrac{b}{c+d+a}=\dfrac{c}{d+a+b}=\dfrac{d}{a+b+c}=\dfrac{1}{3} \\ \dfrac{b+c+d}{\alpha a}=\dfrac{c+d+a}{\alpha b}=\dfrac{d+a+b}{\alpha c}=\dfrac{a+b+c}{\alpha d}=\dfrac{3}{\alpha} \end{cases} \Rightarrow \frac{1}{3}=\frac{3}{\alpha} \Rightarrow \alpha = 9$$

• ***Right solution:*** Transforming and using ***AM − GM Inequality*** we have

$$S = \sum_{cyc}\left(\frac{a}{b+c+d}+\frac{b+c+d}{9a}\right)+\sum_{cyc}\frac{8}{9}.\frac{b+c+d}{9a} \geq$$

$$\geq 8\cdot\sqrt[8]{\frac{a}{b+c+d}.\frac{b}{c+d+a}.\frac{c}{d+a+b}.\frac{d}{a+b+c}.\frac{b+c+d}{9a}.\frac{c+d+a}{9b}.\frac{d+a+b}{9c}.\frac{a+b+c}{9d}}$$

$$+\ \frac{8}{9}\left(\frac{b}{a}+\frac{c}{a}+\frac{d}{a}+\frac{c}{b}+\frac{d}{b}+\frac{a}{b}+\frac{d}{c}+\frac{a}{c}+\frac{b}{c}+\frac{a}{d}+\frac{b}{d}+\frac{c}{d}\right)$$

$$\geq \frac{8}{3}+\frac{8}{9}\cdot 12\cdot\sqrt[12]{\frac{b}{a}\cdot\frac{c}{a}\cdot\frac{d}{a}\cdot\frac{c}{b}\cdot\frac{d}{b}\cdot\frac{a}{b}\cdot\frac{d}{c}\cdot\frac{a}{c}\cdot\frac{b}{c}\cdot\frac{a}{d}\cdot\frac{b}{d}\cdot\frac{c}{d}} = \frac{8}{3}+\frac{8}{9}.12=\frac{8}{3}+\frac{32}{3}=\frac{40}{3}=13\frac{1}{3}$$

For $a = b = c = d > 0$, Min S = $13\frac{1}{3}$

**Problem 18.** Let be given $a$, $b$, $c$, $d > 0$. Find the minimum value of the expression:

$$S = \left(1+\frac{2a}{3b}\right)\left(1+\frac{2b}{3c}\right)\left(1+\frac{2c}{3d}\right)\left(1+\frac{2d}{3a}\right)$$

## *Solution*

• ***Common mistake:***

$$S = \left(1+\frac{2a}{3b}\right)\left(1+\frac{2b}{3c}\right)\left(1+\frac{2c}{3d}\right)\left(1+\frac{2d}{3a}\right) \geq 2\sqrt{\frac{2a}{3b}}\cdot 2\sqrt{\frac{2b}{3c}}\cdot 2\sqrt{\frac{2c}{3d}}\cdot 2\sqrt{\frac{2d}{3a}} = \frac{64}{9} \Rightarrow \text{Min } S = \frac{64}{9}$$

• ***Causes of the mistake:***

$$\text{Min } S = \frac{64}{9} \Leftrightarrow 1 = \frac{2a}{3b}=\frac{2b}{3c}=\frac{2c}{3d}=\frac{2d}{3a} = \frac{2(a+b+c+d)}{3(a+b+c+d)}=\frac{2}{3} \Rightarrow \text{ illogical}$$

• ***Analysis and solutions:***

Since S is a symmetric expression for $a$, $b$, $c$, $d$ we estimate Min S occurs at ***Free point of incidence:*** $a = b = c = d > 0$, then S = $\left(1+\frac{2}{3}\right)^4 = \frac{625}{81}$

• ***Right solution:*** Using ***AM − GM Inequality*** we have

$$+\begin{cases} 1+\dfrac{2a}{3b} = \dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{a}{3b}+\dfrac{a}{3b} \geq 5\cdot\sqrt[5]{\left(\dfrac{1}{3}\right)^3\cdot\left(\dfrac{a}{3b}\right)^2} = \dfrac{5}{3}\left(\dfrac{a}{b}\right)^{\frac{2}{5}} \\ 1+\dfrac{2b}{3c} = \dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{b}{3c}+\dfrac{b}{3c} \geq 5\cdot\sqrt[5]{\left(\dfrac{1}{3}\right)^3\cdot\left(\dfrac{b}{3c}\right)^2} = \dfrac{5}{3}\left(\dfrac{b}{c}\right)^{\frac{2}{5}} \\ 1+\dfrac{2c}{3d} = \dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{c}{3d}+\dfrac{c}{3d} \geq 5\cdot\sqrt[5]{\left(\dfrac{1}{3}\right)^3\cdot\left(\dfrac{c}{3d}\right)^2} = \dfrac{5}{3}\left(\dfrac{c}{d}\right)^{\frac{2}{5}} \\ 1+\dfrac{2d}{3a} = \dfrac{1}{3}+\dfrac{1}{3}+\dfrac{1}{3}+\dfrac{d}{3a}+\dfrac{d}{3a} \geq 5\cdot\sqrt[5]{\left(\dfrac{1}{3}\right)^3\cdot\left(\dfrac{d}{3a}\right)^2} = \dfrac{5}{3}\left(\dfrac{d}{a}\right)^{\frac{2}{5}} \end{cases}$$

$$\Rightarrow S = \left(1+\frac{2a}{3b}\right)\left(1+\frac{2b}{3c}\right)\left(1+\frac{2c}{3d}\right)\left(1+\frac{2d}{3a}\right) \geq \frac{625}{81}\cdot\left(\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{d}\cdot\frac{d}{a}\right)^{\frac{2}{5}} = \frac{625}{81}$$

For $a = b = c = d > 0$, Min S = $\frac{625}{81}$

**4. METHOD OF SPECIALIZING INEQUALITY OF THE SAME DEGREE:**

**Problem 1.** Given $\begin{cases} a,b,c>0 \\ a+b+c=3 \end{cases}$. Prove that $\dfrac{a^3}{(a+b)(a+c)}+\dfrac{b^3}{(b+c)(b+a)}+\dfrac{c^3}{(c+a)(c+b)}\geq\dfrac{3}{4}$

***Solution***

We can transform the above inequality to the 1-degree homogeneous inequality as follows:

$$\frac{a^3}{(a+b)(a+c)}+\frac{b^3}{(b+c)(b+a)}+\frac{c^3}{(c+a)(c+b)}\geq\frac{a+b+c}{4}$$

Using AM – GM inequality, we have:

$$+\begin{cases} \dfrac{a^3}{(a+b)(a+c)}+\dfrac{a+b}{8}+\dfrac{a+c}{8}\geq 3\cdot\sqrt[3]{\dfrac{a^3}{(a+b)(a+c)}\cdot\dfrac{a+b}{8}\cdot\dfrac{a+c}{8}}=\dfrac{3a}{4} \\ \dfrac{b^3}{(b+c)(b+a)}+\dfrac{b+c}{8}+\dfrac{b+a}{8}\geq 3\cdot\sqrt[3]{\dfrac{b^3}{(b+c)(b+a)}\cdot\dfrac{b+c}{8}\cdot\dfrac{b+a}{8}}=\dfrac{3b}{4} \\ \dfrac{c^3}{(c+a)(c+b)}+\dfrac{c+a}{8}+\dfrac{c+b}{8}\geq 3\cdot\sqrt[3]{\dfrac{c^3}{(c+a)(c+b)}\cdot\dfrac{c+a}{8}\cdot\dfrac{c+b}{8}}=\dfrac{3c}{4} \end{cases}$$

$$\Rightarrow \frac{a^3}{(a+b)(a+c)}+\frac{b^3}{(b+c)(b+a)}+\frac{c^3}{(c+a)(c+b)}\geq\frac{a+b+c}{4}=\frac{3}{4} \text{ (q.e.d)}$$

**Problem 2.** Given $\begin{cases} a,b,c>0 \\ a+b+c=3 \end{cases}$. Prove that $\dfrac{a^3}{b(2c+a)}+\dfrac{b^3}{c(2a+b)}+\dfrac{c^3}{a(2b+c)}\geq 1$

***Solution***

We can transform the above inequality to the 1-degree homogeneous inequality as follows:

$\dfrac{a^3}{b(2c+a)}+\dfrac{b^3}{c(2a+b)}+\dfrac{c^3}{a(2b+c)}\geq\dfrac{a+b+c}{3}$. Using AM – GM Inequality we have:

$$+\begin{cases} \dfrac{9a^3}{b(2c+a)}+3b+(2c+a)\geq 3\cdot\sqrt[3]{\dfrac{9a^3}{b(2c+a)}\cdot 3b(2c+a)}=9a \\ \dfrac{9b^3}{c(2a+b)}+3c+(2a+b)\geq 3\cdot\sqrt[3]{\dfrac{b^3}{c(2a+b)}\cdot 3c(2a+b)}=9b \\ \dfrac{9c^3}{a(2b+c)}+3a+(2b+c)\geq 3\cdot\sqrt[3]{\dfrac{c^3}{a(2b+c)}\cdot 3a(2b+c)}=9c \end{cases}$$

$$\Rightarrow 9\left[\frac{a^3}{b(2c+a)}+\frac{b^3}{c(2a+b)}+\frac{c^3}{a(2b+c)}\right]+6(a+b+c)\geq 9(a+b+c)$$

$$\Rightarrow \frac{a^3}{b(2c+a)}+\frac{b^3}{c(2a+b)}+\frac{c^3}{a(2b+c)}\geq\frac{a+b+c}{3}=1 \text{ (q.e.d)}$$

**Problem 3.** Given $\begin{cases} a,b,c>0 \\ a^2+b^2+c^2=1 \end{cases}$. Prove that $\frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b}\ge\frac{1}{3}$

***Solution***

We can transform the above inequality to the 2-degree homogeneous inequality as follows:

$$\frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b}\ge\frac{a^2+b^2+c^2}{3}$$

Using AM – GM Inequality we have:

$$+\begin{cases} \frac{9a^3}{b+2c}+a(b+2c)\ge 2\sqrt{\frac{9a^3}{b+2c}\cdot a(b+2c)}=6a^2 \\ \frac{9b^3}{c+2a}+b(c+2a)\ge 2\sqrt{\frac{9b^3}{c+2a}\cdot b(c+2a)}=6b^2 \\ \frac{9c^3}{a+2b}+c(a+2b)\ge 2\sqrt{\frac{9c^3}{a+2b}\cdot c(a+2b)}=6c^2 \end{cases}$$

$$\Rightarrow 9\left(\frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b}\right)+3(ab+bc+ca)\ge 6(a^2+b^2+c^2)$$

$$\Rightarrow 9\left(\frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b}\right)\ge 3(a^2+b^2+c^2)$$

$$\Rightarrow \frac{a^3}{b+2c}+\frac{b^3}{c+2a}+\frac{c^3}{a+2b}\ge\frac{a^2+b^2+c^2}{3}=\frac{1}{3}$$

**Problem 4.** Given $\begin{cases} a,b,c>0 \\ ab+bc+ca=1 \end{cases}$. Prove that $\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+a)}\ge\frac{9}{2}$

***Solution***

We can transform the above inequality to the 0-degree homogeneous inequality as follows:

$$\frac{c(a+b)+ab}{a(a+b)}+\frac{a(b+c)+bc}{b(b+c)}+\frac{b(c+a)+ca}{c(c+a)}\ge\frac{9}{2}$$

$$\Leftrightarrow \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)\ge\frac{9}{2}$$

$$\Leftrightarrow \left(\frac{a+b}{b}+\frac{b+c}{c}+\frac{c+a}{a}\right)+\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right)\ge\frac{15}{2} \quad (1)$$

We have: LHS (1) = $\frac{a+b}{4b}+\frac{b+c}{4c}+\frac{c+a}{4a}+\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}+\frac{3}{4}\left(\frac{a+b}{b}+\frac{b+c}{c}+\frac{c+a}{a}\right)$

$$\ge 6\cdot\sqrt[6]{\frac{a+b}{4b}\cdot\frac{b+c}{4c}\cdot\frac{c+a}{4a}\cdot\frac{b}{a+b}\cdot\frac{c}{b+c}\cdot\frac{a}{c+a}}+\frac{3}{4}\left(3\cdot\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}+3\right)=\frac{15}{2}$$

**Problem 5.** Given $\begin{cases} a,b,c>0 \\ a+b+c=1 \end{cases}$. Prove that $\dfrac{a}{(b+c)^2}+\dfrac{b}{(c+a)^2}+\dfrac{c}{(a+b)^2}\ge\dfrac{9}{4}$

***Solution***

• ***Lemma:*** $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\ge\dfrac{3}{2},\ \forall a,b,c>0$

$x^2+y^2+z^2\ge\dfrac{1}{3}(x+y+z)^2,\ \forall x,y,z\in\mathbb{R}$

• ***Application:***

We can transform the above inequality to the 0-degree homogeneous inequality as follows:

$$\frac{a(a+b+c)}{(b+c)^2}+\frac{b(a+b+c)}{(c+a)^2}+\frac{c(a+b+c)}{(a+b)^2}\ge\frac{9}{4}$$

$$\Leftrightarrow\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{9}{4}\ (1)$$

We have: LHS (1) $\ge\dfrac{1}{3}\left(\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\right)^2+\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}\ge\dfrac{1}{3}\left(\dfrac{3}{2}\right)^2+\dfrac{3}{2}=\dfrac{9}{4}\Rightarrow$ (q.e.d)

**Problem 6.** Given $\begin{cases} a,b,c>0 \\ a^2+b^2+c^2=3 \end{cases}$. Prove that $\dfrac{ab}{c}+\dfrac{bc}{a}+\dfrac{ca}{b}\ge 3\quad(1)$

***(France Pre-MO 2005)***

***Solution***

$$(1)\Leftrightarrow\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)^2\ge 9\Leftrightarrow\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+2(a^2+b^2+c^2)\ge 3(a^2+b^2+c^2)$$

$\Leftrightarrow\dfrac{a^2b^2}{c^2}+\dfrac{b^2c^2}{a^2}+\dfrac{c^2a^2}{b^2}\ge a^2+b^2+c^2$. Using AM – GM Inequality we have:

$$+\begin{cases}\dfrac{a^2b^2}{c^2}+\dfrac{b^2c^2}{a^2}\ge 2\sqrt{\dfrac{a^2b^2}{c^2}\cdot\dfrac{b^2c^2}{a^2}}=2b^2\\[2ex] \dfrac{b^2c^2}{a^2}+\dfrac{c^2a^2}{b^2}\ge 2\sqrt{\dfrac{b^2c^2}{a^2}\cdot\dfrac{c^2a^2}{b^2}}=2c^2\\[2ex] \dfrac{c^2a^2}{b^2}+\dfrac{a^2b^2}{c^2}\ge 2\sqrt{\dfrac{c^2a^2}{b^2}\cdot\dfrac{a^2b^2}{c^2}}=2a^2\end{cases}$$

$$\Rightarrow\frac{a^2b^2}{c^2}+\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}\ge a^2+b^2+c^2\Rightarrow\text{(q.e.d)}$$

**Problem 7.** Given $\begin{cases} a,b,c>0 \\ abc=1 \end{cases}$. Prove that $\dfrac{a^3}{(1+b)(1+c)}+\dfrac{b^3}{(1+c)(1+a)}+\dfrac{c^3}{(1+a)(1+b)}\geq\dfrac{3}{4}$

***(IMO Shortlist 1998)***

***Solution***

Using AM – GM Inequality we have:

$$+\begin{cases} \dfrac{a^3}{(1+b)(1+c)}+\dfrac{1+b}{8}+\dfrac{1+c}{8}\geq 3\cdot\sqrt[3]{\dfrac{a^3}{(1+b)(1+c)}\cdot\dfrac{1+b}{8}\cdot\dfrac{1+c}{8}}=\dfrac{3a}{4} \\ \dfrac{b^3}{(1+c)(1+a)}+\dfrac{1+c}{8}+\dfrac{1+a}{8}\geq 3\cdot\sqrt[3]{\dfrac{b^3}{(1+c)(1+a)}\cdot\dfrac{1+c}{8}\cdot\dfrac{1+a}{8}}=\dfrac{3b}{4} \\ \dfrac{c^3}{(1+a)(1+b)}+\dfrac{1+a}{8}+\dfrac{1+b}{8}\geq 3\cdot\sqrt[3]{\dfrac{c^3}{(1+a)(1+b)}\cdot\dfrac{1+a}{8}\cdot\dfrac{1+b}{8}}=\dfrac{3c}{4} \end{cases}$$

$$\Rightarrow \frac{a^3}{(a+b)(a+c)}+\frac{b^3}{(b+c)(b+a)}+\frac{c^3}{(c+a)(c+b)}\geq\frac{a+b+c}{2}-\frac{3}{4}\geq\frac{3\cdot\sqrt[3]{abc}}{2}-\frac{3}{4}=\frac{3}{4}$$

**Problem 8.** Given $\begin{cases} a,b,c>0 \\ abc=1 \end{cases}$. Prove that $\dfrac{1}{a^3(b+c)}+\dfrac{1}{b^3(c+a)}+\dfrac{1}{c^3(a+b)}\geq\dfrac{3}{2}$ (1)

***(IMO 1995)***

***Solution***

We can transform the above inequality to a stronger homogeneous one as follows:

$$\frac{abc}{a^3(b+c)}+\frac{abc}{b^3(c+a)}+\frac{abc}{c^3(a+b)}\geq\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\Leftrightarrow\frac{\frac{1}{a^2}}{\frac{1}{b}+\frac{1}{c}}+\frac{\frac{1}{b^2}}{\frac{1}{c}+\frac{1}{a}}+\frac{\frac{1}{c^2}}{\frac{1}{a}+\frac{1}{b}}\geq\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\quad(2)$$

Letting $x=\dfrac{1}{a}, y=\dfrac{1}{b}, z=\dfrac{1}{c}$ for $x,y,z>0$, $xyz=1$.

(2) $\Leftrightarrow \dfrac{x^2}{y+z}+\dfrac{y^2}{z+x}+\dfrac{z^2}{x+y}\geq\dfrac{1}{2}(x+y+z)$. Using AM – GM Inequality we have:

$$+\begin{cases} \dfrac{x^2}{y+z}+\dfrac{y+z}{4}\geq 2\cdot\sqrt{\dfrac{x^2}{y+z}\cdot\dfrac{y+z}{4}}=x \\ \dfrac{y^2}{z+x}+\dfrac{z+x}{4}\geq 2\cdot\sqrt{\dfrac{y^2}{z+x}\cdot\dfrac{z+x}{4}}=y \\ \dfrac{z^2}{x+y}+\dfrac{x+y}{4}\geq 2\cdot\sqrt{\dfrac{z^2}{x+y}\cdot\dfrac{x+y}{4}}=z \end{cases}$$

$$\Rightarrow \frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{1}{2}(x+y+z)\geq\frac{1}{2}\cdot 3\cdot\sqrt[3]{xyz}=\frac{3}{2}\text{ (q.e.d)}$$

**Problem 9.** Let $\begin{cases} a,b,c>0 \\ abc=1 \end{cases}$. Prove that $\dfrac{ab}{a^5+b^5+ab}+\dfrac{bc}{b^5+c^5+bc}+\dfrac{ca}{c^5+a^5+ca}\le 1$

***(IMO Shortlist 1996)***

***Solution***

• ***Lemma:*** $x^5+y^5\ge x^3y^2+x^2y^3=x^2y^2(x+y)$, $\forall x,y>0$

• ***Application:*** Using ***Lemma*** we have

$$+\begin{cases} \dfrac{ab}{a^5+b^5+ab}\le\dfrac{ab}{a^2b^2(a+b)+ab}=\dfrac{1}{ab(a+b)+1}=\dfrac{abc}{ab(a+b)+abc}=\dfrac{c}{a+b+c} \\ \dfrac{bc}{b^5+c^5+bc}\le\dfrac{bc}{b^2c^2(b+c)+bc}=\dfrac{1}{bc(b+c)+1}=\dfrac{abc}{bc(b+c)+abc}=\dfrac{a}{a+b+c} \\ \dfrac{ca}{c^5+a^5+ca}\le\dfrac{ca}{c^2a^2(c+a)+ca}=\dfrac{1}{ca(c+a)+1}=\dfrac{abc}{ca(c+a)+abc}=\dfrac{b}{a+b+c} \end{cases}$$

$$\Rightarrow \dfrac{ab}{a^5+b^5+ab}+\dfrac{bc}{b^5+c^5+bc}+\dfrac{ca}{c^5+a^5+ca}\le\dfrac{a+b+c}{a+b+c}=1$$

**Problem 10.** Given $\begin{cases} a,b,c>0 \\ abc=1 \end{cases}$. Prove that $\dfrac{a^2(b+c)}{b\sqrt{b}+2c\sqrt{c}}+\dfrac{b^2(c+a)}{c\sqrt{c}+2a\sqrt{a}}+\dfrac{c^2(a+b)}{a\sqrt{a}+b\sqrt{b}}\ge 2 \quad (1)$

***Solution***

$$+\begin{cases} \dfrac{a^2(b+c)}{b\sqrt{b}+2c\sqrt{c}}\ge\dfrac{a^2.2\sqrt{bc}}{b\sqrt{b}+2c\sqrt{c}}=\dfrac{2a\sqrt{a}}{b\sqrt{b}+2c\sqrt{c}}=\dfrac{2x}{y+2z} \\ \dfrac{b^2(c+a)}{c\sqrt{c}+2a\sqrt{a}}\ge\dfrac{b^2.2\sqrt{ca}}{c\sqrt{c}+2a\sqrt{a}}=\dfrac{2b\sqrt{b}}{c\sqrt{c}+2a\sqrt{a}}=\dfrac{2y}{z+2x} \\ \dfrac{c^2(a+b)}{a\sqrt{a}+b\sqrt{b}}\ge\dfrac{c^2.2\sqrt{ab}}{a\sqrt{a}+b\sqrt{b}}=\dfrac{2c\sqrt{c}}{a\sqrt{a}+b\sqrt{b}}=\dfrac{2z}{x+2y} \end{cases} \quad \text{in which} \begin{cases} x=a\sqrt{a}>0 \\ y=b\sqrt{b}>0 \\ z=c\sqrt{c}>0 \\ xyz=1 \end{cases}$$

$$\Rightarrow S=\dfrac{a^2(b+c)}{b\sqrt{b}+2c\sqrt{c}}+\dfrac{b^2(c+a)}{c\sqrt{c}+2a\sqrt{a}}+\dfrac{c^2(a+b)}{a\sqrt{a}+b\sqrt{b}}\ge\dfrac{2x}{y+2z}+\dfrac{2y}{z+2x}+\dfrac{2z}{x+2y}=T$$

Letting $\begin{cases} y+2z=m \\ z+2x=n \\ x+2y=p \end{cases} \Rightarrow \begin{cases} x=\dfrac{4n+p-2m}{9} \\ y=\dfrac{4p+m-2n}{9} \\ z=\dfrac{4m+n-2p}{9} \end{cases} \Rightarrow T=\dfrac{2}{9}\left(\dfrac{4n+p-2m}{m}+\dfrac{4p+m-2n}{n}+\dfrac{4m+n-2p}{p}\right)$

$$=\frac{2}{9}\left[4\left(\frac{n}{m}+\frac{m}{p}+\frac{p}{n}\right)+\left(\frac{p}{m}+\frac{m}{n}+\frac{n}{p}\right)-6\right]\ge\frac{2}{9}\left(4\cdot 3\cdot\sqrt[3]{\frac{n}{m}\cdot\frac{m}{p}\cdot\frac{p}{n}}+3\cdot\sqrt[3]{\frac{p}{m}\cdot\frac{m}{n}\cdot\frac{n}{p}}-6\right)=\frac{2}{9}(12+3-6)=2$$

Equality occurs $\Leftrightarrow m=n=p \Leftrightarrow x=y=z=1 \Leftrightarrow a=b=c=1$

**5. NON – SYMMETRIC POINT OF INCIDENCE IN AM – GM INEQUALITY**

The following problems express essence of the term "point of incidence" correctly; means that we can find any "point of incidence" and build up an inequality that reaches it extremum at this point. Since these point are not symmetric so the readers must speculate the point of incidence base on the condition and expression structure given.

**Problem 1.** Given $\begin{cases} a,b,c>0 \\ a+2b+3c\geq 20 \end{cases}$. Prove that S = $a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}\geq 13$

***Proof***

**Predict S = 1 at the point:** $a = 2$, $b = 3$, $c = 4$

Applying AM – GM inequality, we have:

$$\begin{cases} a+\frac{4}{a}\geq 2\sqrt{a\cdot\frac{4}{a}}=4 \\ b+\frac{9}{b}\geq 2\sqrt{b\cdot\frac{9}{b}}=6 \\ c+\frac{16}{c}\geq 2\sqrt{c\cdot\frac{16}{c}}=8 \end{cases} \Rightarrow \begin{cases} \frac{3}{4}\left(a+\frac{4}{a}\right)\geq\frac{3}{4}\cdot 4=3 \\ \frac{1}{2}\left(b+\frac{9}{b}\right)\geq\frac{1}{2}\cdot 6=3 \\ \frac{1}{4}\left(c+\frac{16}{c}\right)\geq\frac{1}{4}\cdot 8=2 \end{cases}$$

$$\Rightarrow \frac{3}{4}a+\frac{1}{2}b+\frac{1}{4}c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}\geq 8 \qquad (1)$$

We have $a + 2b + 3c \geq 20 \Rightarrow \frac{1}{4}a+\frac{b}{2}+\frac{3}{4}c\geq 5$ (2)

Add (1) and (2) $\Rightarrow$ S = $a+b+c+\frac{3}{a}+\frac{9}{2b}+\frac{4}{c}\geq 13$

**Problem 2.** Given $a, b, c > 0$. Prove that

$$S = 30a+3b^2+\frac{2c^3}{9}+36\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 84$$

***Proof***

**Predict S = 84 at the point:** $a = 1$, $b = 2$, $c = 3$

Using AM – GM inequality, we have:

$$\begin{cases} 2.a+1\cdot\frac{b^2}{4}+2\cdot\frac{2}{ab}\geq 5\cdot\sqrt[5]{a^2\cdot\frac{b^2}{4}\left(\frac{2}{ab}\right)^2}=5 \\ 3\cdot\frac{b^2}{4}+2\cdot\frac{c^3}{27}+6\cdot\frac{6}{bc}\geq 11\cdot\sqrt[11]{\left(\frac{b^2}{4}\right)^3\left(\frac{c^3}{27}\right)^2\left(\frac{6}{bc}\right)^6}=11 \\ \frac{c^3}{27}+3.a+3\cdot\frac{3}{ca}\geq 7\cdot\sqrt[7]{\frac{c^3}{27}\cdot a^3\left(\frac{3}{ca}\right)^3}=7 \end{cases} \Rightarrow \begin{cases} 9\left(2a+\frac{b^2}{4}+\frac{4}{ab}\right)\geq 45 \\ \frac{3b^2}{4}+\frac{2c^3}{27}+\frac{36}{bc}\geq 11 \\ 4\left(\frac{c^3}{27}+3a+\frac{9}{ca}\right)\geq 28 \end{cases}$$

$$\Rightarrow 30a+3b^2+\frac{2c^3}{9}+36\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\geq 45+11+28\geq 84$$

**Problem 3.** Given $\begin{cases} a,b,c>0 \\ ac \ge 12\,;\, bc \ge 8 \end{cases}$. Prove that $S = (a+b+c)+2\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\frac{8}{abc} \ge \frac{121}{2}$

***Proof***

**Predict S =** $\frac{121}{2}$ **at the point:** $a = 3,\ b = 2,\ c = 4$

Using AM – GM inequality, we have:

$$\begin{cases} \frac{a}{3}+\frac{b}{2}+\frac{6}{ab} \ge 3\cdot\sqrt[3]{\frac{a}{3}\cdot\frac{b}{2}\cdot\frac{6}{ab}} = 3 \\ \frac{b}{2}+\frac{c}{4}+\frac{8}{bc} \ge 3\cdot\sqrt[3]{\frac{b}{2}\cdot\frac{c}{4}\cdot\frac{8}{bc}} = 3 \\ \frac{c}{4}+\frac{a}{3}+\frac{12}{ca} \ge 3\cdot\sqrt[3]{\frac{c}{4}\cdot\frac{a}{3}\cdot\frac{12}{ca}} = 3 \\ \frac{a}{3}+\frac{b}{2}+\frac{c}{4}+\frac{24}{abc} \ge 4\cdot\sqrt[4]{\frac{a}{3}\cdot\frac{b}{2}\cdot\frac{c}{4}\cdot\frac{24}{abc}} = 4 \end{cases} \Rightarrow \begin{cases} 1\cdot\left(\frac{a}{3}+\frac{b}{2}+\frac{6}{ab}\right) \ge 3 \\ 4\cdot\left(\frac{b}{2}+\frac{c}{4}+\frac{8}{bc}\right) \ge 12 \\ 7\cdot\left(\frac{c}{4}+\frac{a}{3}+\frac{12}{ca}\right) \ge 21 \\ 1\cdot\left(\frac{a}{3}+\frac{b}{2}+\frac{c}{4}+\frac{24}{abc}\right) \ge 4 \end{cases}$$

$3(a+b+c)+\frac{6}{ab}+\frac{32}{bc}+\frac{84}{ca}+\frac{24}{abc} \ge 40$. Also $\begin{cases} ac \ge 12 \\ bc \ge 8 \end{cases}$ therefore $\frac{1}{ac} \le 12; \frac{1}{bc} \le 8$

Hence $40 \le 3S + 26\cdot\frac{1}{bc}+78\cdot\frac{1}{ca} \le 3S+26\cdot\frac{1}{8}+78\cdot\frac{1}{12}$

$\Rightarrow \frac{363}{2} \le 3S \;\Leftrightarrow \frac{121}{2} \le S$ (q.e.d)

**Problem 4.** Let be given $a,b,c>0$ satisfying $a+b+c=1$. Prove that

$$S = a^4b+b^4c+c^4a \le \frac{256}{3125}$$

***Proof***

WLOG, supposing $a = \text{Max}\{a,b,c\} \Rightarrow b^4c \le a^3bc$ and $c^4a \le c^2a^3 \le ca^4$.

Since $\frac{3c}{4} \ge \frac{c}{2}$ then transform the expression $S$ we have

$$S = a^4b+b^4c+\frac{c^4a}{2}+\frac{c^4a}{2} \le a^4b+a^3bc+\frac{ca^4}{2}+\frac{c^2a^3}{2} = a^3b(a+c)+\frac{a^3c}{2}(a+c) = a^3(a+c)\left(b+\frac{c}{2}\right)$$

$$\Rightarrow S = a^4b+b^4c+c^4a \le a^3(a+c)\left(b+\frac{c}{2}\right) \le a^3(a+c)\left(b+\frac{3c}{2}\right)$$

$$S = 4^4\,\frac{a}{4}\cdot\frac{a}{4}\cdot\frac{a}{4}\cdot\frac{a+c}{4}\left(b+\frac{3c}{2}\right) \le 4^4\left[\frac{\frac{a}{4}+\frac{a}{4}+\frac{a}{4}+\frac{a+c}{4}+\left(b+\frac{3c}{2}\right)}{5}\right]^5$$

$$= 4^4.\left(\frac{a+b+c}{5}\right)^5 = \frac{4^4}{5^5} = \frac{256}{3125} \Rightarrow S = a^4b+b^4c+c^4a \le \frac{256}{3125}$$

Equality occurs (corresponding to $a$ = Max ($a$, $b$, $c$)) $\Leftrightarrow \begin{cases} a+b+c=1; c=0 \\ \frac{a}{4}=\frac{a+c}{4}=b+\frac{3c}{4} \end{cases} \Leftrightarrow a=\frac{4}{5}; b=\frac{1}{5}; c=0$

In the general case the inequality becomes equality $\Leftrightarrow$ ($a$, $b$, $c$) is the permutation $\left(\frac{4}{5},\frac{1}{5},0\right)$

**Problem 5.** Given $0\le a\le b\le c\le 1$. Prove that: $a^2(b-c)+b^2(c-b)+c^2(1-c)\le\frac{108}{529}$

***Proof***

Transforming and using ***AM – GM Inequality*** we have

$$a^2(b-c)+b^2(c-b)+c^2(1-c)\le 0+\frac{1}{2}\left[b.b\left(2c-2b\right)\right]+c^2(1-c)$$

$$\le \frac{1}{2}\cdot\left(\frac{b+b+2c-2b}{3}\right)^3+c^2(1-c)=c^2\left(\frac{4}{27}\cdot c+1-c\right)=c^2\left(1-\frac{23}{27}\cdot c\right)$$

$$=\left(\frac{54}{23}\right)^2\left(\frac{23}{54}\cdot c\right)\left(\frac{23}{54}\cdot c\right)\left(1-\frac{23}{27}\cdot c\right)\le\left(\frac{54}{23}\right)^2\cdot\left(\frac{1}{3}\right)^3=\frac{108}{529}$$

Equality occurs $\Leftrightarrow$ a = 0, b = $\frac{12}{23}$, c = $\frac{18}{23}$

**Problem 6.** Let $x$, $y$, $z$, $t\in[0, 1]$. Find the minimum value of

$$S = x^2y+y^2z+z^2t+t^2x-xy^2-yz^2-zt^2-tx^2$$

***Solution***

WLOG suppose $x=\text{Max}\{x,y,z,t\}$. Then we have

$$S=y\left(x^2-z^2+yz-xy\right)+t\left(z^2-x^2+xt-zt\right)=y(x-z)(x+z-y)+t(z-x)(x+z-t)$$

$$\le y(x-z)(x+z-y)+0\le\left[\frac{y+(x-z)+(x+z-y)}{3}\right]^3=\frac{8}{27}\cdot x^3\le\frac{8}{27}\cdot 1^3=\frac{8}{27}$$

For $x=1; y=\frac{2}{3}; z=\frac{1}{3}; t=0$, Max $S=\frac{8}{27}$

**Problem 7.** Let be given $a,b,c\ge 0$ satisfying $a+b+c=3$. Prove that

$$\left(a^2-ab+b^2\right)\left(b^2-bc+c^2\right)\left(c^2-ca+a^2\right)\le 12 \quad (1)$$

***Proof***

WLOG supposing $a\ge b\ge c\ge 0 \Rightarrow \begin{cases}0\le b^2-bc+c^2\le b^2\\ 0\le c^2-ca+a^2\le a^2\end{cases} \Rightarrow \left(b^2-bc+c^2\right)\left(c^2-ca+a^2\right)\le a^2b^2$

$$\text{LHS }(1)\le a^2b^2\left(a^2-ab+b^2\right)=\frac{4}{9}\cdot\frac{3ab}{2}\cdot\frac{3ab}{2}\left(a^2-ab+b^2\right)\le\frac{4}{9}\left[\frac{1}{3}\left(\frac{3ab}{2}+\frac{3ab}{2}+\left(a^2-ab+b^2\right)\right)\right]^3$$

$$=\frac{4}{9}\left(\frac{(a+b)^2}{3}\right)^3\le\frac{4}{9}\left(\frac{(a+b+c)^2}{3}\right)^3=\frac{4}{9}\cdot\frac{(a+b+c)^6}{3^3}=\frac{4}{9}\cdot\frac{3^6}{3^3}=\frac{4}{9}\cdot 3^3=12$$

Equality occurs $\Leftrightarrow \dfrac{3ab}{2}=a^2-ab+b^2 \quad ,c=0\Leftrightarrow a=2,b=1,c=0$

**Problem 8.** Given $\begin{cases}a\ge 7\,;5a\ +7b\ge 70\\ 10a+14b+35c\ge 210\end{cases}$. Find the minimum value of $\mathbf{S}=\ a^3+b^3+c^3$

***Solution***

• ***Lemma:*** $\dfrac{x^3+y^3+z^3}{3}\ge\left(\dfrac{x+y+z}{3}\right)^3 \qquad \forall x,\ y\ ,\ z>0$

• ***Proof:*** $+\begin{cases}x^3+\left(\dfrac{x+y+z}{3}\right)^3+\left(\dfrac{x+y+z}{3}\right)^3\ge 3x\left(\dfrac{x+y+z}{3}\right)^2\\ y^3+\left(\dfrac{x+y+z}{3}\right)^3+\left(\dfrac{x+y+z}{3}\right)^3\ge 3y\left(\dfrac{x+y+z}{3}\right)^2\\ z^3+\left(\dfrac{x+y+z}{3}\right)^3+\left(\dfrac{x+y+z}{3}\right)^3\ge 3z\left(\dfrac{x+y+z}{3}\right)^2\end{cases}$

$$\Rightarrow\ x^3+y^3+z^3+6\left(\frac{x+y+z}{3}\right)^3\ge 3\left(x+y+z\right)\left(\frac{x+y+z}{3}\right)^2\Leftrightarrow x^3+y^3+z^3\ge 3\left(\frac{x+y+z}{3}\right)^3$$

• ***Application:***

$$S=a^3+b^3+c^3=8\left[\left(\frac{a}{7}\right)^3+\left(\frac{b}{5}\right)^3+\left(\frac{c}{2}\right)^3\right]+117\left[\left(\frac{a}{7}\right)^3+\left(\frac{b}{5}\right)^3\right]+218\left(\frac{a}{7}\right)^3$$

$$\ge 8\left[3\left(\frac{\frac{a}{7}+\frac{b}{5}+\frac{c}{2}}{3}\right)^3\right]+117\left[2\left(\frac{\frac{a}{7}+\frac{b}{5}}{2}\right)^3\right]+218\left(\frac{a}{7}\right)^3\ge 8\times 3+117\times 2+218=476$$

With $a=7$, $b=5$, $c=2$ we have Min S = 476

**6. METHOD OF EQUALIZING COEFFICIENTS:**

In five sections above, we can predict the point of incidence of inequalities based on the conditions given in the assumption and mathematical features in the expressions. However, if the point of incidence is not that easy to be predicted, it is necessary to find out a method that can help define the exact point. This is also the content of balancing coefficient technique.

**Problem 1.** Let $x \in [0, 1]$. Find the maximum value of $\mathbf{S} = 13\sqrt{x^2 - x^4} + 9\sqrt{x^2 + x^4}$

***Solution***

For $\alpha, \beta > 0$, using AM – GM inequality we have:

$$+\begin{cases} 13\sqrt{x^2 - x^4} = \dfrac{13}{\alpha} \cdot \sqrt{\alpha^2 x^2 (1 - x^2)} \le \dfrac{13}{\alpha} \cdot \dfrac{\alpha^2 x^2 + (1 - x^2)}{2} = \dfrac{13(\alpha^2 - 1)x^2 + 13}{2\alpha} \\ 9\sqrt{x^2 + x^4} = \dfrac{9}{\beta} \cdot \sqrt{\beta^2 x^2 (1 + x^2)} \le \dfrac{9}{\beta} \cdot \dfrac{\beta^2 x^2 + (1 + x^2)}{2} = \dfrac{9(\beta^2 + 1)x^2 + 9}{2\beta} \end{cases}$$

$$\Rightarrow S = 13\sqrt{x^2 - x^4} + 9\sqrt{x^2 + x^4} \le \left[\frac{13(\alpha^2 - 1)}{2\alpha} + \frac{9(\beta^2 + 1)}{2\beta}\right]x^2 + \frac{13}{2\alpha} + \frac{9}{2\beta}$$

Equality occurs $\Leftrightarrow \begin{cases} \alpha^2 x^2 = 1 - x^2 \\ \beta^2 x^2 = 1 + x^2 \end{cases} \Leftrightarrow (\alpha^2 + 1)x^2 = (\beta^2 - 1)x^2 = 1$

Choose $\alpha, \beta > 0$ such that $\begin{cases} \alpha^2 + 1 = \beta^2 - 1 \\ \dfrac{13(\alpha^2 - 1)}{2\alpha} + \dfrac{9(\beta^2 + 1)}{2\beta} = 0 \end{cases} \Leftrightarrow \alpha = \dfrac{1}{2}; \beta = \dfrac{3}{2}$. Then $S \le \dfrac{13}{2\alpha} + \dfrac{9}{2\beta} = 16$

Equality occurs $\Leftrightarrow (\alpha^2 + 1)x^2 = 1 \Leftrightarrow \dfrac{5}{4}x^2 = 1 \Leftrightarrow x = \dfrac{2\sqrt{5}}{5}$. Thus Max S = 16

**Problem 2.** Let be given $a > 0$ and $x^2 + y^2 + z^2 + \dfrac{9}{16}xy = a^2$.

Find the maximum value of $S = xy + yz + zx$

***Solution***

For $\alpha \in (0;1)$, using AM – GM inequality we have:

$$+\begin{cases} \alpha(x^2 + y^2) \ge 2\alpha.xy \\ (1 - \alpha)x^2 + \dfrac{z^2}{2} \ge 2\sqrt{(1 - \alpha)x^2 \cdot \dfrac{z^2}{2}} \ge \sqrt{2(1 - \alpha)}.xz \\ (1 - \alpha)y^2 + \dfrac{z^2}{2} \ge 2\sqrt{(1 - \alpha)y^2 \cdot \dfrac{z^2}{2}} \ge \sqrt{2(1 - \alpha)}.yz \end{cases}$$

$$\Rightarrow x^2 + y^2 + z^2 \ge 2\alpha.xy + \sqrt{2(1 - \alpha)}(xz + yz) \Rightarrow a^2 \ge \left(2\alpha + \frac{9}{16}\right)xy + \sqrt{2(1 - \alpha)}(xz + yz)$$

Choose $\alpha\in(0;1)$ such that $2\alpha+\frac{9}{16}=\sqrt{2(1-\alpha)}\Leftrightarrow\alpha=\frac{12\sqrt{5}-17}{32}$.

Then we have $a^2\ge\frac{3\sqrt{5}-2}{4}(xy+yz+zx)\Rightarrow S=xy+yz+zx\le\frac{4a^2}{3\sqrt{5}-2}=\frac{4(3\sqrt{5}+2)}{41}a^2$

$\Rightarrow$ Max $S=\frac{4(3\sqrt{5}+2)}{41}a^2$

Equality occurs $\Leftrightarrow\begin{cases}\sqrt{\frac{49-12\sqrt{5}}{32}}\,x=\sqrt{\frac{49-12\sqrt{5}}{32}}\,y=\frac{\sqrt{2}}{2}z\\ x^2+y^2+z^2+\frac{9}{16}xy=a^2\end{cases}\Leftrightarrow\left[\begin{array}{l}x=y=\frac{4a}{\sqrt{90-12\sqrt{5}}};z=\sqrt{\frac{49-12\sqrt{5}}{90-12\sqrt{5}}}\,a\\ x=y=\frac{-4a}{\sqrt{90-12\sqrt{5}}};z=-\sqrt{\frac{49-12\sqrt{5}}{90-12\sqrt{5}}}\,a\end{array}\right.$

**Problem 3.** Given $\begin{cases}a,b,c\ge 0\\ a^2+2b^2+3c^2=1\end{cases}$ Find the minimum value of $S=2a^3+3b^3+4c^3$

***Solution***

Supposing $\alpha,\beta,\gamma\ge 0$ and using AM – GM inequality, we have:

$$+\begin{cases}a^3+a^3+\alpha^3\ge 3\alpha a^2\\ \frac{3}{2}(b^3+b^3+\beta^3)\ge\frac{9}{2}\beta b^2\\ 2(c^3+c^3+\gamma^3)\ge 6\gamma c^2\end{cases}$$

$\Rightarrow 2a^3+3b^3+4c^3+\alpha^3+\frac{3}{2}\beta^3+2\gamma^3\ge 3\alpha.a^2+\frac{9}{4}\beta\cdot 2b^2+2\gamma\cdot 3c^2$

Equality occurs $\Leftrightarrow\begin{cases}a=\alpha,b=\beta,c=\gamma\\ a^2+2b^2+3c^2=1\end{cases}$

Choose $\alpha,\beta,\gamma$ such that $\begin{cases}3\alpha=\frac{9}{4}\beta=2\gamma=k\ge 0\\ \alpha^2+2\beta^2+3\gamma^2=1\end{cases}\Leftrightarrow\begin{cases}3\alpha=\frac{9}{4}\beta=2\gamma=k>0\\ \left(\frac{1}{9}+\frac{32}{81}+\frac{3}{4}\right)k^2=1\end{cases}\Leftrightarrow 3\alpha=\frac{9}{4}\beta=2\gamma=k=\frac{18}{\sqrt{407}}$

Then we have: $2a^3+3b^3+4c^3+\left(\frac{6}{\sqrt{407}}\right)^3+\frac{3}{2}\left(\frac{8}{\sqrt{407}}\right)^3+2\left(\frac{9}{\sqrt{407}}\right)^3\ge\frac{18}{\sqrt{407}}(a^2+2b^2+3c^2)$

$$\Leftrightarrow 2a^3+3b^3+4c^3+\frac{6}{\sqrt{407}}\ge\frac{18}{\sqrt{407}}\Leftrightarrow 2a^3+3b^3+4c^3\ge\frac{12}{\sqrt{407}}$$

For $a=\frac{6}{\sqrt{407}},b=\frac{8}{\sqrt{407}},c=\frac{9}{\sqrt{407}}$, we have Min $S=\frac{12}{\sqrt{407}}$

**Problem 4.** Given $\begin{cases} a,b,c \geq 0 \\ a+b+c=3 \end{cases}$. Find the maximum value of S = $4ab+8bc+6ca$

***Solution***

Assume that $S = ma(b+c)+nb(c+a)+pc(a+b) = (m+n)ab+(n+p)bc+(p+m)ca$

$$\Rightarrow \begin{cases} m+n=4 \\ n+p=8 \\ p+m=6 \end{cases} \Leftrightarrow \begin{cases} m=1 \\ n=3 \\ p=5 \end{cases} \Rightarrow S = 4ab+8bc+6ca = a(b+c)+3b(c+a)+5c(a+b)$$
$$= a(3-a)+3b(3-b)+5c(3-c)$$

$$\Rightarrow S = \frac{81}{4} - \left[\left(a-\frac{3}{2}\right)^2 + 3\left(b-\frac{3}{2}\right)^2 + 5\left(c-\frac{3}{2}\right)^2\right]$$

Take $x = \left|a-\frac{3}{2}\right|; y = \left|b-\frac{3}{2}\right|; z = \left|c-\frac{3}{2}\right| \Rightarrow x+y+z \geq \left|a+b+c-\frac{9}{2}\right| = \frac{3}{2}$

Then: $S = \frac{81}{4} - \left(x^2+3y^2+5x^2\right) \Leftrightarrow x^2+3y^2+5x^2 = \frac{81}{4} - S$. With α, β, $\gamma > 0$ we have

$$+\begin{cases} x^2+\alpha^2 \geq 2\sqrt{x^2\alpha^2} = 2\alpha x \\ 3y^2+3\beta^2 \geq 6\sqrt{y^2\beta^2} = 6\beta y \\ 5z^2+5\gamma^2 \geq 10\sqrt{z^2\gamma^2} = 10\gamma y \end{cases} \Rightarrow \left(\frac{81}{4}-S\right) + \left(\alpha^2+3\beta^2+5\gamma^2\right) \geq 2\left(\alpha x+3\beta y+5\gamma z\right)$$

Choose $\alpha = 3\beta = 5\gamma$

$$\Rightarrow S \leq \frac{81}{4} + \left(\alpha^2+3\beta^2+5\gamma^2\right) - 2\alpha(x+y+z) \leq \frac{81}{4} + \left(\alpha^2+3\beta^2+5\gamma^2\right) - 3\alpha$$

Max $S = \frac{81}{4} + \left(\alpha^2+3\beta^2+5\gamma^2\right) - 3\alpha \Leftrightarrow x=\alpha; y=\beta; z=\gamma$ và $x+y+z=\frac{3}{2}$

$\Rightarrow x+y+z = \alpha+\beta+\gamma = \alpha+\frac{\alpha}{3}+\frac{\alpha}{5} = \frac{3}{2} \Rightarrow \alpha = \frac{45}{46}; \beta = \frac{15}{46}; \gamma = \frac{9}{46}$. Hence:

Max $S = \frac{81}{4} + \left(\alpha^2+3\beta^2+5\gamma^2\right) - 3\alpha = \frac{81}{4} + \frac{23\alpha^2}{15} - 3\alpha = \frac{432}{23}$

The equality occurs $\Leftrightarrow \left|a-\frac{3}{2}\right| = \alpha = \frac{45}{46}; \left|b-\frac{3}{2}\right| = \beta = \frac{15}{46}; \left|c-\frac{3}{2}\right| = \gamma = \frac{9}{46}$

$$\Leftrightarrow a-\frac{3}{2} = -\frac{45}{46}; b-\frac{3}{2} = -\frac{15}{46}; c-\frac{3}{2} = -\frac{9}{46} \Leftrightarrow a = \frac{12}{23}; b = \frac{27}{23}; c = \frac{30}{23}$$

**Problem 5.** Let $a,b,c,m > 0$ be satisfying $ab+bc+ca=1$.

Find the minimum value of $S = m\left(a^2+b^2\right)+c^2$ according to $m$

***Solution***

With $\alpha\in(0;m)$, using inequality AM – GM applied to two positive numbers, we have:

$$+\begin{cases} \alpha a^2+\dfrac{c^2}{2}\ge 2\sqrt{\dfrac{\alpha}{2}}\,ac \\ \alpha b^2+\dfrac{c^2}{2}\ge 2\sqrt{\dfrac{\alpha}{2}}\,bc \\ (m-\alpha)\left(a^2+b^2\right)\ge 2(m-\alpha)ab \end{cases}$$

$$\Rightarrow\ m\left(a^2+b^2\right)+c^2\ge 2\sqrt{\frac{\alpha}{2}}(ac+bc)+2(m-\alpha)ab$$

Choose $\alpha\in(0;m)$ such that $\sqrt{\dfrac{\alpha}{2}}=m-\alpha\ \Leftrightarrow\ \sqrt{\dfrac{\alpha}{2}}=\dfrac{-1+\sqrt{1+8m}}{4}$. Then we have:

$$S=m\left(a^2+b^2\right)+c^2\ge\frac{-1+\sqrt{1+8m}}{2}(ab+bc+ca)=\frac{-1+\sqrt{1+8m}}{2}$$

Equality occurs $\Leftrightarrow\ a=b=\dfrac{1}{\sqrt[4]{1+8m}}$ , $c=\dfrac{-1+\sqrt{1+8m}}{2\cdot\sqrt[4]{1+8m}}$. Thus $\operatorname{Min}S=\dfrac{-1+\sqrt{1+8m}}{2}$

**Problem 6.** Let $a,b,c,m,n>0$ be satisfying $ab+bc+ca=1$.

Find the minimum value of $S=ma^2+nb^2+c^2$ according to $m$, $n$

***Solution***

Supposing $x,y,z>0$ such that $m-x,n-y,1-z>0$. Using AM – GM inequality we have:

$$+\begin{cases} x.a^2+y.b^2\ge 2\sqrt{xy}ab \\ (m-x)a^2+z.c^2\ge 2\sqrt{(m-x)z}\,ac \\ (n-y)b^2+(1-z)c^2\ge 2\sqrt{(n-y)(1-z)}\,bc \end{cases}$$

$$\Rightarrow\ S=ma^2+nb^2+c^2\ge 2\sqrt{xy}\,ab+2\sqrt{(m-x)z}\,ac+2\sqrt{(n-y)(1-z)}\,bc$$

Equality occurs $\Leftrightarrow\begin{cases} xa^2=yb^2 \\ (m-x)a^2=zc^2 \\ (n-y)b^2=(1-z)c^2 \end{cases}\Leftrightarrow\begin{cases} b^2=\dfrac{x}{y}a^2 \\ c^2=\dfrac{m-x}{z}a^2 \\ (n-y)xz=(1-z)(m-x)y \end{cases}$

Choose $x,y,z$ such that $\sqrt{xy}=\sqrt{(m-x)z}=\sqrt{(n-y)(1-z)}=k>0\ \Rightarrow\ (n-y)xz=(1-z)(m-x)y=k^3$

We have: $mn=\left[x+(m-x)\right]\left[y+(n-y)\right]\left[z+(1-z)\right]=k^2(m+n+1)+2k^3$

Letting $f(k)=2k^3+k^2(m+n+1)-mn\ \Rightarrow f'(k)=\ f'(k)=6k^2+2(m+n+1)k>0,\forall k>0$

$\Rightarrow f(k)$ is increasing on $(0;+\infty)\ \Rightarrow$ The equation $f(k)=0$ has unique positive root $k_0>0$

$\Rightarrow\ S=ma^2+nb^2+c^2\ge 2k(ab+bc+ca)=2k_0\ \Rightarrow\ \operatorname{Min}S=2k_0$

**Problem 7.** Let be given $a,b,c$ satisfy $a^2+b^2+c^2=1$. Find the maximum value of

$$P=(a-b)(a-c)(b-c)(a+b+c) \qquad \textbf{(IMO 2006)}$$

***Solution***

We have: $\left[3(a^2+b^2+c^2)\right]^2=\left[2(a-b)^2+2(a-c)(b-c)+(a+b+c)^2\right]^2$

$\geq 8|(a-c)(b-c)|\left[2(a-b)^2+(a+b+c)^2\right]\geq 16\sqrt{2}|(a-c)(b-c)(a-b)(a+b+c)|\geq 16\sqrt{2}P$

$\Rightarrow P\leq \frac{9}{16\sqrt{2}}$. Equality occurs $\Leftrightarrow$ $a=\frac{3\sqrt{3}+\sqrt{6}}{6\sqrt{2}};b=\frac{\sqrt{6}}{6\sqrt{2}};c=\frac{\sqrt{6}-3\sqrt{3}}{6\sqrt{2}}$. Thus $\operatorname{Max}P=\frac{9}{16\sqrt{2}}$.

**Problem 8. (Phan Thanh Viet)** Let $a_1,a_2,\dots,a_n\geq 0$ be satisfying $a_1+a_2+\dots+a_n=1$.

Prove that $\mathrm{S}=(a_1+a_2)(a_1+a_2+a_3).....(a_1+a_2+...+a_{n-1})\geq 4^{n-1}a_1a_2\dots a_n$

***Proof***

With $x_1,x_2,\dots,x_n\geq 0$ and $S_k=x_1+x_2+...+x_k$ ,$\forall k=\overline{1,n}$, using AM – GM inequality we have:

$$a_1+a_2+\dots+a_k=x_1\cdot\frac{a_1}{x_1}+x_2\cdot\frac{a_2}{x_2}+...+x_k\cdot\frac{a_k}{x_k}\geq (x_1+\dots+x_k)\left(\frac{a_1}{x_1}\right)^{\frac{x_1}{S_k}}\left(\frac{a_2}{x_2}\right)^{\frac{x_2}{S_k}}\dots\left(\frac{a_k}{x_k}\right)^{\frac{x_k}{S_k}}$$

It follows: $S=(a_1+a_2)(a_1+a_2+a_3)\dots(a_1+...+a_{n-1})(a_1+...+a_n)^2\dots\geq C.a_1^{C_1}a_2^{C_2}\dots a_n^{C_n}$

$$C_1=\frac{x_1}{S_2}+...+\frac{x_1}{S_{n-1}}+\frac{2x_1}{S_n};C_k=\frac{x_k}{S_k}+...+\frac{x_k}{S_{n-1}}+\frac{2x_k}{S_n},\ \forall k=\overline{2,n-1};\ C_n=\frac{2x_n}{S_n};\ C=\frac{S_2S_3\dots S_{n-1}S_n^2}{x_1^{C_1}x_2^{C_2}\dots x_n^{C_n}}$$

Choose $x_1=\frac{1}{2^{n-1}}$; $x_k=\frac{1}{2^{n+1-k}}\forall k=\overline{2,n}$ so $S_k=\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\frac{1}{2^{n-2}}+\dots+\frac{1}{2^{n+1-k}}=\frac{1}{2^{n-k}}$

$$\Rightarrow C_1=\frac{1}{2^{n-1}}(2^{n-2}+2^{n-3}+...+2+1)=1;\ C_k=\frac{1}{2^{n+1-k}}(2^{n-k}+2^{n-k+1}+....+2+1)=1,\ \forall k=\overline{2,n-1}$$

$$C_n=2\cdot\frac{1}{2}\cdot 1=1\ ;\ C=\frac{2^{1+2+....+(n-2)+2(n-1)}}{2^{1+2+...+(n-2)}}=4^{n-1}.$$ Thus $S\geq 4^{n-1}a_1a_2\dots a_n$ (q.e.d)

Equality occurs $\Leftrightarrow$ $a_1=\frac{1}{2^{n-1}}$; $a_k=\frac{1}{2^{n+1-k}},\forall k=\overline{2,n}$

**Problem 9.** Letting that $x_1,x_2,\dots,x_n$ are $n$ real numbers satisfying $x_1+x_2+\dots+x_n=0$ and $|x_1|+|x_2|+\dots+|x_n|=1$. Find the maximum value of $P=\prod\limits_{1\leq i\leq j\leq n}|x_i-x_j|$

***Solution***

***Case 1:*** For $n=2$, the condition becomes $x_1+x_2=0;|x_1|+|x_2|=1$

$$\Rightarrow x_1=\frac{1}{2};x_2=-\frac{1}{2} \text{ or } x_1=-\frac{1}{2};x_2=\frac{1}{2} \Rightarrow P=|x_1-x_2|=1$$

***Case 2:*** For $n=3$, WLOG supposing $x_1 \le x_2 \le x_3$. We have $\frac{P}{2}=(x_2-x_1)\frac{x_3-x_1}{2}(x_3-x_2)$.

Using AM – GM we have: $\frac{P}{2} \le \left( \frac{(x_2-x_1)+\frac{x_3-x_1}{2}+(x_3-x_2)}{3} \right)^3 = \left( \frac{x_3-x_1}{2} \right)^3 \le \frac{1}{8} \Leftrightarrow P \le \frac{1}{4}$

Equality occurs $\Leftrightarrow \begin{cases} x_1+x_2+x_3=0 \\ |x_1|+|x_2|+|x_3|=1 \\ x_2-x_1=\frac{x_3-x_1}{2}=x_3-x_2 \end{cases} \Leftrightarrow \begin{cases} x_1=-\frac{1}{2} \\ x_2=0 \\ x_3=\frac{1}{2} \end{cases}$. Thus $\max P=\frac{1}{4}$

***Case 3:*** For $n=4$, WLOG supposing $x_1 \le x_2 \le x_3 \le x_4$, then we can predict Max $P$ can be obtained when $x_1=-x_4; x_2=-x_3 \Rightarrow x_2-x_1=x_4-x_3$. Letting $x_3-x_2=a(x_4-x_3)$, P reaches Max when the variables are satisfied with the conditions:

$$x_2-x_1=x_4-x_3=\frac{x_3-x_2}{a}=\frac{x_3-x_1}{a+1}=\frac{x_4-x_2}{a+1}=\frac{x_4-x_1}{a+2}.$$

From these, we come to this solution: WLOG supposing $x_1 \le x_2 \le x_3 \le x_4$

$\Rightarrow P=(x_2-x_1)(x_3-x_2)(x_4-x_1)(x_3-x_2)(x_4-x_2)(x_4-x_3)$

Hence $\frac{P}{a(a+2)(a+1)^2}=(x_2-x_1)\frac{(x_3-x_2)}{a+1}\frac{(x_4-x_1)}{a+2}\frac{(x_3-x_2)}{a}\frac{(x_4-x_2)}{a+1}(x_4-x_3)$

$$\le \left[ \frac{1}{6}\left( (x_2-x_1)\frac{(x_3-x_2)}{a+1}+\frac{(x_4-x_1)}{a+2}+\frac{(x_3-x_2)}{a}+\frac{(x_4-x_2)}{a+1}+(x_4-x_3) \right) \right]^6$$

It follows: $P \le \frac{1}{2^8}\left[ (x_2-x_1)\left(1+\frac{1}{a+1}+\frac{1}{a+2}\right)+\left(\frac{1}{a}+\frac{1}{a+1}+-1\right)(x_3-x_2) \right]^6$

Choose $a$ such that $\left(1+\frac{1}{a+1}+\frac{1}{a+2}\right)=\left(\frac{1}{a}+\frac{1}{a+1}+-1\right) \Leftrightarrow a=\sqrt{2}-1$.

Now $\left(1+\frac{1}{a+1}+\frac{1}{a+2}\right)=\left(\frac{1}{a}+\frac{1}{a+1}+-1\right)=\frac{3\sqrt{2}}{2}$

$\Rightarrow (\sqrt{2}+1)(\sqrt{2}-1)\left(\frac{\sqrt{2}}{2}\right)^2 P \le \frac{1}{2^8}\left[ \frac{3\sqrt{2}}{2}(-x_1-x_2+x_3+x_4)^6 \right]^6 \Rightarrow P \le \frac{1}{2^6}$

Equality occurs

$$\Leftrightarrow \begin{cases} x_1+x_2+x_3+x_4=0 \\ -x_1-x_2+x_3+x_4=|x_1|+|x_2|+|x_3|+|x_4|=1 \\ (x_1-x_2)=(x_4-x_3)=\frac{x_4-x_3}{\sqrt{2}-1}=\frac{x_3-x_1}{\sqrt{2}}=\frac{x_4-x_2}{\sqrt{2}}=\frac{x_4-x_1}{\sqrt{2}+1} \end{cases} \Leftrightarrow \begin{cases} x_4=-x_1=\frac{2-\sqrt{2}}{4} \\ x_3=-x_2=\frac{\sqrt{2}}{4} \end{cases} \quad (1)$$

***Conclusion:*** Max $P=\frac{1}{2^8}$ (occurs when variables are satisfying the condition (1))

***Case 4:*** $n=5$: With such assumption $x_1 \le x_2 \le x_3 \le x_4 \le x_5$, referring to the solutions to case 1 and 2, we can predict P reaches Max when $x_5 = -x_1, x_4 = -x_2, x_3 = 0$

$\Rightarrow x_2 - x_1 = x_5 - x_4, x_3 - x_2 = x_4 - x_3$.

Letting $x_3 - x_2 = a\left(x_2 - x_1\right)$, then $P$ reaches Max when variables are satisfying the condition:

$$\frac{x_2 - x_1}{1} = \frac{x_4 - x_3}{a} = \frac{x_3 - x_2}{a} = \frac{x_3 - x_1}{a+1} = \frac{x_5 - x_3}{a+1} = \frac{x_4 - x_2}{2a} = \frac{x_5 - x_2}{2a+1} = \frac{x_4 - x_1}{2a+1} = \frac{x_5 - x_1}{2a+2} = \frac{x_5 - x_4}{1}$$

From such analysis, the solution to the problem in which $n=5$ will be as follows:

WLOG supposing $x_1 \le x_2 \le x_3 \le x_4 \le x_5$. Now we have:

$$P = \left(x_2 - x_1\right)\left(x_3 - x_1\right)\left(x_4 - x_1\right)\left(x_5 - x_1\right)\left(x_3 - x_2\right)\left(x_4 - x_2\right)\left(x_5 - x_2\right)\left(x_5 - x_2\right)\left(x_5 - x_4\right)$$

Consider the expression: $Q = \dfrac{P}{4a^2\,(a+1)^3\,(2a+1)^2}$. Transform Q in the form:

$$Q = \frac{x_2 - x_1}{1} \cdot \frac{x_3 - x_1}{a+1} \cdot \frac{x_4 - x_1}{2a+1} \cdot \frac{x_5 - x_1}{2a+2} \cdot \frac{x_3 - x_2}{a} \cdot \frac{x_4 - x_2}{2a} \cdot \frac{x_5 - x_2}{2a+1} \cdot \frac{x_4 - x_3}{a} \cdot \frac{x_5 - x_3}{a+1} \cdot \frac{x_5 - x_4}{1}$$

Using AM – GM applied to 10 non-negative numbers, we have:

$$Q \le \frac{1}{10^{10}}\left[\frac{x_2 - x_1}{1} + \frac{x_3 - x_1}{a+1} + \frac{x_4 - x_1}{2a+1} + \frac{x_5 - x_1}{2a+2} + \frac{x_3 - x_2}{a} + \right.$$
$$\left. + \frac{x_4 - x_2}{2a} + \frac{x_5 - x_2}{2a+1} + \frac{x_4 - x_3}{a} + \frac{x_5 - x_3}{a+1} + \frac{x_5 - x_4}{1}\right]^{10}$$

It follows: $Q \le \dfrac{1}{10^{10}}\left[\left(\dfrac{3}{2(a+1)} + \dfrac{1}{2a+1} + 1\right)\left(-x_1 + x_5\right) + \left(-1 + \dfrac{3}{2a} + \dfrac{1}{2a+1}\right)\left(-x_2 + x_4\right)\right]^{10}$ (2)

Choose $a > 0$ such that $\left(\dfrac{3}{2(a+1)} + \dfrac{1}{2a+1} + 1\right) = \left(-1 + \dfrac{3}{2a} + \dfrac{1}{2a+1}\right) = q \Leftrightarrow a = \dfrac{1}{2}, q = \dfrac{5}{2}$

It follows $Q = \dfrac{P}{27/4}$. Now: $Q \le \dfrac{1}{10^{10}}\left[\dfrac{5}{2}\left(-x_1 - x_2 + x_4 + x_5\right)\right]^{10} = \dfrac{1}{2^{10}} \Leftrightarrow P \le \dfrac{1}{2^{20}}\dfrac{27}{4} = \dfrac{27}{2^{22}}$

Equality occurs $\Leftrightarrow$

$$\begin{cases} x_3 = 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ -x_1 - x_2 + x_3 + x_4 + x_5 = \left|x_1\right| + \left|x_2\right| + \left|x_3\right| + \left|x_4\right| + \left|x_5\right| = 1 \\ \left(x_2 - x_1\right) = \dfrac{x_3 - x_1}{3/2} = \dfrac{x_4 - x_1}{2} = \dfrac{x_5 - x_1}{3} = \dfrac{x_3 - x_2}{1/2} = \dfrac{x_4 - x_2}{1} = \dfrac{x_5 - x_2}{2} = \dfrac{x_4 - x_3}{1/2} = \dfrac{x_5 - x_3}{3/2} = \dfrac{x_5 - x_4}{1} \end{cases}$$

$\Leftrightarrow x_1 = -x_5 = -\dfrac{3}{8}$ ; $x_2 = -x_4 = -\dfrac{1}{8}$ ; $x_3 = 0$ (3)

***Conclusion:*** Max $P = \dfrac{27}{2^{22}}$ (occurs when variables are satisfying the condition (3))

**Remarks.** In general cases ($n \ge 6$) the solution of the problem is still an open question

**7. APPLYING AM – GM TO INEQUALITY OF DIFFERENT DEGREE**

**The un-homogeneous inequalities considering in positive real set:**

**Problem 1. (Tran Phuong)** Let be given $a, b, c > 0$. Prove that

$$\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}+abc \le \frac{a^7}{b^2c^2}+\frac{b^7}{c^2a^2}+\frac{c^7}{a^2b^2}+\frac{1}{a^2b^2c^2}$$

***Proof***

Transfer and using AM – GM Inequality we have

$$+\begin{cases} \dfrac{bc}{a^2}=\sqrt[3]{\dfrac{b^7}{a^2c^2}\cdot\dfrac{c^7}{a^2b^2}\cdot\dfrac{1}{a^2b^2c^2}} \le \dfrac{1}{3}\left(\dfrac{b^7}{a^2c^2}+\dfrac{c^7}{a^2b^2}+\dfrac{1}{a^2b^2c^2}\right) \\ \dfrac{ca}{b^2}=\sqrt[3]{\dfrac{c^7}{b^2a^2}\cdot\dfrac{a^7}{b^2c^2}\cdot\dfrac{1}{a^2b^2c^2}} \le \dfrac{1}{3}\left(\dfrac{c^7}{b^2a^2}+\dfrac{a^7}{b^2c^2}+\dfrac{1}{a^2b^2c^2}\right) \\ \dfrac{ab}{c^2}=\sqrt[3]{\dfrac{a^7}{c^2b^2}\cdot\dfrac{b^7}{c^2a^2}\cdot\dfrac{1}{a^2b^2c^2}} \le \dfrac{1}{3}\left(\dfrac{a^7}{c^2b^2}+\dfrac{b^7}{c^2a^2}+\dfrac{1}{a^2b^2c^2}\right) \\ abc=\sqrt[3]{\dfrac{a^7}{c^2b^2}\cdot\dfrac{b^7}{c^2a^2}\cdot\dfrac{c^7}{a^2b^2}} \le \dfrac{1}{3}\left(\dfrac{a^7}{c^2b^2}+\dfrac{b^7}{c^2a^2}+\dfrac{c^7}{a^2b^2}\right) \end{cases}$$

$$\Rightarrow \frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}+abc \le \frac{a^7}{b^2c^2}+\frac{b^7}{c^2a^2}+\frac{c^7}{a^2b^2}+\frac{1}{a^2b^2c^2}$$

**Problem 2. (Tran Phuong)** Let be given $a, b, c > 0$. Prove that

$$\frac{a^9}{bc}+\frac{b^9}{ca}+\frac{c^9}{ab}+\frac{2}{abc} \ge a^5+b^5+c^5+2 \quad (1)$$

***Proof***

Using AM – GM Inequality we have: $\dfrac{a^9}{bc}+abc \ge 2a^5; \dfrac{b^9}{ca}+abc \ge 2b^5; \dfrac{c^9}{ab}+abc \ge 2c^5$

$$\Rightarrow \frac{a^9}{bc}+\frac{b^9}{ca}+\frac{c^9}{ab} \ge 2\left(a^5+b^5+c^5\right)-3abc \ge a^5+b^5+c^5+3\cdot\sqrt[3]{a^5b^5c^5}-3abc$$

Therefore, to prove inequality (1), we have to prove:

$$3\cdot\sqrt[3]{a^5b^5c^5}-3abc+\frac{2}{abc} \ge 2 \Leftrightarrow 3t^5-3t^3+\frac{2}{t^3} \ge 2 \ \ (\text{for } t=\sqrt[3]{abc}>0)$$

$$\Leftrightarrow \frac{(t-1)^2\left(3t^6+6t^5+6t^4+6t^3+6t^2+4t+2\right)}{t^3} \ge 0 \ \ (\text{is always true}) \Rightarrow (\text{q.e.d.})$$

Equality occurs $\Leftrightarrow a=b=c=1$

**Problem 3. (Tran Phuong)** Let be given $a, b, c > 0$. Prove that

$$\frac{a^9}{bc}+\frac{b^9}{ca}+\frac{c^9}{ab}+\frac{3}{abc} \ge a^4+b^4+c^4+3 \quad (1)$$

***Proof***

Using AM – GM Inequality we have:

$$\frac{a^9}{bc}+abc+a^2\ge 3a^4,\frac{b^9}{ca}+abc+b^2\ge 3b^4,\frac{c^9}{ab}+abc+c^2\ge 3c^4$$

$$\Rightarrow \frac{a^9}{bc}+\frac{b^9}{ca}+\frac{c^9}{ab}\ge 3\left(a^4+b^4+c^4\right)-\left(a^2+b^2+c^2\right)-3abc \ (2)$$

Using again AM – GM inequality, we have:

$$\frac{1}{2}\left(1+a^4\right)\ge a^2;\frac{1}{2}\left(1+b^4\right)\ge b^2;\frac{1}{2}\left(1+c^4\right)\ge c^2\Rightarrow \frac{1}{2}\left(a^4+b^4+c^4\right)+\frac{3}{2}\ge a^2+b^2+c^2$$

From (2) $\Rightarrow \frac{a^9}{bc}+\frac{b^9}{ca}+\frac{c^9}{ab}\ge \frac{5}{2}\left(a^4+b^4+c^4\right)-3abc-\frac{3}{2}\ge a^4+b^4+c^4+\frac{9}{2}\cdot\sqrt[3]{a^4b^4c^4}-3abc-\frac{3}{2}$

Therefore, to prove inequality (1), we only have to prove:

$$\frac{9}{2}\cdot\sqrt[3]{a^4b^4c^4}-3abc-\frac{3}{2}+\frac{3}{abc}\ge 3 \Leftrightarrow \frac{9}{2}\cdot t^4-3t^3+\frac{3}{t^3}\ge\frac{9}{2} \ \text{(for } t=\sqrt[3]{abc}>0\text{)}$$

$$\Leftrightarrow \frac{3}{2}(t-1)^2(t+1)\left(3t^4+t^3+4t^2+2t+2\right)\ge 0 \ \text{(is always true)} \Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow a=b=c=1$

**Problem 4. (Tran Nam Dung)** Let be given $a$, $b$, $c \ge 0$. Prove that

$$2\left(a^2+b^2+c^2\right)+abc+8\ge 5(a+b+c) \ (1)$$

***Solution***

• ***Lemma:*** *(Schur inequality)*

$$x^3+y^3+z^3+3xyz\ge xy(x+y)+xy(y+z)+zx(z+x) \ , \ \forall x,y,z\ge 0 \ (*)$$

• ***Proof:*** WLOG supposing $x\ge y\ge z\ge 0$. We have:

LHS (*) – RHS (*) = $x(x-y)^2+z(y-z)^2+(z+x-y)(x-y)(y-z)\ge 0$

• ***Application:*** Using AM – GM inequality and Schur inequality we have:

6.[LHS (1) – RHS (1)] $=12\left(a^2+b^2+c^2\right)+6abc+48-30(a+b+c)$

$$=12\left(a^2+b^2+c^2\right)+3(2abc+1)+45-5\cdot 2\cdot 3(a+b+c)$$

$$\ge 12\left(a^2+b^2+c^2\right)+9\cdot\sqrt[3]{(abc)^2}+45-5\left[(a+b+c)^2+9\right]$$

$$=\frac{9abc}{\sqrt[3]{abc}}+3\left(a^2+b^2+c^2\right)-6(ab+bc+ca)+2\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$\ge\frac{9abc}{\sqrt[3]{abc}}+3\left(a^2+b^2+c^2\right)-6(ab+bc+ca)\ge\frac{27abc}{a+b+c}+3(a+b+c)^2-12(ab+bc+ca)$$

$$=\frac{3}{a+b+c}\left[9abc+(a+b+c)^3-4(ab+bc+ca)(a+b+c)\right]$$

$$=\frac{3}{a+b+c}\left[a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)\right]\geq 0 \Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow$ $a=b=c=1$

**Problem 5. (Darij Grinberg)** Let be given $a$, $b$, $c > 0$. Prove that

$$a^2+b^2+c^2+2abc+1\geq 2(ab+bc+ca)$$

***Proof***

Using AM – GM Inequality and Schur inequality we have:

$$a^2+b^2+c^2+2abc+1-2(ab+bc+ca)\geq a^2+b^2+c^2+3a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}-2(ab+bc+ca)$$

$$\geq a^{\frac{2}{3}}b^{\frac{2}{3}}\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)+b^{\frac{2}{3}}c^{\frac{2}{3}}\left(b^{\frac{2}{3}}+c^{\frac{2}{3}}\right)+c^{\frac{2}{3}}a^{\frac{2}{3}}\left(c^{\frac{2}{3}}+a^{\frac{2}{3}}\right)-2(ab+bc+ca)$$

$$=a^{\frac{2}{3}}b^{\frac{2}{3}}\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^2+b^{\frac{2}{3}}c^{\frac{2}{3}}\left(b^{\frac{1}{3}}-c^{\frac{1}{3}}\right)^2+c^{\frac{2}{3}}a^{\frac{2}{3}}\left(c^{\frac{1}{3}}-a^{\frac{1}{3}}\right)^2\geq 0$$

Equality occurs $\Leftrightarrow$ $a=b=c=1$

**Problem 6. (APMO 2004)** Let be given $a$, $b$, $c > 0$. Prove that

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca) \quad (1)$$

***Proof***

Using AM – GM Inequality we have:

LHS (1) – RHS (1) = $(a^2+2)(b^2+2)(c^2+2)-9(ab+bc+ca)$

$$=4(a^2+b^2+c^2)+2\left[(a^2b^2+1)+(b^2c^2+1)+(c^2a^2+1)\right]+(a^2b^2c^2+1)+1-9(ab+bc+ca)$$

$$\geq 4(a^2+b^2+c^2)+4(ab+bc+ca)+2abc+1-9(ab+bc+ca)$$

$$\geq a^2+b^2+c^2+2abc+1-2(ab+bc+ca)$$

Using the results of problem 5 we have (q.e.d.). Equality occurs $\Leftrightarrow$ $a=b=c=1$

**Problem 7.** Let be given $a$, $b$, $c$, $d > 0$. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}+\frac{1}{d+a}\right)\geq 16\,\text{Max}\left\{\frac{1}{1+abcd}\ ;\ \frac{1}{ac+bd}\right\} \quad (1)$$

***Solution***

Using AM – GM Inequality we have:

$$LHS(1)=\left(\frac{1}{ab}+\frac{1}{cd}\right)+\left[\frac{a+b}{ab(c+d)}+\frac{c+d}{cd(a+b)}\right]+\frac{a+b}{ab(d+a)}+\frac{a+b}{ab(b+c)}+\frac{c+d}{cd(b+c)}+\frac{c+d}{cd(d+a)}$$

$$\geq \frac{4}{\sqrt{abcd}}+\frac{a+b}{ab(d+a)}+\frac{a+b}{ab(b+c)}+\frac{c+d}{cd(b+c)}+\frac{c+d}{cd(d+a)}$$. Similarly we have:

$$LHS(1)\geq \frac{4}{\sqrt{abcd}}+\frac{b+c}{bc(a+b)}+\frac{b+c}{bc(c+d)}+\frac{a+d}{ad(a+b)}+\frac{a+d}{ad(c+d)}$$. It follows:

$$2\cdot LHS(1)\geq \frac{8}{\sqrt{abcd}}+\left[\frac{a+b}{ab(d+a)}+\frac{a+d}{ad(a+b)}\right]+\left[\frac{a+b}{ab(b+c)}+\frac{b+c}{bc(a+b)}\right]+$$

$$+\left[\frac{c+d}{cd(b+c)}+\frac{b+c}{bc(c+d)}\right]+\left[\frac{c+d}{cd(d+a)}+\frac{a+d}{ad(c+d)}\right]$$

$$\geq \frac{8}{\sqrt{abcd}}+\frac{2}{a\sqrt{bd}}+\frac{2}{b\sqrt{ac}}+\frac{2}{c\sqrt{bd}}+\frac{2}{d\sqrt{ac}}=\frac{8}{\sqrt{abcd}}+\frac{2}{\sqrt{bd}}\left(\frac{1}{a}+\frac{1}{c}\right)+\frac{2}{\sqrt{ac}}\left(\frac{1}{b}+\frac{1}{d}\right)$$

$$\geq \frac{8}{\sqrt{abcd}}+\frac{4}{\sqrt{abcd}}+\frac{4}{\sqrt{abcd}}=\frac{16}{\sqrt{abcd}} \Rightarrow LHS(1)\geq \frac{8}{\sqrt{abcd}}$$

On the other hand, using AM – GM Inequality we have:

$$\frac{1+abcd}{2}\geq \sqrt{abcd};\frac{ac+bd}{2}\geq \sqrt{abcd}\Rightarrow \sqrt{abcd}\leq \frac{\operatorname{Min}\{1+abcd\,;ac+bd\}}{2}$$

$$\Rightarrow LHS(1)\geq \frac{16}{\operatorname{Min}\{1+abcd\ ;\ ac+bd\}}=16\operatorname{Max}\left\{\frac{1}{1+abcd}\ ;\ \frac{1}{ac+bd}\right\}$$ (q.e.d.)

**Problem 8. (Le Trung Kien)**

Prove that $a^3+b^3+c^3+4(a+b+c)+9abc\geq 8(ab+bc+ca)$, $\forall a,b,c\geq 0$ (1)

***Solution***

***Lemma:*** $a^4+b^4+c^4+abc(a+b+c)\geq ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)$ (1)

***Proof:*** WLOG supposing $a\geq b\geq c$. We have:

(1) $\Leftrightarrow (b^2+c^2-a^2)(b-c)^2+(c^2+a^2-b^2)(c-a)^2+(a^2+b^2-c^2)(a-b)^2\geq 0$

$\Leftrightarrow c^2(b-c)^2+a^2(a-b)^2+(c^2+a^2-b^2)(a-b)(b-c)\geq 0$ (is always true)

***Application:*** Using AM – GM inequality we have:

$$4(a+b+c)+\frac{4(ab+bc+ca)^2}{a+b+c}\geq 8(ab+bc+ca)$$

So it suffices to prove that: $a^3+b^3+c^3+9abc\geq \dfrac{4(ab+bc+ca)^2}{a+b+c}$

$\Leftrightarrow a^4+b^4+c^4+abc(a+b+c)+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\geq 4(a^2b^2+b^2c^2+c^2a^2)$ (*)

Using lemma we have:

LHS (*) $\geq 2\left[ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\right]\geq 4(a^2b^2+b^2c^2+c^2a^2)\Rightarrow$ (q.e.d)

Equality occurs $\Leftrightarrow a=b=c=1$ or $(a,b,c)\sim(2;2;0)$

**8. SPECIALIZATION IN INEQUALITY OF DIFFERENT DEGREE**

**Problem 1.** Given $\begin{cases} a,b,c>0 \\ a+b+c=3abc \end{cases}$. Prove that $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\ge 3$ (1)

***Proof***

$a+b+c=3abc \Rightarrow \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=3$. Using AM – GM Inequality we have:

$$2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+3=\left(\frac{1}{a^3}+\frac{1}{b^3}+1\right)+\left(\frac{1}{b^3}+\frac{1}{c^3}+1\right)+\left(\frac{1}{c^3}+\frac{1}{a^3}+1\right)\ge 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=9 \Rightarrow \text{(q.e.d)}$$

**Problem 2.** Given $\begin{cases} a,b,c>0 \\ a\sqrt{\frac{b}{c}}+b\sqrt{\frac{c}{a}}+c\sqrt{\frac{a}{b}}=3 \end{cases}$. Prove that $\frac{a^6}{b^3}+\frac{b^6}{c^3}+\frac{c^6}{a^3}\ge 3$ (1)

***Proof***

Using AM – GM Inequality we have:

$$\left(\frac{a^6}{b^3}+\frac{b^6}{c^3}+4\right)+\left(\frac{b^6}{c^3}+\frac{c^6}{a^3}+4\right)+\left(\frac{c^6}{a^3}+\frac{a^6}{b^3}+4\right)\ge 6\left(a\sqrt{\frac{b}{c}}+b\sqrt{\frac{c}{a}}+c\sqrt{\frac{a}{b}}\right)$$

$$\Leftrightarrow 2\left(\frac{a^6}{b^3}+\frac{b^6}{c^3}+\frac{c^6}{a^3}\right)+12\ge 18 \Leftrightarrow \frac{a^6}{b^3}+\frac{b^6}{c^3}+\frac{c^6}{a^3}\ge 3 \quad \text{(q.e.d)}$$

**Problem 3.** Given $\begin{cases} a,b,c,d>0 \\ ab+bc+cd+da=1 \end{cases}$. Prove that $\frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\ge\frac{1}{3}$ (1)

***(IMO Shortlist 1990)***

***Proof***

Using AM – GM inequality we have:

$$+\begin{cases} \frac{36a^3}{b+c+d}+2(b+c+d)+6a+3\ge 4\cdot\sqrt[4]{\frac{36a^3}{b+c+d}\cdot 2(b+c+d)\cdot 6a\cdot 3}=24a \\ \frac{36b^3}{c+d+a}+2(c+d+a)+6b+3\ge 4\cdot\sqrt[4]{\frac{36b^3}{c+d+a}\cdot 2(c+d+a)\cdot 6b\cdot 3}=24b \\ \frac{36c^3}{d+a+b}+2(d+a+b)+6c+3\ge 4\cdot\sqrt[4]{\frac{36c^3}{d+a+b}\cdot 2(d+a+b)\cdot 6c\cdot 3}=24c \\ \frac{36d^3}{a+b+c}+2(a+b+c)+6d+3\ge 4\cdot\sqrt[4]{\frac{36d^3}{a+b+c}\cdot 2(a+b+c)\cdot 6d\cdot 3}=24d \end{cases}$$

$$\Rightarrow \frac{a^3}{b+c+d}+\frac{b^3}{c+d+a}+\frac{c^3}{d+a+b}+\frac{d^3}{a+b+c}\ge\frac{a+b+c+d}{3}-\frac{1}{3}$$

On the other hand, $(a+b+c+d)^2\ge 4(ab+bc+cd+da)=4\Rightarrow a+b+c+d\ge 2$

$\Rightarrow$ LHS (1) $\ge \frac{a+b+c+d}{3}-\frac{1}{3}\ge\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$ (q.e.d)

**9. BEST SOLUTIONS TO FOUR TRIGONOMETRIC INEQUALITIES**

Let ABC be a triangle. Prove that

$$\mathbf{T_1} = \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}\quad;\quad \mathbf{T_2} = \cos A+\cos B+\cos C\le\frac{3}{2}$$

$$\mathbf{T_3} = \sin A+\sin B+\sin C\le\frac{3\sqrt{3}}{2}\quad;\quad \mathbf{T_4} = \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\le\frac{3\sqrt{3}}{2}$$

***Proof***

$$T_1 = \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right).1+\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}$$

$$\le\frac{1}{2}\left[\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2+1^2\right]+\frac{1}{2}\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}\right)-\sin\frac{A}{2}\sin\frac{B}{2}=\frac{3}{2}$$

$$T_2 = \cos A+\cos B+\cos C=(\cos A+\cos B).1+\sin A\sin B-\cos A\cos B$$

$$\le\frac{1}{2}\left[(\cos A+\cos B)^2+1^2\right]+\frac{1}{2}\left(\sin^2 A+\sin^2 B\right)-\cos A\cos B\le\frac{3}{2}$$

$$T_3 = \frac{2}{\sqrt{3}}\left(\sin A\cdot\frac{\sqrt{3}}{2}+\sin B\cdot\frac{\sqrt{3}}{2}\right)+\sqrt{3}\left(\frac{\sin A}{\sqrt{3}}\cdot\cos B+\frac{\sin B}{\sqrt{3}}\cdot\cos A\right)$$

$$\le\frac{1}{\sqrt{3}}\left[\left(\sin^2 A+\frac{3}{4}\right)+\left(\sin^2 B+\frac{3}{4}\right)\right]+\frac{\sqrt{3}}{2}\left[\left(\frac{\sin^2 A}{3}+\cos^2 B\right)+\left(\frac{\sin^2 B}{3}+\cos^2 A\right)\right]=\frac{3\sqrt{3}}{2}$$

$$T_4 = \frac{2}{\sqrt{3}}\left(\cos\frac{A}{2}\cdot\frac{\sqrt{3}}{2}+\cos\frac{B}{2}\cdot\frac{\sqrt{3}}{2}\right)+\sqrt{3}\left(\frac{\cos\frac{A}{2}}{\sqrt{3}}\cdot\sin\frac{B}{2}+\frac{\cos\frac{B}{2}}{\sqrt{3}}\cdot\sin\frac{A}{2}\right)$$

$$\le\frac{1}{\sqrt{3}}\left[\left(\cos^2\frac{A}{2}+\frac{3}{4}\right)+\left(\cos^2\frac{B}{2}+\frac{3}{4}\right)\right]+\frac{\sqrt{3}}{2}\left[\left(\frac{\cos^2\frac{A}{2}}{3}+\sin^2\frac{B}{2}\right)+\left(\frac{\cos^2\frac{B}{2}}{3}+\sin^2\frac{A}{2}\right)\right]=\frac{3\sqrt{3}}{2}$$

***Comment:*** In my colleagues' opinions, the common solutions to the four above problems are the most unprecedented and the shortest among what have ever demonstrated in the world. In all other publications, these inequalities are proved based on graphs of ***Jensen Inequality*** but the above solutions originally apply a simple inequality $2xy\le x^2+y^2$ and the concept "point of incidence", which will be referred to in detail in the book. These solutions were published in the other books in Viet Nam, also written by the author in 2000.

***Trivial classical method:***

- $\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\le 2\sin\frac{x+y}{2},\quad \forall x,y\in(0,\pi)$
- $\sin x+\sin y+\sin z+\sin t=2\sin\frac{x+y}{2}\cos\frac{x-y}{2}+2\sin\frac{z+t}{2}\cos\frac{z-t}{2}\le 2\sin\frac{x+y}{2}+2\sin\frac{z+t}{2}$

$$=4\sin\frac{x+y+z+t}{4}\cos\frac{x+y-z-t}{4}\le 4\sin\frac{x+y+z+t}{4}\quad,\ \forall x,y,z,t\in(0;\pi)$$

- Take $x=A, y=B, z=C, t=\frac{A+B+C}{3}\Rightarrow \sin A+\sin B+\sin C\le 3\sin\frac{A+B+C}{3}=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}$

### 10. SELECTIVE PROBLEMS IN USING POINT OF INCIDENCE FOR AM – GM

**Problem 1. (Tran Phuong)** Let be given ΔABC with the lengths of the sides $a$, $b$, $c$.

Prove that: $$\sqrt[3]{\left(\frac{3a}{3b+3c-a}\right)^2}+\sqrt[3]{\left(\frac{3b}{3c+3a-b}\right)^2}+\sqrt[3]{\left(\frac{3c}{3a+3b-c}\right)^2}\ge\frac{3}{5}\cdot\sqrt[3]{45}\quad(1)$$

***Proof***

Let $$\begin{cases}\dfrac{3a}{3b+3c-a}=\dfrac{1}{x}\\[2mm] \dfrac{3b}{3c+3a-b}=\dfrac{1}{y}\\[2mm] \dfrac{3c}{3a+3b-c}=\dfrac{1}{z}\end{cases}\Rightarrow\begin{cases}x=\dfrac{3b+3c-a}{3a}\\[2mm] y=\dfrac{3c+3a-b}{3b}\\[2mm] x=\dfrac{3a+3b-c}{3c}\end{cases}\Rightarrow\begin{cases}x+\dfrac{4}{3}=\dfrac{a+b+c}{a}\\[2mm] y+\dfrac{4}{3}=\dfrac{a+b+c}{b}\\[2mm] z+\dfrac{4}{3}=\dfrac{a+b+c}{c}\end{cases}$$

$$\Rightarrow\frac{1}{x+\frac{4}{3}}+\frac{1}{y+\frac{4}{3}}+\frac{1}{z+\frac{4}{3}}=\frac{a+b+c}{a+b+c}=1\Leftrightarrow\left(x+\frac{4}{3}\right)^{-1}+\left(y+\frac{4}{3}\right)^{-1}+\left(z+\frac{4}{3}\right)^{-1}=1$$

$$(1)\Leftrightarrow\frac{1}{\sqrt[3]{x^2}}+\frac{1}{\sqrt[3]{y^2}}+\frac{1}{\sqrt[3]{z^2}}\ge\frac{3}{5}\cdot\sqrt[3]{45}\,.$$

Using $AM-GM$ Inequality for 9 terms we have:

$$\begin{cases}x^5\cdot\left(\dfrac{5}{3}\right)^4\le\left[\dfrac{5\left(x+\frac{4}{3}\right)}{9}\right]^9=\left(\dfrac{5}{3}\right)^9\cdot\left(\dfrac{x+\frac{4}{3}}{3}\right)^9\\[2mm] y^5\cdot\left(\dfrac{5}{3}\right)^4\le\left[\dfrac{5\left(y+\frac{4}{3}\right)}{9}\right]^9=\left(\dfrac{5}{3}\right)^9\cdot\left(\dfrac{y+\frac{4}{3}}{3}\right)^9\\[2mm] z^5\cdot\left(\dfrac{5}{3}\right)^4\le\left[\dfrac{5\left(z+\frac{4}{3}\right)}{9}\right]^9=\left(\dfrac{5}{3}\right)^9\cdot\left(\dfrac{z+\frac{4}{3}}{3}\right)^9\end{cases}\Leftrightarrow\begin{cases}x^5\le\left(\dfrac{5}{3}\right)^5\cdot\left(\dfrac{x+\frac{4}{3}}{3}\right)^9\\[2mm] y^5\le\left(\dfrac{5}{3}\right)^5\cdot\left(\dfrac{y+\frac{4}{3}}{3}\right)^9\\[2mm] z^5\le\left(\dfrac{5}{3}\right)^5\cdot\left(\dfrac{z+\frac{4}{3}}{3}\right)^9\end{cases}$$

$$\Leftrightarrow\begin{cases}x\le\dfrac{5}{3}\cdot\left(\dfrac{x+\frac{4}{3}}{3}\right)^{\frac{9}{5}}\\[2mm] y\le\dfrac{5}{3}\cdot\left(\dfrac{y+\frac{4}{3}}{3}\right)^{\frac{9}{5}}\\[2mm] z\le\dfrac{5}{3}\cdot\left(\dfrac{z+\frac{4}{3}}{3}\right)^{\frac{9}{5}}\end{cases}\Leftrightarrow\begin{cases}\dfrac{1}{x}\ge\dfrac{3}{5}\cdot\left(\dfrac{x+\frac{4}{3}}{3}\right)^{\frac{-9}{5}}\\[2mm] \dfrac{1}{y}\ge\dfrac{3}{5}\cdot\left(\dfrac{y+\frac{4}{3}}{3}\right)^{\frac{-9}{5}}\\[2mm] \dfrac{1}{z}\ge\dfrac{3}{5}\cdot\left(\dfrac{z+\frac{4}{3}}{3}\right)^{\frac{-9}{5}}\end{cases}\Leftrightarrow\begin{cases}\dfrac{1}{\sqrt[3]{x^2}}\ge\sqrt[3]{\dfrac{9}{25}}\left(\dfrac{x+\frac{4}{3}}{3}\right)^{\frac{-6}{5}}\\[2mm] \dfrac{1}{\sqrt[3]{y^2}}\ge\sqrt[3]{\dfrac{9}{25}}\left(\dfrac{y+\frac{4}{3}}{3}\right)^{\frac{-6}{5}}\\[2mm] \dfrac{1}{\sqrt[3]{z^2}}\ge\sqrt[3]{\dfrac{9}{25}}\left(\dfrac{z+\frac{4}{3}}{3}\right)^{\frac{-6}{5}}\end{cases}$$

Letting $m=\left(x+\frac{4}{3}\right)^{-1};\, n=\left(y+\frac{4}{3}\right)^{-1};\, p=\left(z+\frac{4}{3}\right)^{-1}\Rightarrow m+n+p=1$

Using $AM-GM$ Inequality for 6 terms we have:

$$+\begin{cases} 5.(3m)^{\frac{6}{5}}+1=(3m)^{\frac{6}{5}}+(3m)^{\frac{6}{5}}+(3m)^{\frac{6}{5}}+(3m)^{\frac{6}{5}}+(3m)^{\frac{6}{5}}+1\geq 18m \\ 5.(3n)^{\frac{6}{5}}+1=(3n)^{\frac{6}{5}}+(3n)^{\frac{6}{5}}+(3n)^{\frac{6}{5}}+(3n)^{\frac{6}{5}}+(3n)^{\frac{6}{5}}+1\geq 18n \\ 5.(3p)^{\frac{6}{5}}+1=(3p)^{\frac{6}{5}}+(3p)^{\frac{6}{5}}+(3p)^{\frac{6}{5}}+(3p)^{\frac{6}{5}}+(3p)^{\frac{6}{5}}+1\geq 18p \end{cases}$$

$$\Rightarrow \frac{1}{\sqrt[3]{x^2}}+\frac{1}{\sqrt[3]{y^2}}+\frac{1}{\sqrt[3]{z^2}}\geq\sqrt[3]{\frac{9}{25}}\left[(3m)^{\frac{6}{5}}+(3n)^{\frac{6}{5}}+(3p)^{\frac{6}{5}}\right]\geq\sqrt[3]{\frac{9}{25}}\cdot\left[\frac{18(m+n+p)-3}{5}\right]=\frac{3}{5}\sqrt[3]{45}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 2. (Tran Phuong)** Let be given $a$, $b$, $c > 0$. Prove that

$$\sqrt[4]{\frac{a^3+b^3}{c^3}}+\sqrt[4]{\frac{b^3+c^3}{a^3}}+\sqrt[4]{\frac{c^3+a^3}{b^3}}\geq\sqrt[12]{2}\left(\sqrt[3]{\frac{a^2+b^2}{c^2}}+\sqrt[3]{\frac{b^2+c^2}{a^2}}+\sqrt[3]{\frac{c^2+a^2}{b^2}}\right)$$

***Proof***

• ***Lemma:*** $\sqrt[3]{\frac{a^3+b^3}{2}}\geq\sqrt{\frac{a^2+b^2}{2}}$ (It is for you to prove it)

• ***Application:*** Using ***AM – GM Inequality*** and the lemma we have

$$\begin{cases} \underbrace{\sqrt[4]{\frac{a^3+b^3}{2c^3}}+\sqrt[4]{\frac{a^3+b^3}{2c^3}}+...+\sqrt[4]{\frac{a^3+b^3}{2c^3}}}_{8\text{ terms}}+1\geq 9\cdot\sqrt[3]{\frac{1}{c^2}\left(\frac{a^3+b^3}{2}\right)^{\frac{2}{3}}}\geq 9\cdot\sqrt[3]{\frac{a^2+b^2}{2c^2}} \\ \underbrace{\sqrt[4]{\frac{b^3+c^3}{2a^3}}+\sqrt[4]{\frac{b^3+c^3}{2a^3}}+...+\sqrt[4]{\frac{b^3+c^3}{2a^3}}}_{8\text{ terms}}+1\geq 9\cdot\sqrt[3]{\frac{1}{a^2}\left(\frac{b^3+c^3}{2}\right)^{\frac{2}{3}}}\geq 9\cdot\sqrt[3]{\frac{b^2+c^2}{2a^2}} \\ \underbrace{\sqrt[4]{\frac{c^3+a^3}{2b^3}}+\sqrt[4]{\frac{c^3+a^3}{2b^3}}+...+\sqrt[4]{\frac{c^3+a^3}{2b^3}}}_{8\text{ terms}}+1\geq 9\cdot\sqrt[3]{\frac{1}{b^2}\left(\frac{c^3+a^3}{2}\right)^{\frac{2}{3}}}\geq 9\cdot\sqrt[3]{\frac{c^2+a^2}{2b^2}} \end{cases}$$

$$\Rightarrow 8\left(\sqrt[4]{\frac{a^3+b^3}{2c^3}}+\sqrt[4]{\frac{b^3+c^3}{2a^3}}+\sqrt[4]{\frac{c^3+a^3}{2b^3}}\right)+3\geq 9\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)$$

$$=8\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)+\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)$$

$$\geq 8\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)+3.\sqrt[9]{\frac{a^2+b^2}{2c^2}\cdot\frac{b^2+c^2}{2a^2}\cdot\frac{c^2+a^2}{2b^2}}$$

$$\geq 8\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)+3.\sqrt[9]{\frac{2ab}{2c^2}\cdot\frac{2bc}{2a^2}\cdot\frac{2ca}{2b^2}}$$

$$=8\left(\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\right)+3.$$ It follows

$$\sqrt[4]{\frac{a^3+b^3}{2c^3}}+\sqrt[4]{\frac{b^3+c^3}{2a^3}}+\sqrt[4]{\frac{c^3+a^3}{2b^3}}\ge\sqrt[3]{\frac{a^2+b^2}{2c^2}}+\sqrt[3]{\frac{b^2+c^2}{2a^2}}+\sqrt[3]{\frac{c^2+a^2}{2b^2}}\Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow a=b=c>0$

**Problem 3. (Vo Quoc Ba Can)** Let be given $a$, $b$, $c>0$. Prove that

$$\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^2+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^2+b^2}}\ge\frac{9\cdot\sqrt[3]{abc}}{a+b+c}\quad(1)$$

***Proof***

Using $AM-GM$ Inequality we have:

$$\frac{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)}{a^2+bc}=\frac{a(b^2+c^2)}{a^2+bc}+b+c\ge3\cdot\sqrt[3]{\frac{abc(b^2+c^2)}{a^2+bc}}$$

$$\Rightarrow\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}\ge\frac{3(a^2+bc)\cdot\sqrt[3]{abc}}{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)}$$

$$\Rightarrow LHS(1)=\sum_{cyc}\sqrt[3]{\frac{a^2+bc}{b^2+c^2}}\ge\frac{3(a^2+b^2+c^2+ab+bc+ca)\cdot\sqrt[3]{abc}}{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)}\quad(2)$$

Using Schur Inequality we have:

$$(a+b+c)(a^2+b^2+c^2+ab+bc+ca)-3\left[a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\right]$$

$$=a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)\ge0$$

$$\Rightarrow\ \frac{3(a^2+b^2+c^2+ab+bc+ca)}{a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)}\ge\frac{9}{a+b+c}\quad(3)$$

From (2) and (3) follow $LHS(1)\ge\dfrac{9\cdot\sqrt[3]{abc}}{a+b+c}$ (q.e.d.)

Equality occurs $\Leftrightarrow a=b=c\ge0$

**Problem 4.** Let be given $a$, $b$, $c>0$. Prove that

$$\sqrt{\frac{a(b+c)}{a^2+bc}}+\sqrt{\frac{b(c+a)}{b^2+ca}}+\sqrt{\frac{c(a+b)}{c^2+ab}}\le\sqrt{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)}\quad(1)$$

***Proof***

$$(1)\Leftrightarrow\left(\sum_{cyc}\sqrt{\frac{a(b+c)}{a^2+bc}}\right)^2\le\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{a(b+c)}{a^2+bc}+2\sum_{cyc}\sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}}\le3+\sum_{cyc}\frac{b+c}{\sqrt{bc}}$$

$$\Leftrightarrow \sum_{cyc} \frac{a(b+c)}{a^2+bc} + 2\sum_{cyc} \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} - \sum_{cyc} \frac{b+c}{\sqrt{bc}} - 3 \le 0 \quad (2)$$

Using $AM-GM$ Inequality we have: $\sum_{cyc} \frac{a(b+c)}{a^2+bc} \le \sum_{cyc} \frac{a(b+c)}{2\sqrt{a^2bc}} = \sum_{cyc} \frac{b+c}{2\sqrt{bc}}$

On the other hand, $(a^2+bc)(b^2+ca) - ab(a+c)(b+c) = c(a-b)^2(a+b) \ge 0$

$$\Rightarrow \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \le 1 \quad \Rightarrow \sum_{cyc} \sqrt{\frac{ab(a+c)(b+c)}{(a^2+bc)(b^2+ca)}} \le 3$$

$$\Rightarrow LHS(2) \le \sum_{cyc} \frac{b+c}{2\sqrt{bc}} + 6 - \sum_{cyc} \frac{b+c}{\sqrt{bc}} - 3 = \sum_{cyc} \left(1 - \frac{b+c}{2\sqrt{bc}}\right) = -\sum_{cyc} \frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{2\sqrt{bc}} \le 0$$

$\Rightarrow$ (2) is true $\Rightarrow$ (q.e.d.). Equality occurs $\Leftrightarrow$ $a = b = c > 0$

**Problem 5.** Let be given $a$, $b$, $c$, $d > 0$. Prove that

$$\frac{1}{a^3+b^3} + \frac{1}{a^3+c^3} + \frac{1}{a^3+d^3} + \frac{1}{b^3+c^3} + \frac{1}{b^3+d^3} + \frac{1}{c^3+d^3} \ge \frac{243}{2(a+b+c+d)^3} \quad (1)$$

***Proof***

WLOG supposing $a \ge b \ge c \ge d \ge 0 \Rightarrow a^3+b^3 \le \left(a+\frac{d}{3}\right)^3 + \left(b+\frac{d}{3}\right)^3$; $a^3+d^3 \le \left(a+\frac{d}{3}\right)^3$

$$\Rightarrow \begin{cases} \dfrac{1}{a^3+b^3} \ge \dfrac{1}{\left(a+\frac{d}{3}\right)^3 + \left(b+\frac{d}{3}\right)^3} \\ \dfrac{1}{a^3+c^3} \ge \dfrac{1}{\left(a+\frac{d}{3}\right)^3 + \left(c+\frac{d}{3}\right)^3} \\ \dfrac{1}{a^3+b^3} \ge \dfrac{1}{\left(b+\frac{d}{3}\right)^3 + \left(c+\frac{d}{3}\right)^3} \end{cases} ; \begin{cases} \dfrac{1}{a^3+d^3} \ge \dfrac{1}{\left(a+\frac{d}{3}\right)^3} \\ \dfrac{1}{b^3+d^3} \ge \dfrac{1}{\left(b+\frac{d}{3}\right)^3} \\ \dfrac{1}{c^3+d^3} \ge \dfrac{1}{\left(c+\frac{d}{3}\right)^3} \end{cases}$$

$$\Rightarrow LHS(1) \ge \sum_{cyc} \frac{1}{x^3+y^3} + \sum_{cyc} \frac{1}{x^3} \quad \text{for } x = a+\frac{d}{3}, y = b+\frac{d}{3}, z = c+\frac{d}{3}$$

We need to prove $\sum_{cyc} \frac{1}{x^3+y^3} + \sum_{cyc} \frac{1}{x^3} \ge \frac{243}{2(x+y+z)^3} \Leftrightarrow \sum_{cyc} \left(\frac{2}{x^3+y^3} + \frac{1}{x^3} + \frac{1}{y^3}\right) \ge \frac{243}{(x+y+z)^3}$

Using ***AM – GM Inequality*** we have $\frac{2}{x^3+y^3} + \frac{1}{x^3} + \frac{1}{y^3} \ge 3 \cdot \sqrt[3]{\frac{2}{x^3y^3(x^3+y^3)}}$

$$= 3 \cdot \sqrt[3]{\frac{2}{(xy)^3(x^2-xy+y^2)(x+y)}} \ge 3 \cdot \sqrt[3]{\frac{2}{\left(\frac{3xy+x^2-xy+y^2}{4}\right)^4 (x+y)}} = \frac{24}{(x+y)^3}$$

$$\Rightarrow \sum_{cyc}\left(\frac{2}{x^3+y^3}+\frac{1}{x^3}+\frac{1}{y^3}\right) \geq 24\sum_{cyc}\frac{1}{(x+y)^3} \geq 24\cdot\frac{3}{(x+y)(y+z)(z+x)}$$

$$\geq \frac{72}{\left[\frac{(x+y)+(y+z)+(z+x)}{3}\right]^3} = 72.\frac{27}{\left[2(x+y+z)\right]^3} = \frac{243}{(x+y+z)^3} \Rightarrow \text{(q.e.d.)}$$

Equality occurs $\Leftrightarrow$ $(a,b,c,d)$ is a permutation of (1; 1; 1; 0)

**Problem 6.** Let be given $a$, $b$, $c > 0$ satisfying the condition $a+b+c=1$

Find the maximum value of $S=\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab}$

***Solution***

$$S=\frac{a}{a+bc}+\frac{b}{b+ca}+\frac{\sqrt{abc}}{c+ab} = \frac{1}{1+\frac{bc}{a}}+\frac{1}{1+\frac{ca}{b}}+\frac{\sqrt{ab/c}}{1+\frac{ab}{c}}$$

Taking $\frac{bc}{a}=\tan^2\frac{A}{2};\frac{ca}{b}=\tan^2\frac{B}{2}$ for $0<A,\, B<\pi$

Then $1=a+b+c=\sqrt{\frac{ab}{c}}\cdot\sqrt{\frac{ca}{b}}+\sqrt{\frac{bc}{a}}\cdot\sqrt{\frac{ab}{c}}+\sqrt{\frac{ca}{b}}\cdot\sqrt{\frac{bc}{a}}$

It follows $\sqrt{\frac{ab}{c}}=\frac{1-\tan\frac{A}{2}\cdot\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}=\cot\left(\frac{A+B}{2}\right)=\tan\frac{C}{2}$ for $\begin{cases}A,B,C>0\\ A+B+C=\pi\end{cases}$

$$\text{S} = \frac{1}{1+\tan^2\frac{A}{2}}+\frac{1}{1+\tan^2\frac{B}{2}}+\frac{\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}=\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\frac{\sin C}{2}=1+\frac{1}{2}(\cos A+\cos B+\sin C)$$

$$=1+\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos A+\frac{\sqrt{3}}{2}\cos B\right)+\frac{1}{2\sqrt{3}}\left(\sqrt{3}\sin A.\cos B+\sqrt{3}\sin B.\cos A\right)$$

$$\leq 1+\frac{1}{2\sqrt{3}}\left[\left(\frac{3}{4}+\cos^2A\right)+\left(\frac{3}{4}+\cos^2B\right)\right]+\frac{1}{4\sqrt{3}}\left[\left(3\sin^2A+\cos^2B\right)+\left(3\sin^2B+\cos^2A\right)\right]$$

$$=1+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}\left(\cos^2A+\sin^2A\right)+\frac{\sqrt{3}}{4}\left(\cos^2B+\sin^2B\right)=1+\frac{3\sqrt{3}}{4}$$

Equality occurs $\Leftrightarrow$ $A=B=\frac{\pi}{6}, C=\frac{2\pi}{3}$

$$\Leftrightarrow \sqrt{\frac{bc}{a}}=\sqrt{\frac{ca}{b}}=\tan\frac{\pi}{12}=2-\sqrt{3},\ \sqrt{\frac{ab}{c}}=\tan\frac{\pi}{3}=\sqrt{3} \Leftrightarrow a=b=2\sqrt{3}-3,\, c=7-4\sqrt{3}$$

## §4.18. INEQUALITY PROOF BY ANALYSIS OF SUM OF SQUARES (S.O.S)

### I. Express non-negative polynomial in terms of sum of squares

***1. Definition:*** Polynomial $F(a_1,a_2,...,a_n)$ is an $n$-variable function $(a)=(a_1,a_2,...,a_k,...,a_n)$ and $\sum!F(a_1,a_2,...,a_n)$ is sum of $n!$ expressions coming from $F(a_1,a_2,...,a_n)$ by all permutations of all variables $(a)=(a_1,a_2,...,a_n)$.

• Polynomial $F(a_1,a_2,...,a_n)$ is an $m$-degree homogeneous function

$\Leftrightarrow$ $F(ta_1,ta_2,...,ta_n)=t^m.F(a_1,a_2,...,a_n)$

• Polynomial $F(a_1,a_2,...,a_n)\geq 0$ $\forall a_1,a_2,...,a_n\in \mathrm{D}$ $\Leftrightarrow$ $F(a_1,a_2,...,a_n)$ is called as non-negative in domain D.

***2. Identity Hurwitz – Muirhead and inequality AM – GM (Cauchy)***

Consider the special case: $F(a_1,a_2,...,a_n)=a_1^{\alpha_1}a_2^{\alpha_2}....a_n^{\alpha_n}$ for $a_k\geq 0,\alpha_k\geq 0$

Let $[\alpha]=[\alpha_1,\alpha_2,...\alpha_n]=\frac{1}{n!}\sum!F(a_1,a_2,...,a_n)=\frac{1}{n!}\sum!a_1^{\alpha_1}a_2^{\alpha_2}....a_n^{\alpha_n}$

Specially, $[1,0,0,...0]=\frac{(n-1)!}{n!}(a_1+a_2+...+a_n)=\frac{a_1+a_2+...+a_n}{n}=A(a)$: Arithmetic Mean (AM)

$$\left[\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right]=\frac{n!}{n!}a_1^{\frac{1}{n}}a_2^{\frac{1}{n}}....a_n^{\frac{1}{n}}=\sqrt[n]{a_1a_2....a_n}=g(a)\text{: Geometric Mean (GM)}$$

Consider transformation: $A(a^n)-g(a^n)=\frac{a_1^n+a_2^n+...+a_n^n}{n}-a_1a_2....a_n=[n,0,0,...0]-[1,1,1,...,1]$

$$=[n,0,0,...0]-[n-1,1,0,...,0]+[n-1,1,0,...,0]-[n-2,1,1,0,...,0]+$$

$$+[n-2,1,1,0,...,0]-[n-3,1,1,1,0,...,0]+...+[3,1,1,...,1,0,0]-[2,1,1,...,1,0]+[2,1,1,...,1,0]-[1,1,1,...,1]$$

$$=\frac{1}{2n!}\left[\sum!(a_1^{n-1}-a_2^{n-1})(a_1-a_2)+\sum!(a_1^{n-2}-a_2^{n-2})(a_1-a_2)a_3+\sum!(a_1^{n-3}-a_2^{n-3})(a_1-a_2)a_3a_4+...\right]$$

As $(a_p-a_q)(a_p^r-a_q^r)\geq 0$ $\forall a_p,a_q\geq 0$ so $A(a^n)-g(a^n)\geq 0$

Let $a_1=x_1^2;...;a_n=x_n^2$ and $(i_1,i_2,...,i_n)$ as a permutation of $(1,2,...n)$. Then we have:

$$n\left[A(a^n)-g(a^n)\right]=(a_1^n+a_2^n+...+a_n^n)-na_1a_2....a_n=x_1^{2n}+...+x_n^{2n}-nx_1^2x_2^2...x_n^2=$$

$$=\frac{1}{2(n-1)!}\left[\sum!(x_{i_1}^{2n-2}-x_{i_2}^{2n-2})(x_{i_1}^2-x_{i_2}^2)+\sum!(x_{i_1}^{2n-4}-x_{i_2}^{2n-4})(x_{i_1}^2-x_{i_2}^2)x_{i_3}^2+...+\sum!(x_{i_1}^2-x_{i_2}^2)^2x_{i_3}^2x_{i_4}^2...x_{i_n}^2\right]$$

Besides,

$$\left(x_{i_1}^{2n-2k}-x_{i_2}^{2n-2k}\right)\left(x_{i_1}^2-x_{i_2}^2\right)x_{i_3}^2...x_{i_{k+1}}^2=\left(x_{i_1}^2-x_{i_2}^2\right)^2\left[x_{i_1}^{2n-2k-2}+x_{i_1}^{2n-2k-4}x_{i_2}^2+...+x_{i_2}^{2n-2k-2}\right]x_{i_3}^2...x_{i_{k+1}}^2$$

$$=\left(x_{i_1}^2-x_{i_2}^2\right)^2\left[\sum_{j=1}^{n-k}\left(x_{i_1}^{n-k-j}x_{i_2}^{j-1}\right)^2\right]x_{i_3}^2...x_{i_{k+1}}^2=\sum_{j=1}^{n-k}\left[\left(x_{i_1}^2-x_{i_2}^2\right)x_{i_1}^{n-k-j}x_{i_2}^{j-1}x_{i_3}...x_{i_{k+1}}\right]^2$$

$$\Rightarrow n\left[A(a^n)-g(a^n)\right]=s_1^2+s_2^2+...+s_m^2\geq 0\Rightarrow A(a^n)\geq g(a^n)\Rightarrow A(a)\geq g(a)$$

***3. Special cases:*** • $n = 2$: $x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2$

• $n = 3$: $x_1^6 + x_2^6 + x_3^6 - 3x_1^2x_2^2x_3^2 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\left[(x_1^2 - x_2^2)^2 + (x_2^2 - x_3^2)^2 + (x_3^2 - x_1^2)^2\right]$

• $n = 4$: $x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4x_1x_2x_3x_4 = (x_1^2 - x_2^2)^2 + (x_3^2 - x_4^2)^2 + 2(x_1x_2 - x_3x_4)^2$

• $n = 5$: $x_1^{10} + x_2^{10} + x_3^{10} + x_4^{10} + x_5^{10} - 5x_1^2x_2^2x_3^2x_4^2x_5^2 =$

$$= \frac{1}{2.4!}\left[\sum!(x_{i_1}^8 - x_{i_2}^8)(x_{i_1}^2 - x_{i_2}^2) + \sum!(x_{i_1}^6 - x_{i_2}^6)(x_{i_1}^2 - x_{i_2}^2)x_{i_3}^2 + \sum!(x_{i_1}^4 - x_{i_2}^4)(x_{i_1}^2 - x_{i_2}^2)x_{i_3}^2x_{i_4}^2 + \sum!(x_{i_1}^2 - x_{i_2}^2)^2 x_{i_3}^2x_{i_4}^2x_{i_5}^2\right]$$

$$= \frac{1}{24!}\left[\sum!\sum_{j=1}^{4}\left[(x_{i_1}^2 - x_{i_2}^2)x_{i_1}^{4-j}x_{i_2}^{j-1}\right]^2 + \sum!\sum_{j=1}^{3}\left[(x_{i_1}^2 - x_{i_2}^2)x_{i_1}^{3-j}x_{i_2}^{j-1}x_{i_3}\right]^2 + \sum!\sum_{j=1}^{2}\left[(x_{i_1}^2 - x_{i_2}^2)x_{i_1}^{2-j}x_{i_2}^{j-1}x_{i_3}x_{i_4}\right]^2 + \sum!\left[(x_{i_1}^2 - x_{i_2}^2)x_{i_3}x_{i_4}x_{i_5}\right]^2\right]$$

• $n = 6$: $x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 - 6x_1x_2x_3x_4x_5x_6 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\left[(x_1^2 - x_2^2)^2 + (x_2^2 - x_3^2)^2 + (x_3^2 - x_1^2)^2\right] +$

$$+ \frac{1}{2}(x_4^2 + x_5^2 + x_6^2)\left[(x_4^2 - x_5^2)^2 + (x_5^2 - x_6^2)^2 + (x_6^2 - x_4^2)^2\right] + 3(x_1x_2x_3 - x_4x_5x_6)^2$$

***4. Theorem and examples:***

***4.1. Theorem Hilbert 1:***

If polynomial $F(a_1, a_2, ..., a_n)$ is homogeneous and non-negative in domain D, i.e.:

$F(a_1, a_2, ..., a_n) \geq 0 \ \forall a_1, a_2, ..., a_n \in \mathrm{D}$ then it is possible to express

$F(a_1, a_2, ..., a_n) = p_1^2 + p_2^2 + ... + p_m^2$ in which $p_k$ is a certain real rational function.

***4.2. Theorem Hilbert 2:*** If polynomial $F(a_1, a_2, ..., a_n)$ is non-homogeneous and non-negative in domain $a_1 \geq 0, a_2 \geq 0, ...a_n \geq 0, a_1 + a_2 + ... + a_n \leq 1$ then

$F(a_1, a_2, ..., a_n) = \sum c_k a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} (1 - a_1 - a_2 - ... - a_n)^{\alpha_{n+1}}$

in which $\alpha_1, ...\alpha_n, \alpha_{n+1}$ are non-negative integers and $c_k > 0$.

***4.3. Comment:*** The Theorems Hilbert 1 & 2 are qualitative, helping us believe that there exists a way to prove a polynomial as non-negative by analysis of sum of squares (as for homogeneous functions) or into sum of non-negative quantities (as for non-homogeneous functions). It is, however, due to this obscure existence that it has little value in effect of proof of elementary math inequalities. Take 3 extreme examples below to show that the analysis into of squares would be actually difficult without belief in the existence of the analysis. Examples 1&2 have short solutions but lack naturalness, so to find solutions would be nothing more than words puzzle in which only a result is a meaningful word.

**Example 1:** $a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \geq 2(a^3b + b^3c + c^3a) \ \forall a, b, c \in \mathbb{R}$

***Proof:*** $2\left[a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 - 2(a^3b + b^3c + c^3a)\right] =$

$$= (a^2 - ab + ac - c^2)^2 + (b^2 - bc + ba - a^2)^2 + (c^2 - ca + cb - b^2)^2 \geq 0$$

**Example 2:** $\left(a^2+b^2+c^2\right)^2 \ge 3\left(a^3b+b^3c+c^3a\right) \quad \forall a,b,c\in \mathbb{R} \quad (1)$

***1^st^ proof:***

$(1) \Leftrightarrow a^4+b^4+c^4+2\left(a^2b^2+b^2c^2+c^2a^2\right)\ge 3\left(a^3b+b^3c+c^3a\right) \Leftrightarrow \sum_{cyc} a^4+2\sum_{cyc}a^2b^2 \ge 3\sum_{cyc}a^3b$

$$\Leftrightarrow \text{S} = \left(\sum_{cyc}a^4-\sum_{cyc}a^2b^2\right)+3\left(\sum_{cyc}a^2b^2-\sum_{cyc}a^2bc\right)-3\left(\sum_{cyc}a^3b-\sum_{cyc}a^2bc\right)\ge 0$$

Transform: $\sum_{cyc}a^4-\sum_{cyc}a^2b^2=\frac{1}{2}\sum_{cyc}(a^2-b^2)^2$ ; $3\left(\sum_{cyc}a^2b^2-\sum_{cyc}a^2bc\right)=\frac{1}{2}\sum_{cyc}(ab+ac-2bc)^2$

$$3\left(\sum_{cyc}a^3b-\sum_{cyc}a^2bc\right)=3\left(\sum_{cyc}b^3c-\sum_{cyc}a^2bc\right)=-3\sum_{cyc}bc(a^2-b^2)$$

$$=-3\sum_{cyc}bc(a^2-b^2)+\sum_{cyc}(ab+bc+ca)(a^2-b^2)=\sum_{cyc}(a^2-b^2)(ab+ac-2bc)$$

Then $\text{S} = \frac{1}{2}\sum_{cyc}(a^2-b^2)^2+\frac{1}{2}\sum_{cyc}(ab+ac-2bc)^2-\sum_{cyc}(a^2-b^2)(ab+ac-2bc)$

$$= \frac{1}{2}\sum_{cyc}(a^2-b^2-ab-ac+2bc)^2\ge 0 \Rightarrow \text{(q.e.d)}$$

***2^nd^ proof:*** $4\left(a^2+b^2+c^2-ab-bc-ca\right)\left[\left(a^2+b^2+c^2\right)^2-3\left(a^3b+b^3c+c^3a\right)\right]=$

$$=\left[\left(a^3+b^3+c^3\right)-5\left(a^2b+b^2c+c^2a\right)+4\left(ab^2+bc^2+ca^2\right)\right]^2+$$

$$+3\left[\left(a^3+b^3+c^3\right)-\left(a^2b+b^2c+c^2a\right)-2\left(ab^2+bc^2+ca^2\right)+6abc\right]^2\ge 0$$

## II. METHOD OF ANALYZING SUM OF SQUARES (S.O.S.)

***1. Introduction:*** To prove a homogeneous polynomial $F\left(a_1,a_2,...,a_n\right)\ge 0$, according to theorem Hilbert there exists an expression $F\left(a_1,a_2,...,a_n\right)=p_1^2+p_2^2+...+p_m^2$ in which $p_k$ is a certain real rational function, but to specify a certain expression would be very difficult. The following method of analyzing sum of squares helps us surmount the abstraction of theorem Hilbert, particularly to symmetrical functions $F\left(a,b,c\right)$. In Vietnam, this method was studied and then stated by Tran Phuong, Tran Tuan Anh and Anh Cuong in August, 2004. In 2006, Pham Kim Hung gave it the official name of S.O.S.

***2. About S.O.S.***

Let consider inequality $A\ge B$ in which $A$, $B$ are expressions of variables $a,b,c$.

Supposing we can turn the inequality $A-B\ge 0$ into:

$S=A-B=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ **(1)**. Considers the following propositions:

***2.1. Proposition 1:*** If $S_a\ge 0; S_b\ge 0; S_c\ge 0$ then $S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$

(1) is obviously true in this case. Therefore, we hereinafter consider other conditions.

***2.2. Proposition 2:*** Letting that $a,b,c,S_a,S_b,S_c$ are real numbers satisfying 2 conditions:

**i)** $S_a+S_b\ge 0; S_b+S_c\ge 0; S_c+S_a\ge 0$.

**ii)** If $a\le b\le c$ or $a\ge b\ge c$ then $S_b\ge 0$

Then we have $S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ (1)

***Proof***

WLOG suppose $a\le b\le c$, then $S_b\ge 0$ and

(1) $\Leftrightarrow$ $S_b\left[(c-b)+(b-a)\right]^2+S_a(b-c)^2+S_c(a-b)^2\ge 0$

$\Leftrightarrow$ $(S_a+S_b)(b-c)^2+(S_b+S_c)(a-b)^2+2S_b(c-b)(b-a)\ge 0$ (is always true)

***2.3. Proposition 3:*** Letting that $a,b,c,S_a,S_b,S_c$ are real numbers satisfying condition:

If $a\le b\le c$ or $a\ge b\ge c$ then $S_a\ge 0, S_c\ge 0$ and $S_a+2S_b\ge 0; S_c+2S_b\ge 0$

Then we have $S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ (1)

***Proof***

WLOG suppose $a\le b\le c$ and $S_a\ge 0, S_c\ge 0$ ; $S_a+2S_b\ge 0; S_c+2S_b\ge 0$

If $S_b\ge 0$ then (1) is always true.

If $S_b<0$ then $S=(S_b+S_a)(c-b)^2+(S_b+S_c)(b-a)^2+2S_b(c-b)(b-a)$

$$\ge (S_b+S_a)(c-b)^2+(S_b+S_c)(b-a)^2+S_b\left[(c-b)^2+(b-a)^2\right]$$

$$=(S_a+2S_b)(c-b)^2+(S_c+2S_b)(b-a)^2\ge 0$$

***2.4. Proposition 4:*** Letting that $a,b,c,S_a,S_b,S_c$ are real numbers satisfying condition:

If $a\le b\le c$ then $S_a\ge 0, S_b\ge 0$ and $b^2S_c+c^2S_b\ge 0$

Then we have $S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ (1)

***Proof***

WLOG suppose $a\le b\le c$ and $S_a\ge 0, S_b\ge 0$; $b^2S_c+c^2S_b\ge 0$

$$S=S_a(b-c)^2+(b-a)^2\left[S_b\left(\frac{c-a}{b-a}\right)^2+S_c\right]\ge S_a(b-c)^2+(b-a)^2\left[S_b\left(\frac{c}{b}\right)^2+S_c\right]$$

$$=S_a(b-c)^2+\left(c^2S_b+b^2S_c\right)\left(\frac{b-a}{b}\right)^2\ge 0$$

***2.5. Proposition 5:*** Letting that $a,b,c,S_a,S_b,S_c$ are real numbers satisfying 2 conditions:

**i)** $S_a+S_b>0 \ \vee\ S_b+S_c>0 \ \vee\ S_c+S_a>0$

**ii)** $S_aS_b+S_bS_c+S_cS_a\ge 0$

Then we have $S=S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ (1)

***Proof***

WLOG supposing $S_b + S_c > 0$. Letting $u = b - a$; $v = c - b$, then:

$$S = S_a(b-c)^2 + S_b\left[(c-b)+(b-a)\right]^2 + S_c(a-b)^2$$

$$= (S_b + S_c)(a-b)^2 + 2S_b(c-b)(b-a) + (S_a + S_b)(b-c)^2$$

$$= (S_b + S_c)\left(u + \frac{S_b}{S_b + S_c}v\right)^2 + \frac{S_aS_b + S_bS_c + S_cS_a}{S_b + S_c}\cdot v^2 \geq 0 \Rightarrow \text{(q.e.d)}$$

***2.6. Proposition 6:*** Letting that $a, b, c, S_a, S_b, S_c$ are real numbers satisfying conditions:

**i)** $a \leq b \leq c$ or $a \geq b \geq c$

**ii)** There exist $\alpha > 0$ such that $S_a + \alpha^2 S_c + (\alpha+1)^2 S_b \geq 0$

**iii)** $\begin{cases} |a-b| \geq \alpha|c-b| \\ S_c + S_b \geq 0 \\ S_c + \dfrac{\alpha+1}{\alpha}S_b \geq 0 \end{cases} \vee \begin{cases} |c-b| \geq \alpha|a-b| \\ S_a + S_b \geq 0 \\ S_a + \dfrac{\alpha+1}{\alpha}S_b \geq 0 \end{cases}$

Then we have: $S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0 \quad (1)$

***Proof***

WLOG suppose $a \leq b \leq c$ and

$\exists\alpha > 0$ such that $S_a + \alpha^2 S_c + (\alpha+1)^2 S_b \geq 0$ and $\begin{cases} |a-b| \geq \alpha|c-b| \\ S_c + S_b \geq 0 \\ S_c + \dfrac{\alpha+1}{\alpha}S_b \geq 0 \end{cases}$

Consider following possibilities:

- If $b = c$ then (1) $\Leftrightarrow (S_b + S_c)(a-c)^2 \geq 0 \Leftrightarrow S_b + S_c \geq 0$ (is always true)
- If $b \neq c$ then (1) $\Leftrightarrow S_a + S_b\left(\dfrac{b-a}{c-b}+1\right)^2 + S_c\left(\dfrac{b-a}{c-b}\right)^2 \geq 0$

$\Leftrightarrow f(t) = S_a + S_b(t+1)^2 + S_c t^2 \geq 0$ for $t = \dfrac{b-a}{c-b} \geq \alpha$

We have: $f(t) - f(\alpha) = (t-\alpha)\left[S_b(t+2+\alpha) + S_c(t+\alpha)\right]$

$$= (t-\alpha)\left[(S_b + S_c)t + S_b(1+\alpha) + S_c\alpha + S_b\right] \geq (t-\alpha)\left[(S_b + S_c)\alpha + S_b(1+\alpha) + S_c\alpha + S_b\right]$$

$$= 2(t-\alpha)\left[S_c\alpha + S_b(1+\alpha)\right] = \frac{2(t-\alpha)}{\alpha}\left[S_c + \frac{\alpha+1}{\alpha}S_b\right] \geq 0, \quad \forall t \geq \alpha > 0$$

It follows: $f(t) \geq f(\alpha) = S_a + \alpha^2 A_c + (\alpha+1)^2 S_b \geq 0$

Equality occurs $\Leftrightarrow \begin{cases} t = \alpha \\ S_a + \alpha^2 A_c + (\alpha+1)^2 S_b = 0 \end{cases} \Leftrightarrow \begin{cases} b - a = \alpha(c-b) \\ S_a + \alpha^2 A_c + (\alpha+1)^2 S_b = 0 \end{cases}$

***Consequence:*** Letting that $a,b,c,S_a,S_b,S_c$ are real numbers satisfying conditions:

**i)** $a \le b \le c$ or $a \ge b \ge c$

**ii)** $S_a + S_c + 4S_b \ge 0$

**iii)** $\begin{cases} |a-b| \ge |c-b| \\ S_c + S_b \ge 0 \\ S_c + 2S_b \ge 0 \end{cases} \vee \begin{cases} |c-b| \ge |a-b| \\ S_a + S_b \ge 0 \\ S_a + 2S_b \ge 0 \end{cases}$

Then we have: $S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \ge 0$

**3. Theorem on the expression of S.O.S.**

***3.1. Theorem:***

Supposing that two homogeneous permutation polynomials $A$, $B$ have the same degree and the same number of variables. If so, the difference of these two polynomials can be written in the form of S.O.S., that is:

$$\sum_{cyc} a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} - \sum_{cyc} a_1^{\beta_1} a_2^{\beta_2} ... a_n^{\beta_n} = \sum P_{ij}(a)(a_i - a_j)^2$$

In which $\alpha_1 + ... + \alpha_n = \beta_1 + ... + \beta_n = m$ and $a = \{a_i\}_{i=1}^n$. In this case, $A$ and $B$ are also said to have S.O.S relation.

***Proof***

First of all, we prove the following lemma

**Lemma:** Supposing $a = \{a_i\}_{i=1}^n$ and $\alpha_1 + \alpha_2 + ... + \alpha_n = m$, then:

$$\sum_{cyc} a_1^m - \sum_{cyc} a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} = \sum P_{ij}(a)(a_i - a_j)^2$$

***Proof:*** We will proof the lemma by inducing based on k, which is the number of elements except 0 belonging to the set $\{\alpha_i\}_{i=1}^n$ ($n, m$ are constant).

If $k = 1$, the theorem is obviously true.

If $k = 2$, we have to prove the existence of the expression $\sum_{cyc} a_1^m - \sum_{cyc} a_1^t a_2^{m-t} = \sum P_{ij}(a)(a_i - a_j)^2$

This can be proved with notice that: $ta^m + (m-t)b^m - ma^t b^{m-t} = P(a,b)(a-b)^2$

Indeed, it is first of all worth noticing that the equation $f(x) = tx^n + (m-t) - mx^t = 0$ has one double root which is 1, because $f(1) = f'(1) = 0$. Therefore, $f(x)$ can be expressed in the form $Q(x)(x-1)^2, \deg(Q) = m-2$

Letting $x = \frac{a}{b}$, then we have: $b^m f\left(\frac{a}{b}\right) = ta^m + (m-t)b^m - ma^t b^{m-t} = b^{m-2} Q\left(\frac{a}{b}\right)(a-b)^2$.

However, $b^{m-2}Q\left(\frac{a}{b}\right)$ is a polynomial having 2 variables $a,b$ because $Q$ is a $(n-2)$-degree polynomial. Supposing that our proposition is already true with $k$, the number of the elements except for 0 in the set $\{\alpha_i\}_{i=1}^{n}$, with $k+1$, we can transform this into the case of $k$ as follows:

$$a_1^{\alpha_1}a_2^{\alpha_2}...a_{k+1}^{\alpha_{k+1}} = -\frac{\alpha_1 a_1^{\alpha_1+\alpha_2}+\alpha_2 a_2^{\alpha_1+\alpha_2}-(\alpha_1+\alpha_2)a_1^{\alpha_1}a_2^{\alpha_2}}{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{\alpha_{k+1}}$$

$$\frac{\alpha_1}{\alpha_1+\alpha_2}a_1^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{\alpha_{k+1}}+\frac{\alpha_2}{\alpha_1+\alpha_2}a_2^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{\alpha_{k+1}}$$

Note that with $k=2$: $\dfrac{\alpha_1 a_1^{\alpha_1+\alpha_2}+\alpha_2 a_2^{\alpha_1+\alpha_2}-(\alpha_1+\alpha_2)a_1^{\alpha_1}a_2^{\alpha_2}}{\alpha_1+\alpha_2}=H_{12}(a)(a_1-a_2)^2$, we have:

$$a_1^{\alpha_1}a_2^{\alpha_2}...a_{k+1}^{\alpha_{k+1}} = Q_{12}(a)(a_1-a_2)^2+\frac{\alpha_1}{\alpha_1+\alpha_2}a_1^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{\alpha_{k+1}}+\frac{\alpha_2}{\alpha_1+\alpha_2}a_2^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{\alpha_{k+1}}$$

Therefore, $\sum\limits_{cyc}a_1^m-\sum\limits_{cyc}a_1^{\alpha_1}a_2^{\alpha_2}...a_{k+1}^{\alpha_{k+1}}$

$$=-\sum_{cyc}Q_{12}(a)(a_1-a_2)^2+\sum_{cyc}a_1^m-\frac{\alpha_1}{\alpha_1+\alpha_2}\sum_{cyc}a_1^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}+\frac{\alpha_2}{\alpha_1+\alpha_2}\sum_{cyc}a_2^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}$$

$$\sum_{cyc}a_1^m-\sum_{cyc}a_1^{\alpha_1}a_2^{\alpha_2}...a_{k+1}^{\alpha_{k+1}}$$

$$=-\sum_{cyc}Q_{12}(a)(a_1-a_2)^2+\frac{\alpha_1}{\alpha_1+\alpha_2}\left(\sum_{cyc}a_1^m-\sum_{cyc}a_1^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}\right)+\frac{\alpha_2}{\alpha_1+\alpha_2}\left(\sum_{cyc}a_1^m-\sum_{cyc}a_2^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}\right)$$

On the other hand, two expressions $\sum\limits_{cyc}a_1^m-\sum\limits_{cyc}a_1^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}$ and $\sum\limits_{cyc}a_1^m-\sum\limits_{cyc}a_2^{\alpha_1+\alpha_2}a_3^{\alpha_3}...a_{k+1}^{k+1}$ can be written in the form of S.O.S, from which we have $\sum\limits_{cyc}a_1^m-\sum\limits_{cyc}a_1^{\alpha_1}a_2^{\alpha_2}...a_{k+1}^{\alpha_{k+1}}$ can also be written in the form of S.O.S.

According to the induction principle, the theorem has been proved.

***Application:*** Using the lemma and following inequality, we get q.e.d.

$$\sum_{cyc}a_1^{\alpha_1}a_2^{\alpha_2}...a_n^{\alpha_n}-\sum_{cyc}a_1^{\beta_1}a_2^{\beta_2}...a_n^{\beta_n}=\left(\sum_{cyc}a_1^{\alpha_1}a_2^{\alpha_2}...a_n^{\alpha_n}-\sum_{cyc}a_1^m\right)+\left(\sum_{cyc}a_1^m-\sum_{cyc}a_1^{\beta_1}a_2^{\beta_2}...a_n^{\beta_n}\right)$$

***3.2. Comment:*** Generally speaking, most symmetrical or permutation inequalities with polynomials (or fractions) of the same degree and having 3 variables can all be transformed into 3-variable homogeneous permutation polynomials, i.e. can be written in the form of S.O.S. Therefore, the method of S.O.S can be used to solve nearly all the 3-variable homogeneous inequalities. The next question is: "How can we transform them into typical form of S.O.S?". Naturally, the algorism mentioned above is a useful suggestion to transforming a polynomial into the typical form of S.O.S. However, what about other algebraic form? To answer this question, let's consider the following definition:

Two polynomials A and B are considered to have a "close" S.O.S relationship if expression $(A-B)$ can be written directly in the form of S.O.S without using intermediary expressions. In this case, we can alter the expressions to clarify the S.O.S relationship via "close" intermediary relationship.

Below are some common types of expressions:

**i)** $P(a,b,c)A(a,b,c)-Q(a,b,c)B(a,b,c)$, in which $A$ and $B$, $P$ and $Q$ are "close" to each other. To transform the given expression into S.O.S form, we will use the intermediary quantity $Q(a,b,c)A(a,b,c)$

$$P(a,b,c)A(a,b,c)-Q(a,b,c)A(a,b,c)+Q(a,b,c)A(a,b,c)-Q(a,b,c)B(a,b,c)$$

$$=A(a,b,c)\left[P(a,b,c)-Q(a,b,c)\right]+Q(a,b,c)\left[A(a,b,c)-B(a,b,c)\right]$$

**ii)** Or in the fractional form:

$$\frac{A(a,b,c)}{P(a,b,c)}-\frac{B(a,b,c)}{Q(a,b,c)}=\frac{A(a,b,c)-B(a,b,c)}{P(a,b,c)}+\frac{B(a,b,c)\left[Q(a,b,c)-P(a,b,c)\right]}{P(a,b,c)Q(a,b,c)}$$

**iii)** In root form:

$$\sqrt{A(a,b,c)P(a,b,c)}-\sqrt{B(a,b,c)Q(a,b,c)}=\frac{A(a,b,c)P(a,b,c)-B(a,b,c)Q(a,b,c)}{\sqrt{A(a,b,c)P(a,b,c)}+\sqrt{B(a,b,c)Q(a,b,c)}}$$

**iv)** Specially cases:

$$a^2+b^2+c^2-ab-ac-bc=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}$$

$$a^3+b^3+c^3-3abc=\frac{(a+b+c)}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$a^2b+b^2c+c^2a-ab^2-bc^2-ca^2=\frac{(a-b)^3+(b-c)^3+(c-a)^3}{3}$$

$$a^3+b^3+c^3-a^2b-b^2c-c^2a=\frac{(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2}{3}$$

$$a^4+b^4+c^4-a^3b-b^3c-c^3a$$
$$=\frac{(3a^2+2ab+b^2)(a-b)^2+(3b^2+2bc+c^2)(b-c)^2+(3c^2+2ca+a^2)(a-c)^2}{4}$$

$$a^3b+b^3c+c^3a-ab^3-bc^3-ca^3=\frac{a+b+c}{3}\left[(b-a)^3+(c-b)^3+(a-c)^3\right]$$

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2=\frac{(a-b)^2(a+b)^2+(b-c)^2(b+c)^2+(c-a)^2(c+a)^2}{2}$$

## 4. Illustrative Problems

**Problem 1.** Prove that $(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right]\ge\frac{9}{4},\ \forall a,b,c>0$ (1)

***(Iranian Mathematical Olympiad 1996)***

***Proof***

WLOG supposing $c\ge b\ge a>0$.

Letting $\begin{cases}b+c=2x\\c+a=2y\\a+b=2z\end{cases}\Leftrightarrow\begin{cases}a=-x+y+z\\b=x-y+z\\c=x+y-z\end{cases}\Rightarrow x,y,z$ are the length of sides of a triangle.

Since $c\ge b\ge a>0$, $x\ge y\ge z>0$. The inequality (1) becomes:

$$\left(2xy+2yz+2zx-x^2-y^2-z^2\right)\left(\frac{1}{4x^2}+\frac{1}{4y^2}+\frac{1}{4z^2}\right)\ge\frac{9}{4}$$

$$\Leftrightarrow\left(2xy+2yz+2zx-x^2-y^2-z^2\right)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\ge 9\ \Leftrightarrow\ \sum_{cyc}(x-y)^2\left(\frac{2}{xy}-\frac{1}{z^2}\right)\ge 0\ (4)$$

Letting $S_x=\frac{2}{yz}-\frac{1}{x^2},S_y=\frac{2}{zx}-\frac{1}{y^2},S_z=\frac{2}{xy}-\frac{1}{z^2}$

The inequality (4) $\Leftrightarrow$ $S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2\ge 0$ (5)

As $x\ge y\ge z>0$ and $y+z>x$, so $S_x\ge 0$ and $S_y\ge 0$

We will prove $y^2S_y+z^2S_z\ge 0\Leftrightarrow y^3+z^3\ge xyz\ \Leftrightarrow\ (y+x)\left(y^2+z^2-yz\right)\ge xyz$ (6)

We have $y+z>x$ and $y^2+z^2-yz\ge yz\ \Rightarrow$ (6) is true $\Rightarrow\ y^2S_y+z^2S_z\ge 0$

Using ***Proposition 4*** we have q.e.d.

**Problem 2.** Given $a,b,c>0$ be satisfying $\min\{a,b,c\}\ge\frac{1}{4}\max\{a,b,c\}$. Prove that

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right]\ge\frac{9}{4}+\frac{1}{16}\sum_{cyc}\frac{(a-b)^2}{(a+b)^2}\quad(1)$$

***Proof***

WLOG supposing $c\ge b\ge a\ge\frac{1}{4}c>0$. Letting $\begin{cases}b+c=2x\\c+a=2y\\a+b=2z\end{cases}\Leftrightarrow\begin{cases}a=-x+y+z\\b=x-y+z\\c=x+y-z\end{cases}$

$\Rightarrow x,y,z$ are the length of sides of a triangle.

As $c\ge b\ge a\ge\frac{1}{4}c>0$, so $x\ge y\ge z>0$ and $4(-x+y+z)\ge x+y-z\Rightarrow 3y+5z\ge 5x$

The inequality (1) $\Leftrightarrow \left(2xy+2yz+2zx-x^2-y^2-z^2\right)\left(\frac{1}{4x^2}+\frac{1}{4y^2}+\frac{1}{4z^2}\right)\geq \frac{9}{4}+\frac{1}{16}\sum_{cyc}\frac{(x-y)^2}{z^2}$

$\Leftrightarrow \left(2xy+2yz+2zx-x^2-y^2-z^2\right)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geq 9+\frac{1}{4}\sum_{cyc}\frac{(x-y)^2}{z^2} \quad \Leftrightarrow \sum_{cyc}(x-y)^2\left(\frac{2}{xy}-\frac{5}{4z^2}\right)\geq 0$

Letting $S_x=\frac{2}{yz}-\frac{5}{4x^2}, S_y=\frac{2}{zx}-\frac{5}{4y^2}, S_z=\frac{2}{xy}-\frac{5}{4z^2}$

We have to prove $S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2\geq 0$

As $x\geq y\geq z>0$ and $3y+5z\geq 5x$, so $S_x>0$ and $8y\geq 5x \Rightarrow S_y\geq 0$

We will prove $y^2S_y+z^2S_z\geq 0 \Leftrightarrow \frac{2y^2}{xz}+\frac{2z^2}{xy}\geq \frac{5}{2}\Leftrightarrow 4\left(y^3+z^3\right)\geq 5xyz$

We have $3y+5z\geq 5x$, it is suffices to prove that $4(y^3+z^3)\geq (3y+5z)yz$

$\Leftrightarrow (y-z)(4y^2+yz-4z^2)\geq 0$ (is always true). Thus $y^2S_y+z^2S_z\geq 0$

We have $x-z\geq \frac{y}{z}(x-y)\geq 0$. Hence, $S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2\geq S_y(z-x)^2+S_z(x-y)^2$

$\geq S_y.\frac{y^2}{z^2}.(x-y)^2+S_z(x-y)^2=\frac{(x-y)^2(y^2S_y+z^2S_z)}{z^2}\geq 0 \Rightarrow$ q.e.d.

Equality occurs $\Leftrightarrow$ $x=y=z$ or $y=z=\frac{5}{8}x$.

**Problem 3.** Given 3 positive real numbers $a$, $b$, $c$ are satisfying $abc\geq 1$. Prove that

$$S=\frac{a^5-a^2}{a^5+b^2+c^2}+\frac{b^5-b^2}{b^5+c^2+a^2}+\frac{c^5-c^2}{c^5+a^2+b^2}\geq 0$$

***(IMO 45 – 2005)***

***Proof***

Since $abc\geq 1$, $S\geq \frac{a^5-a^2abc}{a^5+\left(b^2+c^2\right)abc}+\frac{b^5-b^2abc}{b^5+\left(c^2+a^2\right)abc}+\frac{c^5-c^2abc}{c^5+\left(a^2+b^2\right)abc}$

$$=\sum_{cyc}\frac{a^4-a^2bc}{a^4+\left(b^2+c^2\right)bc}\geq \sum_{cyc}\frac{2a^4-a^2\left(b^2+c^2\right)}{2a^4+\left(b^2+c^2\right)^2}$$

Letting $x=a^2, y=b^2, z=c^2 \Rightarrow S\geq \frac{2x^2-x\left(y+z\right)}{2x^2+\left(y+z\right)^2}+\frac{2y^2-y\left(z+x\right)}{2y^2+\left(z+x\right)^2}+\frac{2z^2-z\left(x+y\right)}{2z^2+\left(x+y\right)^2}$

$=\sum_{cyc}\left(x-y\right)\left(\frac{x}{2x^2+\left(y+z\right)^2}-\frac{y}{2y^2+\left(z+x\right)^2}\right)=\sum_{cyc}\left(x-y\right)^2\frac{x^2+y^2-xy+z^2+2z\left(x+y\right)}{\left[2x^2+\left(y+z\right)^2\right]\left[2y^2+\left(z+x\right)^2\right]}\geq 0$

Equality occurs $\Leftrightarrow$ $x=y=z\Leftrightarrow a=b=c$

**Problem 4.** Find the maximum coefficient $k$ ($k > 0$) such that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+k\frac{ab+bc+ca}{a^2+b^2+c^2}\ge k+\frac{3}{2} \quad ,\ \forall a,b,c>0 \quad (1)$$

***Solution***

***By a general correspondence:*** (1) $\Leftrightarrow \sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right)\ge k\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right)$ (2)

We have: $\sum_{cyc}\left(\frac{a}{b+c}-\frac{1}{2}\right)=\frac{1}{2}\sum_{cyc}\frac{(a-b)+(a-c)}{b+c}=\frac{1}{2}\sum_{cyc}(a-b)\left(\frac{1}{b+c}-\frac{1}{c+a}\right)=\frac{1}{2}\sum_{cyc}\frac{(a-b)^2}{(b+c)(c+a)}$

(2) $\Leftrightarrow \sum_{cyc}\frac{(a-b)^2}{(b+c)(c+a)}\ge k\frac{2\left(a^2+b^2+c^2-ab-bc-ca\right)}{a^2+b^2+c^2}\Leftrightarrow\sum_{cyc}\frac{(a-b)^2}{(b+c)(c+a)}\ge k\sum_{cyc}\frac{(a-b)^2}{a^2+b^2+c^2}$

$\Leftrightarrow \sum_{cyc}\left[\frac{(a-b)^2}{(b+c)(c+a)}-k\frac{(a-b)^2}{a^2+b^2+c^2}\right]\ge 0\Leftrightarrow\sum_{cyc}(a-b)^2\left[\frac{a^2+b^2+c^2}{(b+c)(c+a)}-k\right]\ge 0$ (3)

***The necessary condition:***

Taking $c=b\ne a$ into (3) $\Rightarrow (a-b)^2\left[\frac{a^2+2b^2}{2b(a+b)}-k\right]\ge 0 \Rightarrow k\le\frac{a^2+2b^2}{2b(a+b)}=u(a,b)$

We have: $u(a,b)=\dfrac{\left(\frac{a}{b}\right)^2+2}{2\left(\frac{a}{b}+1\right)}=\dfrac{t^2+2}{2t+2}\Rightarrow (2t+2)u=t^2+2\Leftrightarrow f(t)=t^2-2ut+2-2u=0$

$u\in$ the domain of the function $\Leftrightarrow f(t)=0$ has root $\Leftrightarrow \Delta_t'=u^2+2u-2\ge 0\Rightarrow u\ge\frac{\sqrt{3}-1}{2}$

***The sufficient condition:***

We will prove $k=\frac{\sqrt{3}-1}{2}=\operatorname{Min}u(a,b)$ is the best coefficient of inequality (1)

Indeed, with $k=\frac{\sqrt{3}-1}{2}$ we have: (1) $\Leftrightarrow \sum_{cyc}(a-b)^2\left[\frac{a^2+b^2+c^2}{(b+c)(c+a)}-k\right]\ge 0$ (4)

Letting $S_a=\frac{a^2+b^2+c^2}{(c+a)(b+a)}-k; S_b=\frac{a^2+b^2+c^2}{(a+b)(c+b)}-k; S_c=\frac{a^2+b^2+c^2}{(b+c)(a+c)}-k$

WLOG supposing $a\ge b\ge c\Rightarrow S_a\le S_b\le S_c$

Consider $S_a+S_b=\frac{a^2+b^2+c^2}{(c+a)(b+a)}+\frac{a^2+b^2+c^2}{(a+b)(c+b)}-2k=\frac{\left(a^2+b^2+c^2\right)(a+b+2c)}{(a+b)(b+c)(c+a)}-2k$

Letting $v=\frac{a+b}{2}\Rightarrow S_a+S_b\ge\dfrac{\left[\frac{(a+b)^2}{2}+c^2\right](2v+2c)}{2v(b+c)(c+a)}-2k\ge\dfrac{\left(2v^2+c^2\right)(v+c)}{v\left[\frac{(b+c)+(c+a)}{2}\right]^2}-2k=\frac{2v^2+c^2}{v(v+c)}-2k$

$\Rightarrow S_a+S_b\ge 2\left(\frac{2v^2+c^2}{2v(v+c)}-k\right)=2\left[u(v,c)-k\right]\ge 0$ (since $u(v,c)\ge\frac{\sqrt{3}-1}{2}=k$ )

As $S_a \le S_b \le S_c$ and $S_a + S_b \ge 0$, so $S_b \ge 0; S_b + S_c \ge 0; S_c + S_a \ge 0$

From ***Proposition 2*** it follows: with $k = \frac{\sqrt{3}-1}{2}$ then (4) is true $\forall a, b, c > 0$.

***Conclusion:*** $k = \frac{\sqrt{3}-1}{2}$ is the maximum coefficient to make the inequality (1) is true $\forall a, b, c > 0$.

**Problem 5. (Pham Kim Hung)** Given $a,b,c \ge 0$. Determine the best constant for the following inequality: $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + k\frac{ab+bc+ca}{a^2+b^2+c^2} \ge 9+k$ (1)

***Solution***

We use the following change:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 9 = \frac{(b-c)^2}{bc} + \frac{(c-a)^2}{ca} + \frac{(a-b)^2}{ab}$$

$$1 - \frac{ab+bc+ca}{a^2+b^2+c^2} = \frac{(b-c)^2+(c-a)^2+(a-b)^2}{2(a^2+b^2+c^2)}$$

Then the inequality (1) becomes: $\sum_{cyc}\left(\frac{1}{bc} - \frac{k}{2(a^2+b^2+c^2)}\right)(b-c)^2 \ge 0 \Leftrightarrow \sum_{cyc} S_a (b-c)^2 \ge 0$ (2)

In which: $S_a = 2a(a^2+b^2+c^2) - kabc, S_b = 2b(a^2+b^2+c^2) - kabc, S_c = 2c(a^2+b^2+c^2) - kabc$

Now, it is given that $b = c$, then (2) becomes:

$$S_b \ge 0 \Leftrightarrow 2(a^2+2b^2) - kab \ge 0 \Leftrightarrow 2(a-\sqrt{2}b)^2 + (4\sqrt{2}-k)ab \ge 0$$

Taking $a = \sqrt{2}b$ then $k \le 4\sqrt{2}$. We shall prove the inequality (2) with $k = 4\sqrt{2}$.

Supposing $a \ge b \ge c \Rightarrow S_a \ge S_b \ge S_c$. On the other hand:

$$S_b + S_c = 2(b+c)(a^2+b^2+c^2) - 8\sqrt{2}abc \ge 4\sqrt{bc}.(a^2+2bc) - 8\sqrt{2}abc = 4\sqrt{bc}(a-\sqrt{2bc})^2 \ge 0$$

Using the proposition 2, we got the q.e.d.

***Conclusion:*** $k = 4\sqrt{2}$ is the best constant of the inequality.

**Problem 6. (Pham Kim Hung)** Given $a,b,c \ge 0$. Determine the best constant for the following inequality: $\frac{bc}{b^2+c^2+ka^2} + \frac{ca}{c^2+a^2+kb^2} + \frac{ab}{b^2+a^2+kc^2} \le \frac{3}{5}$ (1)

***Solution***

Given that $a = 3, b = 2, c = 2$, it follows: $k \le 3$. We shall prove the inequality when $k = 3$.

Indeed, WLOG, supposing $a \ge b \ge c$ and $a^2+b^2+c^2 = 1$. We use the following change:

$$2(b^2+c^2+3a^2) - 10bc = 5(b-c)^2 + 3(a^2-b^2+a^2-c^2)$$

Thus the inequality (1) $\Leftrightarrow \sum_{cyc}\frac{5(b-c)^2+3(a^2-b^2+a^2-c^2)}{1+2a^2}\geq 0 \Leftrightarrow \sum_{cyc}S_a(b-c)^2\geq 0$

In which: $S_a=5(1+2b^2)(1+2c^2)-6(b+c)^2(1+2a^2)$ ;

$S_b=5(1+2c^2)(1+2a^2)-6(c+a)^2(1+2b^2)$ ; $S_c=5(1+2a^2)(1+2b^2)-6(a+b)^2(1+2c^2)$

Let $x=\sqrt{\frac{a^2+b^2}{2}}$. Obviously, $S_c\geq 0$ and $(a+c)^2(1+2b^2)+(b+c)^2(1+2a^2)\leq 2(x+c)^2(1+2x^2)$

We shall prove $S_a+S_b\geq 0$. Indeed:

$S_a+S_b\geq 10(1+2c^2)(1+2x^2)-6(x+c)^2(1+2x^2)\geq 0 \Leftrightarrow (2x-3c)^2\geq 0$ (true)

Next we shall prove $S_b\geq 0$. Indeed: $S_b=5(1+2c^2)(1+2a^2)-6(c+a)^2(1+2b^2)\geq 0$

$$\Leftrightarrow \frac{3a^2+b^2+c^2}{3b^2+c^2+a^2}\geq \frac{6(a+c)^2}{5(3c^2+a^2+b^2)} \Leftrightarrow \frac{2(a^2-b^2)}{3b^2+c^2+a^2}\geq \frac{a^2+12ac-5b^2-9c^2}{5(3c^2+a^2+b^2)}$$

This is obviously true as: $2(a^2-b^2)\geq a^2+12ac-5b^2-9c^2$, $3b^2+c^2+a^2\leq 5(3c^2+a^2+b^2)$

Using the proposition 2, we got the q.e.d.

***Conclusion:*** $k=3$ is the best constant of the inequality.

**Problem 7.** Given $\begin{cases} a,b,c>0 \\ ab+bc+ca=1 \end{cases}$. Prove that $\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2}\geq \frac{5}{2}$ (1)

***Proof***

The inequality (1) $\Leftrightarrow \sum_{cyc}\frac{(ab+bc+ca)^2+a^2b^2}{(a+b)^2}\geq \frac{5}{2}(ab+bc+ca)$

$$\Leftrightarrow 2\sum_{cyc}\frac{2ab(ab+bc+ca)+(bc+ca)^2}{(a+b)^2}\geq 5(ab+bc+ca)$$

$$\Leftrightarrow 4(ab+bc+ca)\left[\frac{ab}{(a+b)^2}+\frac{bc}{(b+c)^2}+\frac{ca}{(c+a)^2}\right]+2(a^2+b^2+c^2)\geq 5(ab+bc+ca)$$

$$\Leftrightarrow (ab+bc+ca)\left[\frac{4ab}{(a+b)^2}+\frac{4bc}{(b+c)^2}+\frac{4ca}{(c+a)^2}-3\right]+2(a^2+b^2+c^2-ab-bc-ca)\geq 0$$

$$\Leftrightarrow -(ab+bc+ca)\left[\frac{(a-b)^2}{(a+b)^2}+\frac{(b-c)^2}{(b+c)^2}+\frac{(c-a)^2}{(c+a)^2}\right]+(a-b)^2+(b-c)^2+(c-a)^2\geq 0$$

$$\Leftrightarrow \left[1-\frac{ab+bc+ca}{(a+b)^2}\right](a-b)^2+\left[1-\frac{ab+bc+ca}{(b+c)^2}\right](b-c)^2+\left[1-\frac{ab+bc+ca}{(c+a)^2}\right](c-a)^2\geq 0$$

Letting $S_a=1-\frac{ab+bc+ca}{(b+c)^2}; S_b=1-\frac{ab+bc+ca}{(c+a)^2}; S_c=1-\frac{ab+bc+ca}{(a+b)^2}$

WLOG supposing $a \le b \le c \Rightarrow S_a \ge S_b \ge S_c$

We have: $S_b = 1 - \dfrac{ab+bc+ca}{(c+a)^2} = \dfrac{a^2+(a+c)(c-b)}{(c+a)^2} > 0 \Rightarrow S_a \ge S_b > 0$

On the other hand, $b^2A_3 + c^2A_2 = b^2\left[1 - \dfrac{ab+bc+ca}{(a+b)^2}\right] + c^2\left[1 - \dfrac{ab+bc+ca}{(c+a)^2}\right] =$

$$= b^2 \cdot \frac{a^2+(a+b)(b-c)}{(a+b)^2} + c^2 \cdot \frac{a^2+(a+c)(c-b)}{(c+a)^2} = a^2\left[\frac{b^2}{(a+b)^2} + \frac{c^2}{(c+a)^2}\right] + (c-b)\left(\frac{c^2}{a+c} - \frac{b^2}{a+b}\right)$$

$$= a^2\left[\frac{b^2}{(a+b)^2} + \frac{c^2}{(c+a)^2}\right] + (c-b)^2 \cdot \frac{ab+bc+ca}{(a+c)(a+b)} > 0$$

Using ***Proposition 4*** we have (q.e.d). Equality occurs $\Leftrightarrow a = b = c = \dfrac{1}{\sqrt{3}}$

**Problem 8.** Letting that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ be length of sides of a triangle. Prove that

$$3\,\mathrm{Min}\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}, \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad (1)$$

***Proof***

WLOG supposing $\mathrm{Min}\left\{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}, \dfrac{b}{a}+\dfrac{c}{b}+\dfrac{a}{c}\right\} = \dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$, then we have:

$$(1) \Leftrightarrow 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \Leftrightarrow 3\left(a^2c+b^2a+c^2b\right) \ge (a+b+c)(ab+bc+ca)$$

$$\Leftrightarrow 2\left(a^2c+b^2a+c^2b\right) \ge \left(a^2b+b^2c+c^2a\right) + 3abc$$

$$\Leftrightarrow 2\left[\left(a^3+b^3+c^3\right) - \left(a^2c+b^2a+c^2b\right)\right] \le \left(a^3+b^3+c^3\right) - \left(a^2b+b^2c+c^2a\right) + \left(a^3+b^3+c^3\right) - 3abc \quad (2)$$

We have: $+\begin{cases} 2b^3+a^3-3b^2a = (b-a)\left(2b^2-ba-a^2\right) = (b-a)^2(2b+a) \\ 2c^3+b^3-3c^2b = (c-b)\left(2c^2-cb-b^2\right) = (c-b)^2(2c+b) \\ 2a^3+c^3-3a^2c = (a-c)\left(2a^2-ac-c^2\right) = (a-c)^2(2a+c) \end{cases}$

$$\Rightarrow 3\left[\left(a^3+b^3+c^3\right) - \left(a^2c+b^2a+c^2b\right)\right] = (a-b)^2(2b+a) + (b-c)^2(2c+b) + (c-a)^2(2a+c)$$

Similarly, $3\left[\left(a^3+b^3+c^3\right) - \left(a^2b+b^2c+c^2a\right)\right] = (a-b)^2(2a+b) + (b-c)^2(2b+c) + (c-a)^2(2c+a)$

On the other hand, $a^3+b^3+c^3-3abc = \dfrac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$

It follows: $(2) \Leftrightarrow \dfrac{2}{3}\sum_{cyc}(2b+a)(a-b)^2 \le \dfrac{1}{3}\sum_{cyc}(2a+b)(a-b)^2 + \dfrac{1}{2}\sum_{cyc}(a+b+c)(a-b)^2$

$$\Leftrightarrow \sum_{cyc}(a-b)^2\left[\frac{1}{3}(2a+b) + \frac{1}{2}(a+b+c) - \frac{2}{3}(2b+a)\right] \ge 0 \Leftrightarrow \frac{1}{2}\sum_{cyc}(c+a-b)(a-b)^2 \ge 0$$

$$\Leftrightarrow (c+a-b)(a-b)^2 + (a+b-c)(b-c)^2 + (b+c-a)(c-a)^2 \ge 0 \quad (3)$$

Letting $S_a = a+b-c\,;\, S_b = b+c-a\,;\, S_c = c+a-b$

$\Rightarrow\ S_a + S_b = 2b > 0\,;\, S_b + S_c = 2c > 0\,;\, S_c + S_a = 2a > 0$. Beside, we have:

$$S_aS_b + S_bS_c + S_cS_a = (a+b-c)(b+c-a)+(b+c-a)(c+a-b)+(c+a-b)(a+b-c)$$

$$= a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2 = 2(ab+bc+ca)-\left(a^2+b^2+c^2\right)$$

$$= 4ab-(a+b-c)^2 = \left(2\sqrt{ab}+a+b-c\right)\left(2\sqrt{ab}-a-b+c\right)$$

$$= \left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{b}+\sqrt{c}-\sqrt{a}\right)\left(\sqrt{c}+\sqrt{a}-\sqrt{b}\right)$$

Since $\sqrt{a},\sqrt{b},\sqrt{c}$ are length of sides of a triangle, it follows $S_aS_b + S_bS_c + S_cS_a > 0$.

Using ***Proposition 5*** we have (3) is true $\Rightarrow$ (2) is true $\Rightarrow$ (1) is true.

**Problem 9. (Vasile Cirtoaje)** Letting that $a$, $b$, $c$ are the length of 3 sides of a triangle.

Prove that $3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 2\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)+3 \quad (1)$

***Proof***

The inequality (1) $\Leftrightarrow\ 3\left(a^2c+b^2a+c^2b\right) \ge 2\left(b^2c+c^2a+a^2b\right)+3abc$

$$\Leftrightarrow\ 3\left[\left(a^3+b^3+c^3\right)-\left(a^2c+b^2a+c^2b\right)\right] \le 2\left[\left(a^3+b^3+c^3\right)-\left(a^2b+b^2c+c^2a\right)\right]+\left(a^3+b^3+c^3\right)-3abc$$

$$\Leftrightarrow\ \sum_{cyc}(2b+a)(a-b)^2 \le \frac{2}{3}\sum_{cyc}(2a+b)(a-b)^2+\frac{1}{2}\sum_{cyc}(a+b+c)(a-b)^2$$

$$\Leftrightarrow\ (a-b)^2(5a-5b+3c)+(b-c)^2(5b-5c+3a)+(c-a)^2(5c-5a+3b) \ge 0$$

Letting $S_a = 5b-5c+3a\,;\, S_b = 5c-5a+3b\,;\, S_c = 5a-5b+3c$

WLOG suppose $M\text{in}\{a,b,c\} \le b \le Max\{a,b,c\}$. Consider 2 following possibilities:

- $c \ge b \ge a$: We have $S_b = 5c-5a+3b > 0; S_a + S_b = 8b-2a \ge 0; S_b + S_c = 8c-2b \ge 0$

Using ***Proposition 2*** we have q.e.d.

- $a \ge b \ge c$: We have $S_a + S_c + 4S_b = 18c+12b-12a > 12(c+b-a) > 0$

+ If $|a-b| \ge |c-b|$ then $c > a-b \ge b-c \ge 0 \Rightarrow 2c > b$

$\Rightarrow\ S_b + S_c = 8c-2b > 4b-2b = 2b > 0\ \Rightarrow\ S_c + 2S_b > S_b + 2b = 5(c-a+b) > 0$

+ If $|c-b| \ge |a-b|$ then $b-c \ge a-b \ge 0 \Rightarrow 2b \ge a+c > a$

$\Rightarrow\ S_a + S_b = 8b-2a > 4a-2a = 2a > 0\ \Rightarrow\ S_a + 2S_b > S_b + 2a = 5c-3a+3b > 3(c-a+b) > 0$

Using ***Proposition 6*** with $\alpha = 1$ we have q.e.d.

**Problem 10.** Given $x,y,z > 0$. Prove that

$$x^2+y^2+z^2+xy+yz+zx \ge \sqrt{2}\left(x\sqrt{y^2+z^2}+y\sqrt{z^2+x^2}+z\sqrt{x^2+y^2}\right) \quad (1)$$

***Proof***

(1) $\Leftrightarrow\ 2(x^2+y^2+z^2)-2(xy+yz+zx) \ge 2\sqrt{2}\left(x\sqrt{y^2+z^2}+y\sqrt{z^2+x^2}+z\sqrt{x^2+y^2}\right)-4(xy+yz+zx)$

$$\Leftrightarrow \sum_{cyc}(x-y)^2 \ge 2\sum_{cyc}\left(x\sqrt{2(y^2+z^2)}-x(y+z)\right)$$

$$\Leftrightarrow \sum_{cyc}(x-y)^2 \ge 2\sum_{cyc}\frac{x(y-z)^2}{\sqrt{2(y^2+z^2)}+y+z} \Leftrightarrow \sum_{cyc}(x-y)^2 \ge 2\sum_{cyc}\frac{z(x-y)^2}{\sqrt{2(x^2+y^2)}+x+y}$$

Using Cauchy-Schwarz inequality, we have: $2\sum_{cyc}\frac{z(x-y)^2}{\sqrt{2(x^2+y^2)}+x+y} \le \sum_{cyc}\frac{z(x-y)^2}{x+y}$

So it is suffices to prove that: $\sum_{cyc}(x-y)^2 \ge \sum_{cyc}\frac{z(x-y)^2}{x+y} \quad \Leftrightarrow \sum_{cyc}\left(1-\frac{z}{x+y}\right)(x-y)^2 \ge 0$ (2)

Letting $S_x = 1-\frac{x}{y+z}, S_y = 1-\frac{y}{z+x}, S_z = 1-\frac{z}{x+y}$, then (2) $\Leftrightarrow$ $S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2 \ge 0$

WLOG supposing $x \ge y \ge z > 0$, then $S_y, S_z > 0$

We have $x^2S_y + y^2S_x = x^2+y^2-\frac{x^2y}{x+z}-\frac{xy^2}{y+z} \ge x^2+y^2-2xy \ge 0$

Using ***Proposition 4*** we have q.e.d. Equality occurs $\Leftrightarrow$ $x=y=z$.

**Problem 11. (Komal)** Letting that $a,b,c>0$ are satisfying $abc=1$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{a+b+c} \ge \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\frac{2}{a^2+b^2+c^2} \quad (1)$$

***Proof***

The inequality (1) $\Leftrightarrow ab+bc+ca-\frac{3abc}{a+b+c} \ge \frac{2(a^2b^2+b^2c^2+c^2a^2)}{a^2+b^2+c^2}$

$$\Leftrightarrow ab+bc+ca-\frac{9abc}{a+b+c} \ge \frac{2(a^2b^2+b^2c^2+c^2a^2)}{a^2+b^2+c^2}-\frac{6abc}{a+b+c}$$

$$\Leftrightarrow \sum_{cyc}\frac{c(a-b)^2}{a+b+c} \ge \sum_{cyc}\frac{c(a-b)^2(c^2+bc+ca-2ab)}{(a^2+b^2+c^2)(a+b+c)} \Leftrightarrow \sum_{cyc}c(a-b)^2(a^2+b^2+2ab-bc-ca) \ge 0 \quad (2)$$

Letting $S_a = a(b^2+c^2+2bc-ca-ab)$ ; $S_b = b(c^2+a^2+2ca-ab-bc)$ ;

$S_c = c(a^2+b^2+2ab-bc-ca)$. Then (2) $\Leftrightarrow$ $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \ge 0$

WLOG supposing $a \ge b \ge c$, it easily see that $S_b, S_c \ge 0$.

We have $a^2S_b+b^2S_a = ab\left[(a-b)^2(a+b)+2c(a^2+b^2-ab)+c(a^2+b^2)\right] > 0$

Using ***Proposition 4*** we have q.e.d.

**Problem 12. (Vo Quoc Ba Can)**

Given $x,y,z$ satisfy $xy+yz+zx=1$. Prove that $\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}-2(x^2+y^2+z^2) \ge \sqrt{3}-2$ (1)

***Proof***

The inequality (1) $\Leftrightarrow \left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}-x-y-z\right)+x+y+z-\sqrt{3}\geq 2(x^2+y^2+z^2-1)$

$$\Leftrightarrow \sum_{cyc}\frac{(x-y)^2}{y}+\frac{1}{2}\sum_{cyc}\frac{(x-y)^2}{x+y+z+\sqrt{3}}\geq \sum_{cyc}(x-y)^2 \Leftrightarrow \sum_{cyc}(x-y)^2\left[\frac{1}{y}+\frac{1}{2\left(x+y+z+\sqrt{3}\right)}-1\right]\geq 0 \quad (2)$$

Letting $S_x=\frac{1}{z}+\frac{1}{2\left(x+y+z+\sqrt{3}\right)}-1$ ; $S_y=\frac{1}{x}+\frac{1}{2\left(x+y+z+\sqrt{3}\right)}-1$; $S_z=\frac{1}{y}+\frac{1}{2\left(x+y+z+\sqrt{3}\right)}-1$

Now, the inequality (2) $\Leftrightarrow S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2\geq 0$

We have $S_x+S_y+S_z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-3+\frac{3}{2\left(x+y+z+\sqrt{3}\right)}$

$$=\frac{xy+yz+zx}{xyz}-3+\frac{3}{2\left(x+y+z+\sqrt{3}\right)}=\frac{1}{xyz}-3+\frac{3}{2\left(x+y+z+\sqrt{3}\right)}$$

$$\geq \frac{3\sqrt{3}}{(xy+yz+zx)^{\frac{3}{2}}}-3+\frac{3}{2\left(x+y+z+\sqrt{3}\right)}=3\sqrt{3}-3+\frac{3}{2\left(x+y+z+\sqrt{3}\right)}>0$$

Letting $t=\frac{1}{2\left(x+y+z+\sqrt{3}\right)}$, we have:

$$S_xS_y+S_yS_z+S_zS_x=\left(t+\frac{1}{x}-1\right)\left(t+\frac{1}{y}-1\right)+\left(t+\frac{1}{y}-1\right)\left(t+\frac{1}{z}-1\right)+\left(t+\frac{1}{z}-1\right)\left(t+\frac{1}{x}-1\right)$$

$$=3t^2+2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-3\right)t+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}-2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+3$$

$$>\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}-2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+3=\frac{x+y+z+3xyz-2}{xyz}$$

We will prove $\frac{x+y+z+3xyz-2}{xyz}\geq 0\Leftrightarrow x+y+z+3xyz-2\geq 0$ (3)

If $x+y+z\geq 2$ then (3) is obviously true.

If $x+y+z\leq 2$, letting $p=x+y+z \Rightarrow 2\geq p\geq\sqrt{3}$.

Now using Schur inequality, we have: $xyz\geq\frac{4p-p^3}{9}$. Hence:

$$p+3xyz-2\geq p-2+\frac{4p-p^3}{3}=\frac{-p^3+7p-6}{3}=\frac{(2-p)(p-1)(p+3)}{3}\geq 0 \Rightarrow \text{(3) is true}$$

Thus we have $\begin{cases} S_x+S_y+S_z>0 \\ S_xS_y+S_yS_z+S_zS_x>0\end{cases}$, so $S_x(y-z)^2+S_y(z-x)^2+S_z(x-y)^2\geq 0 \Rightarrow$ (q.e.d)

Equality occurs $\Leftrightarrow x=y=z=\frac{1}{\sqrt{3}}$

**Problem 13. (Nguyen Van Thach)**

Given $a,b,c>0$. Prove that $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3\cdot\sqrt{\frac{a^4+b^4+c^4}{a^2+b^2+c^2}}$ (1)

***Proof***

The inequality (1) $\Leftrightarrow \sum_{cyc}\frac{a^4}{b^2}+2\sum_{cyc}\frac{a^2b}{c}\ge\frac{9(a^4+b^4+c^4)}{a^2+b^2+c^2}$

$$\Leftrightarrow \sum_{cyc}\left(\frac{a^4}{b^2}+b^2-2a^2\right)+2\sum_{cyc}\left(\frac{a^2b}{c}+bc-2ab\right)\ge\left(\frac{9(a^4+b^4+c^4)}{a^2+b^2+c^2}-3\sum_{cyc}a^2\right)+\left(2\sum_{cyc}a^2-2\sum_{cyc}ab\right)$$

$$\Leftrightarrow \sum_{cyc}(a-b)^2\left(\frac{a}{b}+1\right)^2+2\sum_{cyc}(c-a)^2\frac{b}{c}\ge\frac{3\sum_{cyc}(a-b)^2(a+b)^2}{a^2+b^2+c^2}+\sum_{cyc}(a-b)^2\Leftrightarrow\sum_{cyc}S_c(a-b)^2\ge 0$$

in which $S_a=\frac{b^2}{c^2}+\frac{2b}{c}+\frac{2a}{b}-\frac{3(b+c)^2}{a^2+b^2+c^2}$ ; $S_b=\frac{c^2}{a^2}+\frac{2c}{a}+\frac{2b}{c}-\frac{3(c+a)^2}{a^2+b^2+c^2}$

$$S_c=\frac{a^2}{b^2}+\frac{2a}{b}+\frac{2c}{a}-\frac{3(a+b)^2}{a^2+b^2+c^2}$$

Obviously, we just need to consider inequality in case $a\ge b\ge c>0$

*** Case 1.** $b-c\ge a-b\Leftrightarrow 2(b-c)\ge a-c\Leftrightarrow 2b\ge a+c$. We have:

$$\begin{aligned}S_a+S_c&=\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{4a}{b}+\frac{2b}{c}+\frac{2c}{a}-\frac{3(a+b)^2+3(b+c)^2}{a^2+b^2+c^2}\\&\ge\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{4a}{b}+4\sqrt{\frac{b}{a}}-\frac{3(a+b)^2+3(b+c)^2}{a^2+b^2+c^2}\\&\ge 10-\frac{3(a+b)^2+3(b+c)^2}{a^2+b^2+c^2}=\frac{7a^2+4b^2+7c^2-6ab-6bc}{a^2+b^2+c^2}\ge 0\end{aligned}$$

$$\begin{aligned}S_a+S_b&=\frac{b^2}{c^2}+\frac{2b}{c}+\frac{2a}{b}+\frac{c^2}{a^2}+\frac{2c}{a}+\frac{2b}{c}-\frac{3(a+c)^2+3(b+c)^2}{a^2+b^2+c^2}\\&\ge 9-\frac{3(a+c)^2+3(b+c)^2}{a^2+b^2+c^2}=\frac{3\left[(2a-b)^2+(b-2c)^2\right]}{2(a^2+b^2+c^2)}\ge 0\end{aligned}$$

$$\begin{aligned}S_a+4S_b&=\frac{4c^2}{a^2}+\frac{8c}{a}+\frac{10b}{c}+\frac{2a}{b}+\frac{b^2}{c^2}-\frac{3(b+c)^2+12(a+c)^2}{a^2+b^2+c^2}\\&\ge\frac{4c}{a}-1+\frac{8c}{a}+\frac{5a}{c}+5+\frac{b^2}{c^2}+\frac{2a}{b}-\frac{3(b+c)^2+12(a+c)^2}{a^2+b^2+c^2}\\&\ge 22-\frac{3(b+c)^2+12(a+c)^2}{a^2+b^2+c^2}=\frac{10a^2+19b^2+7c^2-24ac-6bc}{a^2+b^2+c^2}\ge 0\end{aligned}$$

$$S_a+4S_b+S_c=\frac{4c^2}{a^2}+\frac{10c}{a}+\frac{10b}{c}+\frac{4a}{b}+\frac{b^2}{c^2}+\frac{a^2}{b^2}-\frac{3(b+c)^2+12(a+c)^2+3(a+b)^2}{a^2+b^2+c^2}$$

$$\geq 26-\frac{3(b+c)^2+12(a+c)^2+3(a+b)^2}{a^2+b^2+c^2}$$

$$=\frac{11a^2+20b^2+11c^2-24ab}{a^2+b^2+c^2}\geq 0$$

If $S_b\geq 0$ then we have $\sum_{cyc}S_c(a-b)^2\geq (S_a+S_c)(a-b)^2\geq 0$

If $S_b\leq 0, S_c\geq 0$ then $\sum_{cyc}S_c(a-b)^2\geq (S_a+4S_b)(b-c)^2\geq 0$

If $S_b, S_c\leq 0$ then $\sum_{cyc}S_c(a-b)^2\geq (S_a+4S_b+S_c)(b-c)^2\geq 0$

* **Case 2.** $a-b\geq b-c$. We will prove $S_c\geq 0$

Indeed, consider the function $f(c)=S_c=\frac{a^2}{b^2}+\frac{2a}{b}+\frac{2c}{a}-\frac{3(a+b)^2}{a^2+b^2+c^2}$

Clearly, this function is increasing with $c$, so

+ If $2b\leq a$, we have: $f(c)\geq f(0)=\frac{a^2}{b^2}+\frac{2a}{b}-\frac{3(a+b)^2}{a^2+b^2}\geq 8-\frac{3(a+b)^2}{a^2+b^2}=\frac{5a^2+5b^2-6ab}{a^2+b^2}\geq 0$

+ If $2b\geq a$, we have: $f(c)\geq f(2b-a)=\frac{a^2}{b^2}+\frac{2a}{b}+\frac{4b}{a}-2-\frac{3(a+b)^2}{2a^2+5b^2-4ab}\geq 0$

Thus $S_a, S_c\geq 0$. So if $S_b\geq 0$ then we right have q.e.d.

Consider case $S_b\leq 0$, we have:

$$S_a+2S_b=\frac{2c^2}{a^2}+\frac{4c}{a}+\frac{6b}{c}+\frac{b^2}{c^2}+\frac{2a}{b}-\frac{6(a+c)^2+3(b+c)^2}{a^2+b^2+c^2}$$

$$\geq\frac{8c}{a}-2+\frac{8b}{c}-1+\frac{2a}{b}-\frac{6(a+c)^2+3(b+c)^2}{a^2+b^2+c^2}\geq 12-\frac{6(a+c)^2+3(b+c)^2}{a^2+b^2+c^2}\geq 0$$

$$S_c+2S_b=\frac{2c^2}{a^2}+\frac{6c}{a}+\frac{4b}{c}+\frac{a^2}{b^2}+\frac{2a}{b}-\frac{6(a+c)^2+3(a+b)^2}{a^2+b^2+c^2}$$

$$\geq\frac{\left(2\sqrt{2}+6\right)c}{a}-1+\frac{4b}{c}+\frac{4a}{b}-1-\frac{6(a+c)^2+3(a+b)^2}{a^2+b^2+c^2}\geq 13.6-\frac{6(a+c)^2+3(a+b)^2}{a^2+b^2+c^2}\geq 0$$

$$\Rightarrow \sum_{cyc}S_c(a-b)^2\geq (S_a+2S_b)(b-c)^2+(S_c+2S_b)(a-b)^2\geq 0$$

The inequality is proven. Equality occurs $\Leftrightarrow a=b=c$.

**Problem 14. (Nguyen Van Thach)**

Prove that $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 2\left(\frac{a^2-ab+b^2}{a+b}+\frac{b^2-bc+c^2}{b+c}+\frac{c^2-ca+a^2}{c+a}\right)$ , $\forall a,b,c>0$ (1)

***Proof***

The inequality (1) $\Leftrightarrow \sum_{cyc}\frac{a^2}{b}-(a+b+c)\ge 2\left(\sum_{cyc}\frac{a^2-ab+b^2}{a+b}-\frac{a+b+c}{2}\right)$

$$\Leftrightarrow \sum_{cyc}\frac{(a-b)^2}{b}\ge \frac{3}{2}\sum_{cyc}\frac{(a-b)^2}{a+b}\Leftrightarrow \sum_{cyc}(a-b)^2\left(\frac{1}{b}-\frac{3}{2(a+b)}\right)\ge 0$$

Letting $S_a=\frac{1}{c}-\frac{3}{2(b+c)}$ ; $S_b=\frac{1}{a}-\frac{3}{2(c+a)}$ ; $S_c=\frac{1}{b}-\frac{3}{2(a+b)}$

WLOG we just need to consider the case $a\ge b\ge c$ . Then we have $S_a,S_c\ge 0$.

So if $S_b\ge 0$ then we right have q.e.d.

Consider $S_b\le 0$, we have: $S_a+2S_b=\frac{2}{a}-\frac{3}{a+c}+\frac{1}{c}-\frac{3}{2(b+c)}\ge\frac{3}{a+c}-\frac{3}{a+c}+\frac{3}{4c}-\frac{3}{2(b+c)}\ge 0$

$$S_c+2S_b=\frac{2}{a}+\frac{1}{b}-\frac{3}{a+c}-\frac{3}{2(a+b)}$$

**+ Case 1.** $b-c\ge a-b\Leftrightarrow 2(b-c)\ge a-c\Leftrightarrow 2b\ge a+c$

We have $S_a+4S_b=\frac{4}{a}+\frac{1}{c}-\frac{6}{a+c}-\frac{3}{2(b+c)}\ge\frac{2}{3}\left(\frac{4}{a}+\frac{1}{c}\right)+\frac{4}{3a}+\frac{1}{3c}-\frac{6}{a+c}-\frac{3}{a+3c}$

$\ge\frac{2\cdot 9}{3(a+c)}+\frac{\left(\frac{2}{\sqrt{3}}+1\right)^2}{a+3c}-\frac{6}{a+c}-\frac{3}{a+3c}\ge 0$. Then $\sum_{cyc}S_c(a-b)^2\ge(S_a+4S_b)(b-c)^2\ge 0$

*** Case 2.** $b-c\le a-b\Leftrightarrow a+c\ge 2b$ . Letting $f(c)=S_c+2S_b$ . Clearly, $f$(c) is increasing, so

+ If $2b\le a$ then $f(c)\ge f(0)=\frac{2}{a}-\frac{3}{a}+\frac{1}{b}-\frac{3}{2(a+b)}=\frac{a-b}{ab}-\frac{3}{2(a+b)}=\frac{2a^2-3ab-2b^2}{2ab(a+b)}\ge 0$

+ If $2b\ge a$ then $f(c)\ge f(2b-a)=\frac{2}{a}-\frac{3}{2b}+\frac{1}{b}-\frac{3}{2(a+b)}\ge\frac{2}{a}-\frac{1}{a}-\frac{3}{2a+a}=0$

Thus $f(c)\ge 0$, so we have: $\sum_{cyc}S_c(a-b)^2\ge(S_a+2S_b)(b-c)^2+(S_c+2S_b)(a-b)^2\ge 0$

The inequality is proven. Equality occurs $\Leftrightarrow a=b=c$.

***Comment:*** We can see the use of S.O.S principle in solving three variables inequality, but what about multi-variable ones? In fact, S.O.S proves to have little help, but we still can apply it in many cases, and it is usually accompanied by the induction principle and mixing variables technique. Let's consider some examples to clarify this problem.

**Problem 15 (Vasile Cirtoaje)**

Giving $n$ positive real numbers $a_1, a_2, ..., a_n$ satisfying $a_1 + a_2 + ... + a_n = n$. Prove that:

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right) \geq 4(n-1)(a_1{}^2+a_2{}^2+...+a_n{}^2)+n(n-2)^2 \quad (1)$$

***Proof***

We will prove the inequality by induction. And here we just consider the most complex part, that is proving the inequality with $n+1$ numbers when we already knew that it is true with $n$ or less than $n$ numbers.

Firstly, notice that the inequality is equivalent to:

$$\left[n^2(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n})-n(n-2)^2\right](a_1+...+a_n)^2 \geq 4(n-1)n^2(a_1{}^2+...+a_n{}^2)$$

So if the inequality is true for $a_1+...+a_n = n$ then it is also true for $a_1+...+a_n \leq n$.

Letting $f\left(a_1, a_2, ...a_n\right) = LHS - RHS$.

WLOG supposing $a_1 = \max\left\{a_1, a_2, ..., a_n\right\}$. We have:

$$f(a_1,...,a_n) - f\left(a_1, \frac{a_2+...+a_n}{n-1}, ..., \frac{a_2+...+a_n}{n-1}\right)$$

$$= n^2\left(\frac{1}{a_2}+\cdots+\frac{1}{a_n}-\frac{n^2}{a_2+...+a_n}\right)-4(n-1)\left[a_2{}^2+...+a_n{}^2-\frac{(a_2+...+a_n)^2}{n-1}\right]$$

$$= \frac{n^2}{(a_2+...+a_n)}\sum\frac{(a_i-a_j)^2}{a_ia_j}-4\sum(a_i-a_j)^2 \geq \frac{n^2}{n-1}\sum\frac{(a_i-a_j)^2}{a_ia_j}-4\sum(a_i-a_j)^2 \quad (2)$$

However, we have already had the inequality is true with $n$ variables:

$$(n-1)^2\left(\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geq 4(n-2)(a_2{}^2+...+a_n{}^2)+(n-1)(n-3)^2$$

$$\Leftrightarrow (n-1)\sum\frac{(a_i-a_j)^2}{a_ia_j} \geq \frac{4(n-2)}{n-1}\sum(a_i-a_j)^2 \Leftrightarrow \frac{(n-1)^2}{n-2}\sum\frac{(a_i-a_j)^2}{a_ia_j} \geq 4\sum(a_i-a_j)^2$$

Clearly, $\frac{n^2}{n-1} > \frac{(n-1)^2}{n-2}$, so $\frac{n^2}{n-1}\sum\frac{(a_i-a_j)^2}{a_ia_j}-4\sum(a_i-a_j)^2 \geq 0$

Therefore, we only have to prove that $f(a,d,...,d) \geq 0, a+(n-1)d = n$. This is not so difficult and you can easily find out the answers. The application technique is also the performance of S.O.S.

**Problem 16 (Vasile Cirtoaje)**

Given $n$ positive real numbers $a_1, a_2, ..., a_n$. Prove that

$$a_1{}^n + ... + a_n{}^n + n(n-1)a_1...a_n \geq a_1...a_n(a_1 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right)$$

***Proof***

The proof below uses Suranyi inequality:

$$(n-1)(a_1^n + ... + a_n^n) - \sum a_1\left(a_2^{n-1} + ... + a_n^{n-1}\right) \geq a_1^n + ... + a_n^n - na_1...a_n$$

We will again prove the inequality by induction. And here we just consider the most complex part, that is proving the inequality with $n+1$ numbers when we already knew that it is true with $n$ or less than $n$ numbers.

The inequality $\Leftrightarrow$ $\left(a_1^n + ... + a_n^n - na_1a_2...a_n\right) \geq a_1...a_n\left[(a_1 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) - n^2\right]$

$$\Leftrightarrow (a_1^n + ... + a_n^n - na_1a_2...a_n) \geq a_i..a_j\left[\frac{(a_i - a_j)^2}{a_ia_j}\right]$$

Now, we will transform the expression $a_1^n + ... + a_n^n - na_1...a_n$ into the form of S.O.S. This can be easily carried out via intermediary expression $\sum a_1\left(a_2^{n-1} + ... + a_n^{n-1}\right)$. We have:

$$(n-1)(a_1^n + ... + a_n^n - na_1a_2..a_n)$$

$$= (n-1)(a_1^n + ... + a_n^n) - \sum a_1(a_2^{n-1} + ... + a_n^{n-1}) + \sum a_1\left[a_2^n + ... + a_n^n - (n-1)a_2....a_n\right]$$

$$\geq a_1{}^n + ... + a_n{}^n - na_1...a_n + (n-2)\sum a_1...a_n\frac{(a_i - a_j)^2}{a_ia_j}$$

$$\Rightarrow a_1^n + ... + a_n^n - na_1a_2...a_n \geq \sum a_1...a_n\frac{(a_i - a_j)^2}{a_ia_j} = a_1a_2...a_n\left[(a_1 + ... + a_n)\left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) - n^2\right] \text{ (q.e.d)}$$

## ABSTRACT CONCRETENESS METHOD

### I. The basis of ABC method

When using derivative to prove multi-variable inequality, there is one notable technique that is to investigate the function with one variable while the others are considered as parameters. However, if difficulties raised when we apply this technique to the symmetrical three-variable $a$, $b$, $c$, we should transform it into $f(a,b,c)$ which only contains $a+b+c$, $ab+bc+ca$ and $abc$. Then, we investigate the function with variable $abc$ and 2 parameters $a+b+c$ and $ab+bc+ca$ follow variable abc with 2 parameters $a+b+c$ and $ab+bc+ca$. This is the basis of ABC method. First of all, we will find the domain of $abc$ in the following statements:

**1. Clause 1.** Assume $m\in\left[-\infty,-\sqrt{3}\right]\cup\left[\sqrt{3},+\infty\right]$ and real numbers $a$, $b$, $c$ satisfying

$\begin{cases} ab+bc+ca=1 \\ a+b+c=m \end{cases}$. Then the domain of $abc$ is $\left[\dfrac{(6-2m^2)X_2+m}{9};\dfrac{(6-2m^2)X_1+m}{9}\right]$

**2. Clause 2.** Assume $m\in\left[\sqrt{3},+\infty\right]$ and nonnegative real numbers $a$, $b$, $c$ satisfying

$\begin{cases} ab+bc+ca=1 \\ a+b+c=m \end{cases}$ then the domain of $abc$ is $\left[\text{Max}\left\{0,\dfrac{(6-2m^2)X_2+m}{9}\right\};\dfrac{(6-2m^2)X_1+m}{9}\right]$

***Proof***

Consider the equation: $X^3-mX^2+X-abc=0$ (1)

Then two request from two clauses $\Leftrightarrow$ Find the condition of $abc$ such that

**i)** Clause 1: Equation (1) has three real roots.

**ii)** Clause 2: Equation (1) has three nonnegative roots.

Take $f(X)=X^3-mX^2+X-abc \Rightarrow f'(X)=3X^2-2mX+1$

We have: $f'(X)=0 \Leftrightarrow X_1=\dfrac{m-\sqrt{m^2-3}}{3};X_2=\dfrac{m+\sqrt{m^2-3}}{3} \Rightarrow$ Variation table

| $X$ | $-\infty$ | | $X_1$ | | $X_2$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(X)$ | | $+$ | $0$ | $-$ | $0$ | $+$ | |
| $f(X)$ | | $\nearrow$ | | $\searrow$ | | $\nearrow$ | |

i) (1) has three real roots $\Leftrightarrow \begin{cases} f(X_1)\ge 0 \\ f(X_2)\le 0 \end{cases} \Leftrightarrow \dfrac{m-(2m^2-6)X_2}{9}\le abc\le\dfrac{m-(2m^2-6)X_1}{9}$ (2)

So the domain of $abc$ is $\left[\dfrac{(6-2m^2)X_2+m}{9};\dfrac{(6-2m^2)X_1+m}{9}\right]$

**ii)** Notice that if $abc \ge 0, a+b+c \ge 0,\ ab+bc+ca \ge 0$ then $a,b,c \ge 0$.

Hence (1) has three nonnegative real numbers $\Leftrightarrow$ *abc* must satisfy (2) and $abc \ge 0$.

$$\Leftrightarrow \text{ Max}\left\{0, \frac{m-(2m^2-6)X_2}{9}\right\} \le abc \le \frac{m-(2m^2-6)X_1}{9}$$

And the domain of $abc$ is $\left[\text{Max}\left\{0, \dfrac{(6-2m^2)X_2+m}{9}\right\}; \dfrac{(6-2m^2)X_1+m}{9}\right]$

***Note:*** • Assume that $a+b+c=m \in \left[-\infty,-\sqrt{3}\right] \text{U} \left[\sqrt{3},+\infty\right]$ are constructed from the equality $ab+bc+ca=1$ and the inequality $(a+b+c)^2 \ge 3(ab+bc+ca)=3$ or can be explained from the condition $f(x)$ has three real roots then $f'(X)=3X^2-2mX+1=0$ should have solutions, or $\Delta' = m^2-3 \ge 0 \Leftrightarrow m \in \left[-\infty,-\sqrt{3}\right] \text{U} \left[\sqrt{3},+\infty\right]$

• From clause 1 we can find out a important result is that instead of using the set $(a,b,c)$ with $a,b,c \in \mathbb{R}$ to represent every expression in $\mathbb{R}^3$ satisfying $ab+bc+ca=1$, we can use the set $(a+b+c, ab+bc+ca, abc)$ with the condition

$$a+b+c=m \in \left[-\infty,-\sqrt{3}\right] \text{U} \left[\sqrt{3},+\infty\right] \text{ and } abc \in \left[\frac{(6-2m^2)X_2+m}{9}; \frac{(6-2m^2)X_1+m}{9}\right]$$

• From clause 1 we can find out a important result is that instead of using the set $(a,b,c)$ with $a,b,c \in \mathbb{R}$ to represent every expression in $\mathbb{R}^{+3}$ satisfying $ab+bc+ca=1$, we can use the set $(a+b+c, ab+bc+ca, abc)$ with the condition

$$a+b+c=m \in \left[-\infty,-\sqrt{3}\right] \text{U} \left[\sqrt{3},+\infty\right],\ abc \in \left[\text{Max}\left\{0, \frac{(6-2m^2)X_2+m}{9}\right\}; \frac{(6-2m^2)X_1+m}{9}\right]$$

**3. Clause 3.** For every set $(a_0,b_0,c_0) \in \mathbb{R}^3$ there exist $(x_0,x_0,y_0);(z_0,z_0,t_0) \in \mathbb{R}^3$ such that

$$\begin{cases} a_0+b_0+c_0 = x_0+x_0+y_0 = z_0+z_0+t_0 \\ a_0b_0+b_0c_0+c_0a_0 = x_0x_0+x_0y_0+y_0x_0 = z_0z_0+z_0t_0+t_0z_0 \\ x_0x_0y_0 \le a_0b_0c_0 \le z_0z_0t_0 \end{cases}$$

The equality occurs if and only if $(a_0-b_0)(b_0-c_0)(c_0-a_0)=0$.

***Proof***

Firstly we will prove the clause for all real set $(a_0,b_0,c_0)$ satisfy $a_0+b_0+c_0=m$ and $a_0b_0+b_0c_0+c_0a_0=1$. According to clause 1 we have:

$$s = \frac{(6-2m^2)X_2+m}{9} \le a_0b_0c_0 \le \frac{(6-2m^2)X_1+m}{9} = S$$

We will consider when $a_0b_0c_0$ get its edge then what is the shape of $a_0,b_0,c_0$ ?

Assume that $a_0b_0c_0$ = s. Consider the equation $f(X)=X^3-mX^2+X-s=0$ (1)

We have: $f'(X)=3X^2-2mX+1$ has two solutions: $X_1=\dfrac{m-\sqrt{m^2-3}}{3};X_2=\dfrac{m+\sqrt{m^2-3}}{3}$

In the other hands, $f\left(X_1\right)=0, f'\left(X_1\right)=0$ then the equation $f(X)=0$ must have double roots $X_1$, we will call the double roots is $x_0,x_0$ and another root is $y_0$.

Using Viet Theorem we have: $\begin{cases} x_0+x_0+y_0=m=a_0+b_0+c_0 \\ x_0x_0+x_0y_0+y_0x_0=1=a_0b_0+b_0c_0+c_0a_0 \\ x_0x_0y_0=s\le a_0b_0c_0 \end{cases}$

• When $a_0b_0c_0$ = S, apply the same argument we can prove the existing of $\left(z_0,z_0,t_0\right)$

So we have proved the clause in the case that $a_0+b_0+c_0=m$ and $a_0b_0+b_0c_0+c_0a_0=1$.

Using the same ideas, the reader can have a proof for the existing in the case: $a_0+b_0+c_0=m$ and $a_0b_0+b_0c_0+c_0a_0=-1$. Now let assume that $a_0+b_0+c_0=M$ and $a_0b_0+b_0c_0+c_0a_0=\pm N$, we will prove the existing of the set $\left(x_0,x_0,y_0\right)$ and $\left(z_0,z_0,t_0\right)$. Indeed,

consider the numbers $(a_1,b_1,c_1)=\left(\dfrac{a_0}{\sqrt{|N|}},\dfrac{b_0}{\sqrt{|N|}},\dfrac{c_0}{\sqrt{|N|}}\right)$ satisfying $\begin{cases} a_1+b_1+c_1=\dfrac{M}{\sqrt{|N|}} \\ a_1b_1+b_1c_1+c_1a_1=\pm 1 \end{cases}$.

As the proof above $\left(x_1,x_1,y_1\right),\left(z_1,z_1,t_1\right)$ are exist , then we choose $\left(x_0,x_0,y_0\right)=\left(\sqrt{|N|}x_1,\sqrt{|N|}x_1,\sqrt{|N|}y_1\right)$ and $\left(z_0,z_0,t_0\right)=\left(\sqrt{|N|}z_1,\sqrt{|N|}z_1,\sqrt{|N|}t_1\right)$. Then:

$$\begin{cases} x_0+x_0+y_0=\sqrt{|N|}\left(x_1+x_1+z_1\right)=\sqrt{|N|}(z_1+z_1+t_1)=z_0+z_0+t_0=\sqrt{|N|}\dfrac{M}{\sqrt{|N|}}=M=a_0+b_0+c_0 \\ x_0x_0+x_0y_0+y_0x_0=|N|(x_1x_1+x_1y_1+y_1x_1)=z_0z_0+z_0t_0+t_0z_0=|N|\left(z_1z_1+z_1t_1+t_1z_1\right)=\pm|N|=\pm N \\ x_0x_0y_0=\left(\sqrt{|N|}\right)^3x_1x_1y_1\le\left(\sqrt{|N|}\right)^3a_1b_1c_1=a_0b_0c_0=\left(\sqrt{|N|}\right)^3a_1b_1c_1\le\left(\sqrt{|N|}\right)^3z_1z_1t_1=z_0z_0t_0 \end{cases}$$

**4. Clause 4.** For every set $\left(a_0,b_0,c_0\right)\in\mathbb{R}^{+3}$ we can find two sets $\left(x_0,x_0,y_0\right);\left(z_0,z_0,t_0\right)\in\mathbb{R}^{+3}$ or $\left(0,x_0,y_0\right);\left(z_0,z_0,t_0\right)\in\mathbb{R}^{+3}$ such that

Or $\begin{cases} a_0+b_0+c_0=x_0+x_0+y_0=z_0+z_0+t_0 \\ a_0b_0+b_0c_0+c_0a_0=x_0x_0+x_0y_0+y_0x_0=z_0z_0+z_0t_0+t_0z_0 \\ x_0x_0y_0\le a_0b_0c_0\le z_0z_0t_0 \end{cases}$

The equality occurs if and only if $\left(a_0-b_0\right)\left(b_0-c_0\right)\left(c_0-a_0\right)=0$.

Or $\begin{cases} a_0+b_0+c_0=0+x_0+y_0=z_0+z_0+t_0 \\ a_0b_0+b_0c_0+c_0a_0=x_0y_0=z_0z_0+z_0t_0+t_0z_0 \\ 0\le a_0b_0c_0\le z_0z_0t_0 \end{cases}$

The equality occurs if and only if $a_0b_0c_0=0$ .

***Comments:*** Clauses 3, 4 are not trivial and they are also the roots of ABC method. From these clauses we can deduce that every symmetric expression $f(a,b,c)$ with variables $a,b,c$ can be represented as $g(A,B,C)$ through these three quantities: $A=a+b+c$ ; $B=ab+bc+ca$ ; $C=abc$ .

Hence reduce the number of variables, we can fix $A,B$ and let $C$ free. We hope that the function g achieves global optima when $C$ reaches its boundary. However we will not need to know exactly when $C$ reach its edges, we just need to know abstractly what is the shape of $a,b,c$ ? This sounds like we can find optima of a one variable function without solve its derivative.

Clauses 3, 4 will let us know about that shape of $a$, $b$, $c$ when $C$ reach its edge.

**5. Clause 5:**

Every symmetric polynomial $f$ with variables $a,b,c$ can be represented in the form of polynomial $\varphi$ with variables $abc, ab+bc+ca, a+b+c$ and $\deg\varphi(abc)\le\frac{1}{3}\deg f$ .

***Remarks:*** The readers may understand the degrees of a polynomial with variable $abc$ such that: Assume that $f(a,b,c)=(a+b+c)abc+ab+bc+ca$ are a 4-degree polynomial with variables $a,b,c$ . However when we consider it as a polynomial with variable $abc$, we consider $a+b+c$ and $ab+bc+ca$ are constant, say $m$ and $n$ . Then our polynomial can be rewritten:

$f(a,b,c)=\varphi(abc)=m\,abc+n$ which is a linear polynomial with variable $abc$ .

***Proof***

Since every symmetric three variables polynomial $a$, $b$, $c$ can be represented as sum of basic expression $P(n),Q(m,n),R(m,n,p)$, we only need to prove the representation for $P(n),Q(m,n),R(m,n,p)$.

$$\begin{cases} P(n)=a^n+b^n+c^n \\ Q(m,n)=a^mb^n+a^nb^m+b^mc^n+c^nb^m+c^ma^n+a^mc^n & (m\ge n\ge p\ge 0) \\ R(m,n,p)=a^mb^nc^p+a^mb^pc^n+b^ma^nc^p+b^ma^pc^n+c^ma^nb^p+c^ma^pb^n \end{cases}$$

However, notice that:

- $R$ can be represented by $Q$ and $abc$ : $R=(abc)^pQ(m-p,n-p)$
- $Q$ can be represent by $P$ through the equality: $Q=P(m)P(n)-P(m+n)$

Therefore we only need to prove that $P$ can be represented by $abc, ab+bc+ca, a+b+c$.

We will prove this using induction:

For $n=0; n=1$, the clause are trivially correct.

Assume that we proved that $P(k)$ can be represented as a polynomial form with variables $abc, ab+bc+ca, a+b+c, \forall k \le h-1$. Notice that this also valid for $Q(m,n)$ and $R(m,n,p)$ $\forall m \ge n \ge p \ge 0 : m+n \le h-1$. Now we need to prove the representation for $P(h)$

We have: $P(h) = (a+b+c)P(h-1) - Q(h-1,1)$

Also: $Q(h-1,1) = (ab+bc+ca)P(h-2) - R(h-2,1,1)$.

Hence: $P(h) = (a+b+c)P(h-1) - (ab+bc+ca)P(h-2) + R(h-2,1,1)$.

On the other hand, using induction supposition, all expressions $P(h-1), P(h-2), R(h-2,1,1)$ can be represented by variables $abc, ab+bc+ca, a+b+c$.

This means that our clause also correct for $k = h$.

In conclusion, our clause has been proved completely using induction.

Property $\deg \varphi(abc) \le \frac{1}{3} \deg f$ was deduced trivially, because the quantity $abc$ has three degrees with variables a, $b$, $c$. Hence when we consider $abc$ as a first degree variable then the degree of $abc$ are not greater than $\frac{1}{3}$ the degrees of the polynomial with variables $a, b, c$.

### II. ABC THEOREM

Through two previous problems, we have enough backgrounds to take an adventure in ABC World, Abstract Concreteness.

Consider $f(abc, ab+bc+ca, a+b+c)$ as a one variable function with variable $abc$ on ¡ or ¡ $^+$

**1. First Theorem:** If the function $f(abc, ab+bc+ca, a+b+c)$ is a monotonous function then it gets the maximum and minimum values on ¡ in case $(a-b)(b-c)(c-a)=0$, and on ¡ $^+$ in case $(a-b)(b-c)(c-a)=0$ or $abc=0$.

**2. Second Theorem:** If the function $f(abc, ab+bc+ca, a+b+c)$ is a convex function then it gets the maximum values on ¡ in case $(a-b)(b-c)(c-a)=0$, and on ¡ $^+$ in case $(a-b)(b-c)(c-a)=0$ or $abc=0$.

**3. Third Theorem:** If the function $f(abc, ab+bc+ca, a+b+c)$ is a concave function then it gets the minimum values on ¡ in case $(a-b)(b-c)(c-a)=0$, and on ¡ $^+$ in case $(a-b)(b-c)(c-a)=0$ or $abc=0$.

***Remark:*** All of three theorems can be proved using the third and forth clauses .

**4. Proof for the Theorems:**

We will prove the first theorem using ***Contradiction Proof Method***. Consider $f\left(abc, ab+bc+ca, a+b+c\right)$ as a monotonous function on $\mathbb{R}$ with the only variable $abc$.

WLOG we will only prove for the case $f$ increase and get maximum value. Assume that $f$ get the maximum point at $\left(a_0, b_0, c_0\right)$ where $a_0, b_0, c_0$ pair-wise distinct and obtain $M$ as maximum value. However there exists and triple $\left(z_0, z_0, t_0\right)$ satisfying:

$$M = f\left(a_0b_0c_0, a_0b_0 + b_0c_0 + c_0a_0, a_0b_0c_0\right)$$

$$< f\left(z_0z_0t_0, a_0b_0 + b_0c_0 + c_0a_0, a_0 + b_0 + c_0\right) = f\left(z_0z_0t_0, z_0z_0 + z_0t_0 + z_0t_0, z_0 + z_0 + t_0\right)$$

This contradiction is enough for us to conclude that maximum value occur only in case $(a-b)(b-c)(c-a) = 0$. Consider $f\left(abc, ab+bc+ca, a+b+c\right)$ as a monotonous function on $\mathbb{R}^+$ with the only variable $abc$.

In case $f$ increases and we must find the maximums value then it is very similar to the above proof. In case $f$ increase and we must find the minimum value. Assume that $f$ obtain the minimum as $m$ at the point $\left(a_0, b_0, c_0\right)$ where $a_0, b_0, c_0$ pair-wise distinct and there are no 0 value. Then at least one of two below cases happens:

$$m = f\left(a_0b_0c_0, a_0b_0 + b_0c_0 + c_0a_0, a_0b_0c_0\right)$$

$$> f\left(x_0x_0y_0, a_0b_0 + b_0c_0 + c_0a_0, a_0 + b_0 + c_0\right) = f\left(x_0x_0y_0, x_0x_0 + x_0y_0 + y_0x_0, x_0 + x_0 + y_0\right)$$

$$m = f\left(a_0b_0c_0, a_0b_0 + b_0c_0 + c_0a_0, a_0b_0c_0\right)$$

$$> f\left(0, a_0b_0 + b_0c_0 + c_0a_0, a_0 + b_0 + c_0\right) = f\left(0, x_0y_0, x_0 + x_0\right)$$

This contradiction is enough for us to conclude that maximum value occur only in case $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.

• ***The proofs for the second and the third theorem are almost similar. A reminder is that convex function get maximum, concave function get minimum when variable reach the edge***

**5. Consequences:** We can have some useful consequences from three theorems above:

***5.1. First consequence:*** Assume that $f\left(a+b+c, ab+bc+ca, abc\right)$ is a linear function with variable $abc$, then $f$ get maximum, minimum on $\mathbb{R}$ if and only if $(a-b)(b-c)(c-a) = 0$, on $\mathbb{R}^+$ if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.

***5.2. Second consequence:*** Assume that $f\left(a+b+c, ab+bc+ca, abc\right)$ is a quadratic trinomial with variable $abc$, then $f$ get maximum on $\mathbb{R}$ if and only if $(a-b)(b-c)(c-a) = 0$, on $\mathbb{R}^+$ if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.

***5.3. Third consequence:*** All symmetric three variables polynomial which degrees less than or equal to 5 get maximum, minimum on $\mathbb{R}$ if and only if $(a-b)(b-c)(c-a) = 0$, on $\mathbb{R}^+$ if and only if $(a-b)(b-c)(c-a) = 0$ or $abc = 0$.

***5.4. Fourth consequence:*** All symmetric three variables polynomial which degrees less than or equal to $8$ with the nonnegative coefficient of $a^2b^2c^2$ in the representation form $f(abc, ab+bc+ca, a+b+c)$ get maximum, on ¡ if and only if $(a-b)(b-c)(c-a)=0$, on ¡ $^+$ if and only if $(a-b)(b-c)(c-a)=0$ or $abc=0$.

**_Proof_**

***5.1. First consequence:*** Linear polynomial $mx+y$ is a monotonic function. Therefore according to the first theorem, the linear function $f(a+b+c, ab+bc+ca, abc)$ ,which is also a monotonous function is a linear polynomial get maximum, minimum on ¡ if and only if $(a-b)(b-c)(c-a)=0$, on ¡ $^+$ if and only if $(a-b)(b-c)(c-a)=0$ or $abc=0$.

***5.2. 2$^{nd}$ consequence:*** Quadratic trinomial with non-negative coefficient $m^2x^2+nx+p$ is a convex function on a continuous region. Therefore according to the second theorem, the function $f(a+b+c, ab+bc+ca, abc)$ is a quadratic trinomial with variable $abc$ with the non-negative coefficient also a convex function get maximum on ¡ if and only if $(a-b)(b-c)(c-a)=0$, on ¡ $^+$ if and only if $(a-b)(b-c)(c-a)=0$ or $abc=0$.

***5.3. 3$^{rd}$ consequence:*** According to the fifth clause, symmetric polynomial with three variables $a,b,c$ whose degree are less than or equal to 5 can be represented as a polynomial $f(a+b+c, ab+bc+ca, abc)$ which is 1-degree polynomial with variable $abc$ (since $\deg\varphi(abc) \le \frac{1}{3}\deg f = \frac{5}{3}$ then $\deg\varphi(abc)=1$). Therefore according to the first consequence, the polynomial gets maximum, minimum on ¡ if and only if $(a-b)(b-c)(c-a)=0$, on ¡ $^+$ if and only if $(a-b)(b-c)(c-a)=0$ or $abc=0$.

***5.4. 4$^{th}$ consequence:*** According to the fifth clause, symmetric polynomial with three variables $a,b,c$ of which degree are less than or equal to 8 can be represented as a quadratic trinomial $f(a+b+c, ab+bc+ca, abc)$ with variable $abc$ (since $\deg\varphi(abc) \le \frac{1}{3}\deg f = \frac{8}{3}$ then $\deg\varphi(abc)=2$). Otherwise the coefficient of $a^2b^2c^2$ is nonnegative then according to the second consequence , the polynomial get maximum on ¡ if and only if $(a-b)(b-c)(c-a)=0$, on ¡ $^+$ if and only if $(a-b)(b-c)(c-a)=0$ or $abc=0$.

***6. Remarks:*** Almost inequalities can be represented as a symmetric three variables polynomial; therefore four consequences above are really useful in many cases.

***7. Some addition equalities and bonus conditions***

Take $a = x+y+z, b = xy+yz+xz, c = xyz$. Then we have:

**7.1.** $x^2+y^2+z^2 = a^2-2b$

**7.2.** $x^3+y^3+z^3 = a^3-3ab+3c$

**7.3.** $x^4+y^4+z^4=a^4-4a^2b+2b^2+4ac$

**7.4.** $x^5+y^5+z^5=a^5-5a^3b+5ab^2+5a^2c-5bc$

**7.5.** $x^6+y^6+z^6=a^6-6a^4b+6a^3c+9a^2b^2-12abc+3c^2-2b^3$

**7.6.** $x^7+y^7+z^7=a^7-7a^5b+14a^3b^2+7a^4c^2-7b^3a-21a^2bc+7cb^2+7ac^2$

**7.7.** $(xy)^2+(yz)^2+(zx)^2=b^2-2ac$

**7.8.** $(xy)^3+(yz)^3+(zx)^3=b^3-3abc+3c^2$

**7.9.** $(xy)^4+(yz)^4+(zx)^4=b^4-4b^2ac+2a^2c^2+4c^2b$

**7.10.** $(xy)^5+(yz)^5+(zx)^5=b^5-5ab^3c+5a^2bc^2+5b^2c^2-5ac^3$

**7.11.** $xy(x+y)+yz(y+z)+zx(z+x)=ab-3c$

**7.12.** $xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)=a^2b-2b^2-ac$

**7.13.** $xy(x^3+y^3)+yz(y^3+z^3)+zx(z^3+x^3)=a^3b-3ab^2-a^2c+5bc$

**7.14.** $x^2y^2(x+y)+y^2z^2(y+z)+z^2x^2(z+x)=ab^2-2a^2c-bc$

**7.15.** $x^3y^3(x+y)+y^3z^3(y+z)+z^3x^3(z+x)=ab^3-3a^2bc+5ac^2-cb^2$

**7.16.** $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)=9c^2+(a^3-6ab)c+b^3$

**7.17.** $(x^3y+y^3z+z^3x)(xy^3+yz^3+zx^3)=7a^2c^2+(a^5-5a^3b+ab^2)c+b^4$

**7.18.** $(x^3y^2+y^3z^2+z^3x^2)(x^2y^3+y^2z^3+z^2x^3)=a^4c^2+a^2bc^2+7b^2c^2+b^5-5b^3ac$

**7.19.** $(x^4y+y^4z+z^4x)(xy^4+yz^4+zx^4)=$

$$=b^5-12ab^3c-16a^2bc^2+13b^2c^2+ca^7-7ca^5b+14ca^3b^2+7a^4c^2$$

**7.20.** The 3-degree equation $u^3-au^2+bu-c=0$ has real roots $x,y,z$

$$\Leftrightarrow -27c^2+(18ab-4a^3)c+a^2b^2-4b^3\geq 0 \quad (1)$$

**7.21.** The 3-degree equation $u^3-au^2+bu-c=0$ has positive real roots $x,y,z>0$

$$\Leftrightarrow \begin{cases} -27c^2+(18ab-4a^3)c+a^2b^2-4b^3\geq 0 \\ a>0, b>0, c>0 \end{cases}$$

**7.22.** The 3-degree equation $u^3-au^2+bu-c=0$ has roots $x,y,z$ which are length of sides of a triangle $\Leftrightarrow \begin{cases} -27c^2+(18ab-4a^3)c+a^2b^2-4b^3\geq 0 \\ a^3-4ab+8c>0. \\ a>0, b>0, c>0 \end{cases}$

Now we will go through some specifies examples:

**Problem 1 [Nguyen Anh Cuong]** Give $a,b,c>0$. Prove that:

**1.** $\dfrac{abc}{a^3+b^3+c^3}+\dfrac{2}{3}\ge\dfrac{ab+ac+bc}{a^2+b^2+c^2}$ (1) **2.** $\dfrac{a^3+b^3+c^3}{4abc}+\dfrac{1}{4}\ge\left(\dfrac{a^2+b^2+c^2}{ab+ac+bc}\right)^2$ (2)

***Solution***

**1.** (1) $\Leftrightarrow$ $P=abc\left(a^2+b^2+c^2\right)+\dfrac{2}{3}\left(a^3+b^3+c^3\right)-\left(a^3+b^3+c^3\right)(ab+bc+ca)\ge 0$

$P$ has third degrees so it will get the minimum values when $(a-b)(b-c)(c-a)=0$ or $abc=0$.

• In case $(a-b)(b-c)(c-a)=0$, assume that $a=c$ , the inequality is equivalent to:

$$\frac{a^2b}{2a^3+b^3}+\frac{2}{3}\ge\frac{a^2+2ab}{2a^2+b^2}\Leftrightarrow(a-b)^2\left[\frac{1}{2a^2+b^2}-\frac{2a+b}{3\left(2a^3+b^3\right)}\right]\ge 0\Leftrightarrow(a-b)^4(a+b)\ge 0.$$

• In case $abc=0$, assume that $c=0$, the inequality is equivalent to:

$$\frac{2}{3}\ge\frac{ab}{a^2+b^2}\Leftrightarrow a^2+b^2+3\left(a-b\right)^2\ge 0.$$

**2.** Notice that we can transform (2) to a 7-degree symmetric polynomial with variables $a$, $b$, $c$ but it is only a 1-degree polynomial with variable $abc$, therefore according to the ***1st consequence***, we just need to consider two cases:

• In the case $(a-b)(b-c)(c-a)=0$, assume that $a=c$ the inequality is equivalent to:

$$\frac{2a^3+b^3}{4a^2b}+\frac{1}{4}\ge\left(\frac{2a^2+b^2}{a^2+2ab}\right)^2\Leftrightarrow\left(\frac{2a^3+b^3}{4a^2b}-\frac{3}{4}\right)\ge\left(\frac{2a^2+b^2}{a^2+2ab}\right)^2-1$$

$$\Leftrightarrow\frac{(a-b)^2(2a+b)}{4a^2b}\ge\frac{(a-b)^2\left(3a^2+b^2+2ab\right)}{\left(a^2+2ab\right)^2}\Leftrightarrow(a-b)^2\left[\left((2b-a)^2+a^2\right)\right]\ge 0$$

• In the case $abc=0$, the inequality is trivially correct.

**Problem 2.** Give $a,b,c>0$. Prove that: $(ab+bc+ca)\left[\dfrac{1}{(a+b)^2}+\dfrac{1}{(b+c)^2}+\dfrac{1}{(c+a)^2}\right]\ge\dfrac{9}{4}$ (1)

***(Iran Olympiad 1996)***

***Solution***

Transform (1) to a 6-degrees symmetric polynomial with $a$, $b$, $c$ and a 2-degrees polynomial with variable $abc$ and positive coefficient:

$f(a+b+c,ab+bc+ca,abc)=$

$9\left[(a+b)(b+c)(c+a)\right]^2-4(ab+bc+ca)\left[(a+b)^2(b+c)^2+(b+c)^2(c+a)^2+(c+a)^2(a+b)^2\right](*)$

$=m\left(abc\right)^2+nabc+p(m\ge 0)\le 0$

in which $m,n,p$ are quantities containing constant or $a+b+c,ab+bc+ca$ which we also consider as constant.

We can give more detail for this as following: $(a+b)(b+c)(c+a)$ has the form $mabc+n$ so $9\left[(a+b)(b+c)(c+a)\right]^2$ has the form $m^2(abc)^2+nabc+p$.

$4(ab+bc+ca)\left[(a+b)^2(b+c)^2+(b+c)^2(c+a)^2+(c+a)^2(a+b)^2\right]=4mA$ where $m=ab+bc+ca$ are considered as constant, $A$ is a 4 degrees polynomial so its form is also $mabc+n$.

Therefore the expression (*) has the mentioned form.

Then the LHS function gets the maximum value only in the case $(a-b)(b-c)(c-a)=0$ or $abc=0$

• In the case $(a-b)(b-c)(c-a)=0$, assume that $a=c$ the inequality is equivalent to

$$(a^2+2ab)\left(\frac{1}{4a^2}+\frac{2}{(a+b)^2}\right)\ge\frac{9}{4}\Leftrightarrow(a-b)^2\left[\frac{2a+b}{2a(a+b)^2}-\frac{1}{(a+b)^2}\right]\ge 0\Leftrightarrow b(a-b)^2\ge 0$$

• In the case $abc=0$, assume that $c=0$, the inequality is equivalent to

$$ab\left(\frac{1}{(a+b)^2}+\frac{1}{a^2}+\frac{1}{b^2}\right)\ge\frac{9}{4}\Leftrightarrow(a-b)^2\left[\frac{1}{ab}-\frac{1}{4(a+b)^2}\right]\ge 0\Leftrightarrow(a-b)^2\left(4a^2+4b^2+7ab\right)\ge 0$$

Following is some another example that we can not use $ABC$ Theorem or its consequences directly until we have some algebraic transformation.

**Problem 3.** Given $a,b,c>0$ satisfying $a^2+b^2+c^2=1$. Prove that:

$$\frac{a}{a^3+bc}+\frac{b}{b^3+ac}+\frac{c}{c^3+ab}\ge 3 \quad (1)$$

***(Russia Olympiad 2005)***

***Solution***

***Comments:*** If we transform (1) to a symmetric polynomial then we will obtain a nine degrees polynomial with variable $a$, $b$, $c$; and third degrees with variable $abc$. These cases are not in our consequences.

Therefore we should have some algebraic transformation before think of ABC.

Take $x=\dfrac{bc}{a};y=\dfrac{ac}{b};z=\dfrac{ab}{c}\Rightarrow xy+yz+xz=1$ and (1) $\Leftrightarrow\dfrac{1}{xy+z}+\dfrac{1}{yz+x}+\dfrac{1}{zx+y}\ge 3$ (2)

Transform (2) to a second degrees polynomial with variable $xyz$ with the coefficient for $xyz^2$ is non-negative. We just need to consider two cases as usual.

***Case 1:*** $x=z$. Then (2) $\Leftrightarrow\dfrac{2}{xy+x}+\dfrac{1}{x^2+y}\ge 3$ then $2xy+x^2=1$.

Replace y by $\dfrac{1-x^2}{2x}$ then we must prove that: $\dfrac{2}{\frac{1-x^2}{2}+x}+\dfrac{1}{x^2+\frac{1-x^2}{2x}}\ge 3,\ \forall x\in[0;1]$.

This is no longer too difficult with Fermat Theorem.

***Case 2:*** $z=0$. Then (2) $\Leftrightarrow \dfrac{1}{xy}+\dfrac{1}{x}+\dfrac{1}{y}\geq 3$ where $xy=1$.

We have $\dfrac{1}{x}+\dfrac{1}{y}\geq\dfrac{2}{\sqrt{xy}}\Rightarrow\dfrac{1}{xy}+\dfrac{1}{x}+\dfrac{1}{y}\geq\dfrac{1}{xy}+\dfrac{2}{\sqrt{xy}}=3$.

**Problem 4 [Nguyen Anh Cuong]** Give positive real numbers $x, y, z$. Prove that

$$\sqrt{\frac{x^4+y^4+z^4}{x^2y^2+y^2z^2+z^2x^2}}+\sqrt{\frac{2(xy+yz+xz)}{x^2+y^2+z^2}}\geq 1+\sqrt{2}\quad (1)$$

***Solution***

For this problem, thinking about transform it to a polynomial is really a problem.

In this case we should forget about the consequences and apply theorems for it.

As usual denote that $a=x+y+z, b=xy+yz+zx, c=xyz$, use the equality mentioned on previous part to transform our inequality to : $\sqrt{\dfrac{a^4-2a^2b+2b^2+4ac}{b^2-2ac}}+\sqrt{\dfrac{2b}{a^2-2b}}\geq 1+\sqrt{2}$

The function with variable $c$ is simple a monotonous one. According to the first theorem, its minimum occurs only in cases $(a-b)(b-c)(c-a)=0$ or $abc=0$.

**Case 1:** $x=z$. Then (1) $\Leftrightarrow \sqrt{\dfrac{2x^4+y^4}{x^4+2x^2y^2}}+\sqrt{\dfrac{2(x^2+2xy)}{2x^2+y^2}}\geq 1+\sqrt{2}$ (2)

We can assume that $x=1$ because of its homogeneous property, then

$$(2)\Leftrightarrow\sqrt{\frac{y^4+2}{2y^2+1}}-1\geq\sqrt{2}-\sqrt{\frac{2(2y+1)}{y^2+2}}\Leftrightarrow\frac{(y^2-1)^2}{2y^2+1+\sqrt{(y^4+2)(2y^2+1)}}\geq\frac{\sqrt{2}(y-1)^2}{y^2+2+\sqrt{(y^2+2)(2y+1)}}$$

Notice some evaluation:

$$\sqrt{(y^4+2)(2y^2+1)}\leq\sqrt{2}y^3+\sqrt{2}\Rightarrow 2y^2+1+\sqrt{(y^4+2)(2y^2+1)}\leq 2y^2+1+\sqrt{2}y^3+\sqrt{2}$$

$$\sqrt{(y^2+2)(2y+1)}\geq y+\sqrt{2}\qquad\Rightarrow y^2+2+\sqrt{(y^2+2)(2y+1)}\geq y^2+y+2+\sqrt{2}$$

So it suffices to prove that:

$$\frac{(y+1)^2}{\sqrt{2}y^3+2y^2+\sqrt{2}+1}\geq\frac{\sqrt{2}}{y^2+y+2+\sqrt{2}}\Leftrightarrow y^4+y^3+(5-\sqrt{2})y^2+(5+2\sqrt{2}y)\geq 0$$

***Case 2:*** $z=0$. Then (1) $\Leftrightarrow \sqrt{\dfrac{x^4+y^4}{x^2y^2}}+\sqrt{\dfrac{2xy}{x^2+y^2}}\geq 1+\sqrt{2}$ (2)

We have: $x^4+y^4 \ge \dfrac{\left(x^2+y^2\right)^2}{2} \ge 2x^2y^2 \Rightarrow \sqrt{\dfrac{x^4+y^4}{2x^2y^2}} \ge \dfrac{x^2+y^2}{2xy} \ge 1$, therefore:

$$LHS(2) \ge \left(\sqrt{2}-1\right)\sqrt{\frac{x^4+y^4}{2x^2y^2}}+\left(\frac{x^2+y^2}{2xy}+\sqrt{\frac{2xy}{x^2+y^2}}\right) \ge \left(\sqrt{2}-1\right)+2\cdot\sqrt[4]{\frac{x^2+y^2}{2xy}} \ge 1+\sqrt{2}$$

**Problem 5.** Given $a,b,c\in$ ¡ and $a^2+b^2+c^2=2$. Prove that: $\left|a^3+b^3+c^3-abc\right| \le 2\sqrt{2}$

***(Mongolia MO 1991)***

***Proof***

The expression given is not a normal polynomial. We have two choices in this case; prove two inequalities or take its square. In this part I will square it since it is still valid for ABC Theorem. Indeed the polynomial obtained after squaring is a six degrees polynomial with positive coefficient for $a^2b^2c^2$ and we must find its maximum. According to *ABC*, we refer to two cases:

**Case 1:** $(a-b)(b-c)(c-a)=0$

Given *x*, *y*: $x^2+2y^2=2$. Prove that: $\left|x^3+2y^3-xy^2\right| \le 2\sqrt{2}$ (1)

We have (1) $\Leftrightarrow$ $\left(x^3+2y^3-xy^2\right)^2 \le \left(x^2+2y^2\right)^3$ (2)

If $y=0$ then we have the equality.

When $y\ne 0$, divide two sides for $y^6$ and replace $\dfrac{x}{y}$ by $t$, then (2) $\Leftrightarrow$ $\left(t^3-t+2\right)^2 \le \left(t^2+2\right)^3$

We have: $\left(t^3-t+2\right)^2 \le \left(t^3+2\right)^2 = t^6+4t^3+2 < t^6+6t^4+12t^2+8=\left(t^2+2\right)^3$

**Case 2:** $abc=0$

Give *x*, *y*: $x^2+y^2=2$. Prove that: $\left|x^3+y^3\right| \le 2\sqrt{2}$

Indeed, we have: $\left|x^3+y^3\right| \le \sqrt{\left(x^4+y^4\right)\left(x^2+y^2\right)} \le \sqrt{\left(x^2+y^2\right)^3} = 2\sqrt{2}$

**Problem 6 [Nguyen Anh Cuong]**

Given non-negative numbers $a,b,c$ satisfying $a+b+c=2$. Prove that:

$$0 \le \sqrt{a^2b+b^2c+c^2a}+\sqrt{ab^2+bc^2+ca^2} \le 2$$

***Solution***

These are no longer polynomial, and to transform it to polynomial is really a problem.

There are no better way in mind, square it and see:

$$a^2b+b^2c+c^2a+ab^2+bc^2+ca^2+2\sqrt{\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right)} \le 4 \quad (1)$$

Still not a polynomial, we can transform to a polynomial in the next square step but our polynomial has too big degrees already.

Try to transform it with $A=a+b+c=2, B=ab+bc+ca, C=abc$ :

$$(1) \Leftrightarrow 2B-3C+2\sqrt{9C^2+(8-12B)C+B^3} \le 4 \Leftrightarrow 2\sqrt{9C^2+(8-12B)C+B^3} \le 4-2B+3C$$

$$\Leftrightarrow 4\left(9C^2+\left(8-12B\right)C+B^3\right) \le \left(4-2B+3C\right)^2 = 9C^2+6(4-2B)C+(4-2B)^2$$

$$\Leftrightarrow 27C^2+(8-36B)C+4B^3-(4-2B)^2 \le 0$$

Not it is very nice to apply $ABC$ already, the function with variable $C$ is convex and we are finding its maximum value. As usual consider two cases:

***Case 1:*** $(a-b)(b-c)(c-a)=0$

Given $x, y \ge 0$ satisfying $2x+y=2$. Prove that $2\sqrt{x^3+x^2y+xy^2} \le 2$

$$\Leftrightarrow x^3+x^2(2-2x)+x(2-2x)^2 \le 1,\ \forall x \in [0;1]$$

$$\Leftrightarrow (x-1)\left(3x^2-3x+1\right) \le 0\ ,\ \forall x \in [0;1] \Leftrightarrow 3x^2-3x+1 \ge 0, \forall x \in [0;1]$$

***Case 2:*** $abc=0$

Given $x, y \ge 0$ satisfying: $x+y=2$. Prove that: $\sqrt{x^2y}+\sqrt{xy^2} \le 2$

We have: $\sqrt{x^2y}+\sqrt{xy^2}=\sqrt{xy}\left(\sqrt{x}+\sqrt{y}\right) \le \dfrac{x+y}{2}\sqrt{2\left(x+y\right)}=2$

***Proposed Problem***

**Problem 7.** Give non-negative numbers a, b, c satisfying: $xy+yz+zx=1$.

Prove that: $\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \le \sqrt{2}+\dfrac{9}{4}xyz$

**Problem 8.** Give positive numbers $a,b,c$. Prove that:

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab} \ge \frac{6}{a^2+b^2+c^2+ab+bc+ca}$$

**Problem 9.** [Darij Grinberg- Old and New Inequality]

Give $a,b,c$ as positive numbers. Prove that:

$$\frac{a}{(b+c)^2}+\frac{b}{(c+a)^2}+\frac{c}{(a+b)^2} \ge \frac{9}{4(a+b+c)}$$

**Problem 10.** [Mircea Lascu – Old and New Inequality]

Give $a,b,c$ as positive numbers. Prove that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)$$

**Problem 11.** [Vietnam TST, 1996]

Prove this inequality for all real numbers $a,b,c$

$$(a+b)^4+(b+c)^4+(c+a)^4\geq \frac{4}{7}\left(a^4+b^4+c^4\right)$$

**Problem 12.** [Nguyen Anh Cuong] Give positive $a,b,c$. Prove that

$$\sqrt{\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}}+2\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\geq\sqrt{6}+2$$

**Problem 13.** [Russia MO] Assume that $x,y,z$ are positive numbers of which sum is 3.

Prove that: $\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx$

**Problem 14.** [Vasile Cirtoaje] Give a, b, c ≥ 0 and $a^2+b^2+c^2=3$. Prove that:

$$(2-ab)(2-bc)(2-ca)\geq 1$$

**Problem 15.** [Nguyen Anh Cuong]

Give non-negative numbers $x,y,z$: $x+y+z=2$. Prove that:

$$0\leq\sqrt{x^3y+y^3z+z^3x}+\sqrt{xy^3+yz^3+zx^3}\leq 2$$

**Problem 16.** [Nguyen Anh Cuong] Give $a,b,c\geq 0$ and $a^2+b^2+c^2=1$.

Prove that: $\dfrac{bc}{a-a^3}+\dfrac{ca}{b-b^3}+\dfrac{ab}{c-c^3}\geq\dfrac{5}{2}$

**III. ABC EXTRA**

As you have seen in previous part, almost the evaluation of ABC based on $abc$. Therefore if the problems come with a condition with $abc$ then it is really a problem with ABC. We will consider some problems having this condition and try to solve it with ABC.

A trick we might use is that we can destroy the condition in many ways and refer to a homogeneous inequality. And then we can think of using $ABC$, specified in this problems.

**Problem 1 (Vasile Cirtoaje, MS, 2006)**

Given positive numbers $a,b,c$ satisfying $abc=1$. Prove that:

$$a^2+b^2+c^2+6\ge\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ (1)$$

***Proof***

The inequality (1) $\Leftrightarrow$ $a^2+b^2+c^2+6\ge\frac{3}{2}(a+b+c+ab+bc+ca)$

$\Leftrightarrow 2(a+b+c)^2+12\ge 3(a+b+c)+7(ab+bc+ca)$ (2)

There exits positive numbers $x,y,z$ such that $a=\frac{x^2}{yz},b=\frac{y^2}{zx},c=\frac{z^2}{xy}$.

Then (2) $\Leftrightarrow$ $2\left(\frac{x^3+y^3+z^3}{xyz}\right)^2+12\ge\frac{3\left(x^3+y^3+z^3\right)}{xyz}+7\cdot\frac{x^3y^3+y^3z^3+z^3x^3}{x^2y^2z^2}$

$\Leftrightarrow 2\left(x^3+y^3+z^3\right)^2+12x^2y^2z^2\ge 3\left(x^3+y^3+z^3\right)xyz+7\left(x^3y^3+y^3z^3+z^3x^3\right)$ (3)

Because of homogeneous property (3), we can assume that $x+y+z=1,xy+yz+zx=v,xyz=w$

We have: $x^3+y^3+z^3=1-3v+3w$ ; $x^3y^3+y^3z^3+z^3x^3=v^3-3vw+3w^2$

Then (3) $\Leftrightarrow$ $2(1-3v+3w)^2+12w^2\ge 3(1-3v+3w)w+7\left(v^3-3vw+3w^2\right)$

$$\Leftrightarrow 2(1-3v)^2+9(1-3v)w+7\left(v^3-3vw\right)\ge 0$$

The first degrees polynomial with variable $w$, good condition for *ABC - Theorem* for (3).

**Case 1:** $z=0,y=1$

The inequality (3) is equivalent to: $2\left(x^3+1\right)^2\ge 7x^3$ $\Leftrightarrow$ $2\left(x^3-1\right)^2+x^3\ge 0$

**Case 2:** $y=z=1$

The inequality (3) is equivalent to: $2\left(x^3+2\right)^2+12x^2\ge 3\left(x^3+2\right)x+7\left(2x^3+1\right)$

$\Leftrightarrow 2x^6-3x^4-6x^3+12x^2-6x+1\ge 0\Leftrightarrow\left(2x^4+4x^3+3x^2-4x+1\right)(x-1)^2\ge 0$

**Problem 2.** Given $a,b,c \geq 0$ satisfying: $ab+bc+ca+abc=4$.

Prove that: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a+b+c$ (1)

***Proof***

There exits $x,y,z$ such that: $a=\frac{2x}{y+z}, b=\frac{2y}{x+z}, c=\frac{2z}{x+y}$

Then (1) becomes: $\frac{x+y}{z}+\frac{x+z}{y}+\frac{y+z}{x} \geq 4\left(\frac{x}{y+z}+\frac{y}{x+z}+\frac{z}{x+y}\right)$

Because of homogeneous property we can assume that $x+y+z=1, xy+yz+zx=v, xyz=w$

and rewrite inequality as: $\frac{1-z}{z}+\frac{1-y}{y}+\frac{1-x}{x} \geq 4\left(\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}\right)$

$$\Leftrightarrow \left[xy(1-z)+xz(1-y)+yz(1-x)\right]\left[(1-x)(1-y)(1-z)\right]$$
$$\geq 4xyz\left[x(1-y)(1-z)+y(1-z)(1-x)+z(1-x)(1-y)\right]$$

$$\Leftrightarrow (v-3w)(v-w) \geq 4w(1-2v+3w) \Leftrightarrow 9w^2+4(1-v)w-v^2 \leq 0$$

We can apply $ABC$ now, consider two cases:

In case $z=0$, the inequality is trivially correct.

In case $y=z=1$ (we can consider them as one because of homogeneous property):

$$2(x+1)+\frac{2}{x} \geq 4\left(\frac{x}{2}+\frac{2}{x+1}\right) \Leftrightarrow 2x(x+1)+2(x+1) \geq 8x \Leftrightarrow 2(x-1)^2 \geq 0$$

**Problem 3.** Given $a$, $b$, $c$ satisfying $a \geq 1, b \geq 1, c \geq 1$ and $a+b+c+2=abc$.

Prove that: $a+b+c+3 \geq 6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ (1)

***Proof***

There exits positive numbers $x,y,z$ such that $a=\frac{y+z}{x}, b=\frac{x+z}{y}, c=\frac{x+y}{z}$

where $x,y,z$ are length of sides of a triangle.

$$(1) \Leftrightarrow (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right) \quad (2)$$

Because of homogeneous property, we can assume that $x+y+z=1, xy+yz+zx=n, xyz=p$

$$(2) \Leftrightarrow (x+y+z)\frac{xy+yz+zx}{xyz} \geq 6\left(\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}\right)$$

$$\Leftrightarrow (x+y+z)\frac{xy+yz+zx}{xyz} \geq 6\cdot\frac{x(1-y)(1-z)+y(1-z)(1-x)+z(1-x)(1-y)}{(1-x)(1-y)(1-z)}$$

$$\Leftrightarrow (x+y+z)\frac{xy+yz+zx}{xyz} \geq 6\cdot\frac{x+y+z-2(xy+yz+zx)+3xyz}{1-xyz-(x+y+z)+xy+yz+zx}$$

$$\Leftrightarrow \frac{n}{p} \geq \frac{6(1-2n+3p)}{n-p} \Leftrightarrow n(n-p) \geq 6(1-2n)p+3p^2 \Leftrightarrow 3p^2+(6-11n)p-n^2 \leq 0$$

Since $x, y, z$ are length of sides of a triangle then we can take $x=v+w, y=u+w, z=u+v$.

We have: $a+b+c=2(u+v+w), xy+yz+zx=uv+vw+wu+(u+v+w)^2$ and $xyz=(u+v+w)(uv+vw+wu)-uvw$ then the expression (1) with variables $u,v,w$ still be a 2-degree polynomial with variable $uvw$ and positive coefficient for $(uvw)^2$ and we must find maximum value. We can apply $ABC$ now, consider two cases:

***Case 1:*** $w=0 \Rightarrow z=x+y$. We can assume $y=1$.

Inequality (2) becomes: $2(x+1)\left(\frac{1}{x}+1+\frac{1}{x+1}\right) \geq 6\left(\frac{x}{x+2}+\frac{1}{2x+1}+1\right)$

$$\Leftrightarrow 2\left(3+x+\frac{1}{x}\right) \geq 6\left[\frac{2x^2+2x+2}{(x+2)(2x+1)}+1\right] \Leftrightarrow \frac{x^2+1}{2x} \geq \frac{3(x^2+x+1)}{2x^2+5x+2}$$

$$\Leftrightarrow \frac{(x-1)^2}{2x} \geq \frac{(x-1)^2}{2x^2+5x+2} \Leftrightarrow (x-1)^2(2x^2+3x+2) \geq 0 \text{ (is always true)} \Rightarrow \text{(q.e.d)}$$

***Case 2:*** $v=w \Rightarrow y=z$. We can assume $y=z=1, x \leq 2$

Inequality (2) becomes $\Leftrightarrow (x+2)\left(\frac{1}{x}+2\right) \geq 6\left(\frac{x}{2}+\frac{2}{x+1}\right)$

$$\Leftrightarrow (x+1)(x+2)(2x+1) \geq 3x(x^2+x+4) \Leftrightarrow (2-x)(x-1)^2 \geq 0 \text{ (is always true)} \Rightarrow \text{(q.e.d)}$$

The problem was solved.

However in many cases, we can not destroy condition and refer the old problem to a homogeneous problem, and ABC extra was born for these problems.

**ABC – EXTRA**

**i)** Given $a,b,c$ are all real numbers or positive real numbers. If we fix $abc, a+b+c$ (consider it as constant) then $ab+bc+ca$ have its maximum value or minimum value only when $(a-b)(b-c)(c-a)=0$.

**ii)** Given $a,b,c$ are all positive real numbers. If we fix $abc, a+b+c$ (consider it as constant) then $a+b+c$ have its minimum value only when $(a-b)(b-c)(c-a)=0$.

### *Proof*

**i)** Assume that $a+b+c=1$ and $abc=m$ (the case $a+b+c=n$ can be refer to this cases through a algebraic transformation)

We will prove that $ab+bc+ca$ can obtain maximum and minimum value only when $(a-b)(b-c)(c-a)=0$ in both cases $a,b,c\in\mathbb{R}$ and $a,b,c\in\mathbb{R}^{+}$.

Assume that $ab+bc+ca=S$. Again we refer to the third degrees equation which have $a,b,c$ as its roots:

Take $f(X)=X^3-X^2+SX-m$. We have: $f'(X)=3X^2-2X+S$.

Equation $f'(X)=0$ has two solutions $X_1=\dfrac{1+\sqrt{1-3S}}{3}; X_2=\dfrac{1-\sqrt{1-3S}}{3}$

The equation has three roots if and only if $f(X_2)\geq 0, f(X_1)\leq 0$.

We have:

$f(X_2)\geq 0 \Leftrightarrow (6S-2)X_2+S-9m\geq 0$, assume that its solution range is $R_{X_2}$

$f(X_1)\leq 0 \Leftrightarrow (6S-2)X_1+S-9m\leq 0$, assume that its solution range is $R_{X_1}$

Firstly we prove our theorem for the case $a,b,c\in\mathbb{R}$.

Assume that $S_{\min}, S_{\max}$ are minimum and maximum values in the set $R_{X_1}\cap R_{X_2}$ (the union of these two ranges are not null or $f(X)=0$ has no solution for all values of $S$). Notice that $S_{\min}, S_{\max}$ are two solution of $(6S-2)X_1+S-9m=0$ or $(6S-2)X_2+s-9m=0$ ( these value surely exits or $f(X)=0$ has solutions when $S$ reaches infinity values), then we have $S_{\min}\leq S\leq S_{\max}$. Now let consider $a,b,c$ when $S$ reaches these two values .

The first point we can see is that $f(X)=0$ still have three solutions since $S_{\min}, S_{\max}\in R_{X_1}\cap R_{X_2}$. The second point, because $S_{\min}, S_{\max}$ are solutions of $(6S-2)X_1+S-9m=0$ or $(6S-2)X_2+s-9m=0$, therefore $f(X_1)=0$, or $f(X_2)=0$. In this case the equation $f(X)=0$ has double root, or in the other hands, the shape of $(a,b,c)$ is $(x,x,y)$.

Now we prove our theorem for the case $a,b,c\in\mathbb{R}^{+}$. At first we must have $m\geq 0$. Notice that $0\notin R_{X_1}\cap R_{X_2}$ because there are no real number $a,b,c$ such that $a+b+c=1, ab+bc+ca=0, abc\geq 0$. This means we can separate $R_{X_1}\cap R_{X_2}$ in smalls $[x_i, y_i]$ such chat $x_i y_i>0$. Let $R_3$ are the set of $R_{X_1}\cap R_{X_2}$ ignored the negative ranges. And let $S_{\min}, S_{\max}$ are minimum and maximum values in the set $R_3$ . We have: $S_{\min}\leq S\leq S_{\max}$. Using the same argument as above we also can obtain the result that $S$ reach its edge if and only if $(a-b)(b-c)(c-a)=0$.

**ii)** The readers can ready use the same arguments to prove the ***ABC Extra Theorem 2***.

However in the case for $a+b+c$, we can not limit the upper bound for $a+b+c$ when $a, b, c \geq 0$.

Therefore in this case we can only have $S_{\min}$. And we will come out with the result $a+b+c$ get minimum value if and only if $(a-b)(b-c)(c-a)=0$.

*Let see some example to know how this theorem can help us in a lot of cases:*

**Problem 4 [Hojoo Lee]** Give positive numbers $a,b,c$ satisfying $abc \geq 1$.

Prove that: $1 \geq \dfrac{1}{1+a+b}+\dfrac{1}{1+b+c}+\dfrac{1}{1+c+a}$ (1)

***Proof***

Notice that the LHS function decrease when $a$ increase. Therefore it is enough to prove the inequality when $abc=1$ (for the case $abc=k\geq 1$, we have:

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq \frac{1}{1+\frac{a}{k}+b}+\frac{1}{1+b+c}+\frac{1}{1+c+\frac{a}{k}} \leq 1 \text{ )}$$

The inequality (1) $\Leftrightarrow$

$$f(a,b,c)=(1+a+b)(1+b+c)(1+c+a)-(1+a+b)(1+b+c)-(1+b+c)(1+c+a)-(1+c+a)(1+a+b)\geq 0$$

Notice that the degrees of $ab+bc+ca$ in $f(a,b,c)$ in only one ( $f(a,b,c)$ is only a third degrees polynomial with variables $a,b,c$ ). Therefore when we fix $a+b+c$ then $f(a,b,c)$ has its minimum value when $ab+bc+ca$ has its minimum value, then $(a-b)(b-c)(c-a)=0$. We refer to this problem:

Given $x, y>0$ satisfying $x^2y=1$. Prove that: $\dfrac{1}{1+2x}+\dfrac{2}{1+x+y}\leq 1$ (2)

Replace $y$ with $\dfrac{1}{x^2}$ into (2) we have: $\dfrac{1}{1+2x}+\dfrac{2}{1+x+\frac{1}{x^2}}\leq 1 \Leftrightarrow -\dfrac{2x(x-1)^2(x+1)}{(1+2x)(x^3+x^2+1)}\leq 0$

Hence the proof is completed.

**Problem 5 [Bui Viet Anh]** Give three positive numbers $a,b,c$. Prove that:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+2\sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}\geq 2 \quad (1)$$

***Solution***

For this problem, even when we fix $abc, ab+bc+ca$ or $a+b+c$ we can not obtain a nice function (means monotony, concave or convex). However we can see some relation between $\dfrac{a}{b+c}+\dfrac{b}{c+a}+\dfrac{c}{a+b}$ and $\dfrac{abc}{(a+b)(b+c)(c+a)}$, they are sum and product of $\dfrac{a}{b+c},\dfrac{b}{c+a},\dfrac{c}{a+b}$.

Therefore we take $x=\frac{a}{b+c}, y=\frac{b}{c+a}, z=\frac{c}{b+a}$, the problem is that how to connect $x, y, z$. Here I will give a condition for $x, y, z$, and with this condition we can convert $x, y, z$ to $a,b,c$ again. And then we will come out with the equivalent problem:

Give $x, y, z \geq 0$ satisfying: $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}=2 \Leftrightarrow 2xyz+xy+yz+zx=1$.

Prove that: $x+y+z+2\sqrt{xyz} \geq 2$.

So we will fix $xyz$ and $xy+yz+zx$ and refer to the problem:

Give $a,b \geq 0$ and $2a^2b+a^2+2ab=1$. Find the minimum of: $2a+b+2a\sqrt{b}$.

Replace $b=\frac{1-a^2}{2a^2+2a}=\frac{1-a}{2a}$ $(a \leq 1)$, we need to prove $2a+\frac{1-a}{2a}+\sqrt{2a(1-a)} \geq 2$ (2)

Indeed, the inequality (2) $\Leftrightarrow \frac{1-a}{2a}+\sqrt{\frac{2a}{1-a}}(1-a) \geq 2(1-a) \Leftrightarrow \frac{1}{2a}+\sqrt{\frac{2a}{1-a}} \geq 2$

In the other hands $\sqrt{\frac{2a}{1-a}}=\frac{\sqrt{2}a}{\sqrt{(1-a)a}} \geq \frac{\sqrt{2}a}{\frac{1-a+a}{2}} \geq 2\sqrt{2}a$.

Therefore: $\frac{1}{2a}+\sqrt{\frac{2a}{1-a}} \geq \frac{1}{2a}+2\sqrt{2}a \geq 2\sqrt{\frac{1}{2a}2\sqrt{2}\,a}=2\cdot\sqrt[4]{2} \geq 2$.

The proof is end here.

**Problem 6.** Give positive numbers $x, y, z$. Prove that

$$\frac{2x^2y^2z^2}{x^3y^3+y^3z^3+z^3x^3}+\frac{1}{3} \geq \frac{3xyz}{x^3+y^3+z^3} \quad (1)$$

***Proof***

Take $a=\frac{yz}{x^2}, b=\frac{xz}{y^2}, c=\frac{xy}{z^2}$, the inequality (1) $\Leftrightarrow \frac{2}{a+b+c}+\frac{1}{3} \geq \frac{3}{ab+bc+ca}$

where $a,b,c$ are positive numbers satisfying $abc=1$.

After this algebraic transformation, the problem is quite trivial with $ABC-extra$.

We refer to the only case:

Give positive numbers $x, y: x^2y=1$. Prove that: $\frac{2}{2x+y}+\frac{1}{3} \geq \frac{3}{x^2+2xy}$

$$\Leftrightarrow \frac{2}{2x+\frac{1}{x^2}}-\frac{2}{3}\ge \frac{3}{x^2+\frac{2}{x}}-1 \Leftrightarrow \frac{2x^2}{2x^3+1}-\frac{2}{3}\ge \frac{3x}{x^3+2}-1$$

$$\Leftrightarrow \frac{(x+2)(x-1)^2}{x^3+2}\ge \frac{2(x+1)(x-1)^2}{3(2x^3+1)} \Leftrightarrow (4x^4+10x^3-x+2)(x-1)^2\ge 0$$

The readers might try this problems with $ABC-Theorem$, and you might see the efficient of $ABC-extra$.

**Proposed Problems**

**Problem 7.** Given $a,b,c>0$. Prove that: $\frac{x+y+z}{3}+\sqrt[3]{xyz}\ge \frac{2}{3}\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)$

**Problem 8.** Give positive numbers $a,b,c$ of which product is 1. Prove that

$$a^2+b^2+c^2+3\ge a+b+c+ab+bc+ca$$

**Problem 9.** Prove this inequality for non-negative numbers $a,b,c$:

$$a^2+b^2+c^2+2abc+1\ge 2(ab+bc+ca)$$

**Problem 10.** Give positive numbers $a,b,c$ of which product is 1.

Prove that $a+b+c\ge \frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$

**Problem 11. [Vietnam MO 1996]**

Given $a,b,c\ge 0$ satisfying $a+b+c+abc=4$. Prove that $a+b+c\ge ab+bc+ca$

**Problem 12 [Le Trung Kien, Vo Quoc Ba Can]**

Given $a,b,c\ge 0$ satisfying $ab+bc+ca+6abc=9$. Prove that: $a+b+c+3abc\ge 6$

**Problem 13.** Give positive numbers $x,y,z$ satisfying: $xyz\ge 1$. Prove that

$$\frac{1}{2+x}+\frac{1}{2+y}+\frac{1}{2+z}\le 1$$

**Problem 14.** Give positive numbers $a,b,c$ of which product is 1. Prove that

$$3^6(a^2+1)(b^2+1)(c^2+1)\le 8(a+b+c)^6$$

**Problem 15 [Nguyen Anh Cuong]**

Give positive numbers $a,b,c$ satisfying: $(a+b)(b+c)(c+a)=8$.

Find the maximum and minimum values of: $P=\sqrt{abc}+\sqrt{a+b+c}$

**Problem 16.** Given $a,b,c$ satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Prove that $a+b+c-3\le \frac{9}{4}(abc-1)$

### IV. ABC – OPEN THE WINDOW

In this part, we will again consider a class of problem that the old ABC can not handle. That is the class in which variables are bounded in close sets. For these problems, there are some classic ideas to solve that I will introduce again in this part, and later is a new convention in ABC to handle this close set

**Problem:** Give $x_1, x_2, ..., x_n \in [a,b]$. Prove that $f(x_1, x_2, ..., x_n) \geq C$

Where $a, b, C$ are constant given.

In these cases, the equality of inequality usually occurs when variables reach theirs edge, means $a$ or $b$. In these cases, we will try to prove that $f(x_1, x_2, ..., x_n) \geq \min\{f(a, x_2, ..., x_n), f(b, x_2, ..., x_n)\}$. At last we will obtain the minimum value for our function, occurring when some variables are equal to $a$, and the rest get $b$ as their values . We will consider some example to make this idea clearer:

**Problem 1.** Give $a,b,c \in [0,1]$. Find the maximum value of: $f(a,b,c) = a+b+c-ab-bc-ca$

***Solution***

We will prove that $f(a,b,c) \leq \max\{f(0,b,c), f(1,b,c)\}$ (*).

Following is a direct proof using algebraic only:

$[f(a,b,c) - f(0,b,c)][f(a,b,c) - f(1,b,c)] = c(c-1)(1-a-b)^2 \leq 0$, from here we have (*)

Or using a bit more advance view, we can find out the result faster with a notice that $f(a,b,c)$ is a monotonic function when we only consider variable $a$. Then we can find out (*) immediately.

Apply the same thing for the rest variables $b, c$, we will find out this property :
$f(a,b,c) \leq \max\{f(0,0,0); f(0,0,1); f(0,1,1); f(1,1,1)\}$.

From here we can conclude that the maximum value of $f(a, b, c)$ is 1.

The inequality occurs , for example, when: $a = b = 0, c = 1$.

**Problem 2 [Nguyen Anh Cuong]**

Given $a,b,c \in [1,2]$. Find the maximum value of $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}$

***Solution***

We will consider $b, c$ as constant, and $a$ is variable: $g''(a,b,c) = \frac{2b}{(c+a)^3} + \frac{2c}{(b+a)^3} > 0$

Therefore $g$ is a convex with variable $a$ on $\mathbb{R}^+$, i.e. $g$ will get its maximum value when $a$ reaches its edges.

From here we can deduce what we hope: $g(a,b,c) \le \max\{g(1,b,c), g(2,b,c)\}$.

Apply the same thing for the rest variables $b,c$, we will find out a property:

$g(a,b,c) \le \max\{g(1,1,1); g(2,1,1); g(2;2;1); g(2,2,2)\}$.

From here we can conclude that the maximum value of $g$ is $\frac{19}{12}$, occurs when $(a,b,c) = (2,2,1)$

*Through two examples above, the readers may have some familiar feeling when meet the kind of problems again. Now I will represent another method to kill these problems, an $ABC$ approach.*

*In above parts, we see that $ABC$ can only apply for problems when domain is $\mathbb{R}$ or $\mathbb{R}^+$, so to apply $ABC$, we must resize the given domain to $\mathbb{R}$ or $\mathbb{R}^+$.*

**Problem 1.** Given $a,b,c \in [0,1]$. Find the maximum value of: $f(a,b,c) = a+b+c-ab-bc-ca$

***Solution***

Again we meet the problem in section one, now we need to $[0;1]$ to $\mathbb{R}^+$.

Notice that if we transform $a \to \frac{1}{u}$, we will obtain the domain $[1;+\infty]$ for $u$. It is still not $\mathbb{R}^+$, we continue the next step , map $u \to x+1$ and we will obtain the domain $[0,+\infty]$ for $x$. So we have transform $a \to \frac{1}{x+1}$ to pull $[0;1] \to [0;+\infty]$. From these ideas we can come out with a solution:

Take $a = \frac{1}{1+x}, b = \frac{1}{1+y}, c = \frac{1}{1+z}$ where $x,y,z \in \mathbb{R}^+$. The expression given becomes:

$$f(a,b,c) = g(x,y,z) = \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} - \frac{1}{(1+x)(1+y)} - \frac{1}{(1+y)(1+z)} - \frac{1}{(1+z)(1+x)}$$

$$= \frac{(1+y)(1+z)+(1+z)(1+x)+(1+x)(1+y)-3-x-y-z}{(1+x)(1+y)(1+z)} = \frac{xy+yz+zx+x+y+z}{xyz+xy+yz+zx+x+y+z+1}$$

Now we can easily see that $g(x,y,z) \le 1$, no need to apply $ABC$ anymore, of if we want we still can apply $ABC$ to refer to two variables problems.

There are two advantages for us when pull a small domain to $\mathbb{R}^+$ like this. First thing is that we can $ABC$ theorem as above problem; second thing is that we can evaluate the inequality's variable with 0 which is always easier.

**Problem 3** Given $x, y, z \in [1,2]$. Prove that: $g(x,y,z) = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \le 10$

Again we use the resize domain technique then apply ABC for it.

Firstly we resize $[1,2] \to \mathbb{R}^+$ by the way: $x \to \frac{1}{x-1} - 1 = a$.

Then take $x = \frac{a+2}{a+1}, y = \frac{b+2}{b+1}, z = \frac{c+2}{c+1},\ a,b,c \in \mathbb{R}^+$. We need to prove this problem:

$$f(a,b,c) = \left(\frac{a+2}{a+1}+\frac{b+2}{b+1}+\frac{c+2}{c+1}\right)\left(\frac{a+1}{a+2}+\frac{b+1}{b+2}+\frac{c+1}{c+2}\right) \le 10$$

$$\Leftrightarrow \frac{M.N}{(a+1)(b+1)(c+1)(a+2)(b+2)(c+2)} \le 10$$

where $M = (a+2)(b+1)(c+1)+(a+1)(b+2)(c+1)+(a+1)(b+1)(c+2)$

and $\quad N = (a+1)(b+2)(c+2)+(a+2)(b+1)(c+2)+(a+2)(b+2)(c+1)$

Is it a big hurt when the inequality: $MN - 10(a+1)(b+1)(c+1)(a+2)(b+2)(c+2) \le 0$ can not solved by $ABC$ theorem easily, because the coefficient of $a^2b^2c^2$ will be negative (the reader can easily mental calculate it). However the ABC Theorem now has more than it was be, we can apply *ABC Extra* to solve this case. Fix $abc$ and $a+b+c$, and we will se that the expression obtained is the first degree polynomial with variable $ab+bc+ca$. Therefore we can conclude that the inequality obtain its maximum in the case $(a-b)(b-c)(c-a) = 0$.

Then we only need to prove the inequality: $f(a,a,b) = \left(2\cdot\frac{a+2}{a+1}+\frac{b+2}{b+1}\right)\left(2\cdot\frac{a+1}{a+2}+\frac{b+1}{b+2}\right) \le 10$

$$\Leftrightarrow \left(2\cdot\frac{a+2}{a+1}-\frac{b+2}{b+1}\right)\left(2\cdot\frac{a+1}{a+2}-\frac{b+1}{b+2}\right) \ge 0 \Leftrightarrow (ab+3b+2)(ab+3a+2) \ge 0$$

So the problem has been proved completely.

**Following is some apply problems:**

**Problem 1.** Give $a,b,c \in [1,2]$. Find the maximum of: $A = \frac{a^3+b^3+c^3}{3abc}$

**Problem 2. [VN TST 2006]**

Give $x, y, z \in [1,2]$. Prove that: $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \ge 6\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)$

## VI. THE ABC GENERALIZATION

So we have dealt with two weak points of $ABC$. The third weak points we can easily find out is that $ABC$ can only apply for three variables problems, so how about many variables problems ??? . We will together solve this problem to come out with the ABC generalization. Firstly, we should be familiar with some concept used in the ABC generalization.

### 1. ABC-able

***a. Definition:***

Consider a three variables problem $f(a,b,c)$.

We call $f(a,b,c)$ an ABC-able expression if the $f(a,b,c)\geq 0$ can be applied ABC theorem to refer to two cases:

**i)** $(a-b)(b-c)(c-a)=0$.

**ii)** $abc=0$ (This condition only occurs when apply ABC-able for variable $abc$)

***b. An approach using ABC-able:***

To prove that an expression $f(a,b,c)$ is ABC-able, we will transform the expression $f(a,b,c)$ to expression $g$ with three variables $A=a+b+c, B=ab+bc+ca, C=abc$. The expression $f(a,b,c)$ is an ABC-able expression if $g(A,B,C)$ is a convex function for variable $A,B$ or $C$.

***Example:*** Prove this expression is ABC-able

$$f(a,b,c)=a^3+b^3+c^3+3abc-ab(a+b)-bc(b+c)-ca(c+a)$$

***Solution***

Take $A=a+b+c, B=ab+bc+ca, C=abc$, we have:

$$f(a,b,c)=g(A,B,C)=A^3-3AB+3C+3C-AB+3C=A^3-4AB+9C$$

Consider $g(A,B,C)$ only with variable $C$. We have: $g'=9, g''=0$ therefore $g$ is a convex function with variable C. Therefore $f(a,b,c)$ is ABC-able.

### 2. The ABC Generalization

Consider a symmetric $n$ variables function $f(a_1,a_2,...,a_n)$, where $f$ has minimum and $n\geq 3$.
We will consider $f(a_1,a_2,...,a_n)$ as a three variables function $g(a_1,a_2,a_3)$ and $a_4,a_5,...,a_n$ are considered as constant.

If $g$ is ABC-able then inequality $f(a_1,a_2,...,a_n)\geq 0$ can be refer to two cases :

$i)$ $m$ variables are equal and the rest $n-m$ variables are equal.

$ii)$ One variable is equal to 0 (This condition only occurs when apply ABC-able for variable $abc$)

The inequality is completely proved if it is correct in two cases given.

***Proof***

We will assume that $f$ has minimum as given and its minimum occurs when $(x_1, x_2, ..., x_n)$ (*)
If $x_1 x_2 ... x_n = 0$ or $x_i$ can only obtain one in two fixed values then our theorem was proved. If $x_i$ can obtain at least three different non-zero values, we assume that $(x_1, x_2, x_3)$ are a set that its elements are different in pairs and different from 0. We will fix $x_4, x_5, ..., x_n$ as constant and consider the function $f(x_1, x_2, x_3, x_4, ..., x_n)$ as the function $g(x_1, x_2, x_3)$.

As given in theorem, $g$ is ABC-able, so it get it minimum value only when
$(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = 0$ or $x_1 x_2 x_3 = 0$. This also means that there exists a set $(a, b, c)$ so that $g(x_1, x_2, x_3) > g(a, b, c)$ or $f(x_1, x_2, x_3, x_4, ..., x_n) \ge f(a, b, c, x_4, ..., x_n)$.

This is a contradiction with (*). In conclusion, the ABC generalization has been proved.

**3. Problems with the ABC generalization**

**Problem 1 [Nguyen Anh Cuong]:**

Prove that $\dfrac{3(a^4 + b^4 + c^4 + d^4)}{4abcd} \ge 1 + \dfrac{3(a^2 + b^2 + c^2 + d^2)}{ab + ac + ad + bc + bd + cd}$ , $\forall a, b, c, d > 0$ (1)

***Solution***

The inequality (1) $\Leftrightarrow$ $f(a,b,c,d) = 3(a^4 + b^4 + c^4 + d^4)(ab + ac + ad + bc + bd + cd)$

$$-4abcd(ab + ac + ad + bc + bd + cd) - 12abcd(a^2 + b^2 + c^2 + d^2) \ge 0 \quad (2)$$

Notice that if we fix $d$ then $f$ is a three variables symmetric polynomial, more than that $f$ is ABC-able. Therefore according to ABC Theorem we only need to consider two cases (we will consider the inequality in the form (1)):

**i)** $a = 0$: The inequality is trivial.

**ii)** $a = b = x, c = d = y$:

Then (1) $\Leftrightarrow$ $\dfrac{3(x^4 + y^4)}{2x^2 y^2} \ge 1 + \dfrac{6(x^2 + y^2)}{x^2 + y^2 + 4xy} \Leftrightarrow \dfrac{3(x + y)^2 (x - y)^2}{2x^2 y^2} \ge \dfrac{2(x - y)^2}{x^2 + y^2 + 4xy}$

Apply the inequality $(x + y)^2 \ge 4xy$; $x^2 + y^2 \ge 2xy$ we will prove the above inequality.

**iii)** $a = b = c = x, d = y$:

Then (1) $\Leftrightarrow$ $\dfrac{3(3x^4 + y^4)}{4x^3 y} \ge 1 + \dfrac{3(3x^2 + y^2)}{3x^2 + 3xy} \Leftrightarrow \dfrac{3(x - y)^2 (3x^2 + 2xy + y^2)}{4x^3 y} \ge \dfrac{3(x - y)^2}{3x^2 + 2xy}$

Apply the inequality $3x^2 + 2xy \ge 3x^2$ and $3x^2 + 2xy + y^2 \ge 4xy$ we will prove for this case.

In conclusion, we have completed the proof.

**Problem 2 [Nguyen Anh Cuong]** Give $a_1, a_2, ..., a_n \geq 0$ satisfying $a_1 + a_2 + ... + a_n = n$.

$$\text{Prove that: } \frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} + \frac{2n\sqrt{n-1}}{a_1^2 + a_2^2 + ... + a_n^2} \geq n + 2\sqrt{n-1} \quad (1)$$

***Solution***

Take $f(a_1, a_2, ..., a_n) = LHS(1)$. Fix $a_4, a_5, ..., a_n$ we will obtain a symmetric three variables expression which is also ABC-able. Therefore we only need to consider two cases:

**i)** For $a_1 = 0$: The inequality is trivial.

**ii)** $m$ variables are equal to $x$ and $(n - m)$ variables are equal to $y$, we have: $mx + (n-m)y = n$.

Prove that: $\dfrac{m}{x} + \dfrac{n-m}{y} + \dfrac{2n\sqrt{n-1}}{mx^2 + (n-m)y^2} \geq n + 2\sqrt{n-1}$

Refer to: $\dfrac{mx + (n-m)y}{n}\left(\dfrac{m}{x} + \dfrac{n-m}{y}\right) + \dfrac{2\sqrt{n-1}\left[mx + (n-m)y\right]^2}{n\left[mx^2 + (n-m)y^2\right]} \geq n + 2\sqrt{n-1}$

$$\Leftrightarrow \frac{m(n-m)(x-y)^2}{nxy} \geq \frac{2\sqrt{n-1}\, m(n-m)(x-y)^2}{nmx^2 + n(n-m)y^2} \Leftrightarrow \left[mx^2 + (n-m)y^2 - 2\sqrt{n-1}xy\right](x-y)^2 \geq 0$$

The inequality is correct because: $mx^2 + (n-m)y^2 \geq 2\sqrt{m(n-m)}\,xy \geq 2\sqrt{n-1}\,xy$.

Therefore the problem was completely proved.

**Problem 3 [Nguyen Anh Cuong]** Give $a_1, a_2, ..., a_n \geq 0$. Prove that:

$$\frac{\sum\limits_{1 \leq i < j \leq n} \left(a_i - a_j\right)^2}{n^3(a_1 + a_2 + ... + a_n)} \leq \frac{a_1 + a_2 + .. + a_n}{n} - \sqrt[n]{a_1 a_2 ... a_n}$$

***Solution***

We have met this problem in S.O.S technique; however this problem can not be solved completely using S.O.S. Here we will give a better solution, using ABC Generalization.

$$\text{Letting } f(a_1, a_2, ..., a_{n-1}, a_n) = \frac{a_1 + a_2 + ... + a_n}{n} - \sqrt[n]{a_1 a_2 ... a_n} - 2n\frac{\sum\limits_{1 \leq i < j \leq n} \left(a_i - a_j\right)^2}{a_1 + a_2 + ... + a_n}$$

Here fixing $a_1 + a_2 + ... + a_n$ and $\sum\limits_{1 \leq i < j \leq n} a_i a_j$ is not a good idea because the degrees of $a_1 a_2 ... a_n$ will be too big. For this reason we will fix $a_1 + a_2 + ... + a_{n-1}$ and $a_1 a_2 ... a_{n-1}$, clearly that $f$ will become a quadratic trinomial with variables $a_1, a_2, ... a_{n-1}$ or a linear polynomial with variable $\sum\limits_{1 \leq i < j \leq n} a_i a_j$. Therefore $f$ is ABC-able with any three variables in the set $\{a_i\}_{i=1}^{n-1}$.

Here we prove that the expression is $ABC$-able with variable $ab+bc+ca$ so we have only one case to consider: $m$ variables are equal, the rest $n-m$ variables are equal.

Assume that $\left(a_1,a_2,...,a_n\right)=(x^n,x^n,...,x^n,y^n,y^n,...y^n)$ ($m$ variable $x$, $n-m$ variable $y$).

Assume that $x \ge y$ ($x \le y$ are consider similarly). We refer to the problem:

Give $x$, $y \ge 0$. Prove that: $\dfrac{m(n-m)\left(x^n-y^n\right)^2}{n^3mx^n+n^3(n-m)y^n} \le \dfrac{mx^n+(n-m)y^n}{n}-x^m y^{n-m}$

Due to the homogeneous property, we can consider $y=1, x\ge 1$ and we will need to prove that:

$$f(x)=n^2\left(mx^n+n-m\right)^2-n^3x^m\left(mx^n+n-m\right)-m(n-m)\left(x^n-1\right)^2 \ge 0$$

We have referred to a one variable problem, but to prove it is still a big hurt.

Actually we must face a problem is that variable $m$ is dynamic, non-continuous in $[0,n]$...

Through the first and second example, we have a notice that $m$ may "prefer" $0, 1, n-1, n$.

A forsee feeling is that we may always push $m$ to these values, then how easy our work will be ...

And so great it is, this foresee is really correct. I will represent this result to the readers:

*Result 1:* In ¡ or ¡ $^+$, when two values $M_1=a+b+c, M_2=ab+bc+ca$ are fixed then $M_3=abc$ get it maximum value when $(a,b,c)=(x,x,y)$ or its permutation then $x\le y$, and if $M_3$ get it minimum value when $(a,b,c)=(z,z,t)$ or its permutation then $z\ge t$.

***Proof***

We will consider the case when $M_3$ get its maximum or minimum value when there are two equal variables and refer to the problem:

$M_1=2x+y$ and $M_2=x^2+2xy$. Find the maximum value of: $M_3=x^2y$.

Replace $y=M_1-2x$, we will have: $3x^2-2M_1x+M_2=0$ and

$$M_3=x^2\left(M_1-2x\right)=-2x^3+M_1x^2=\frac{2\left(3M_2-M_1^2\right)x+M_1M_2}{9} \text{ (notice that } 3M_2-M_1^2\le 0\text{)}$$

Therefore, the maximum and minimum value of $M_3$ depends only the maximum and minimum value of $x$. Here more specify is that $M_3$ will get its maximum and minimum value when $x$ get minimum and maximum values in order:

$$x=\frac{M_1-\sqrt{M_1^2-3M_2}}{3}\le\frac{M_1}{3}\le y \text{ and } x=\frac{M_1+\sqrt{M_1^2-3M_2}}{3}\ge\frac{M_1}{3}\ge y.$$

With a same idea, we also hope for this result:

*Result 2:* In $\mathbb{R}^+$, when two values $M_1 = a+b+c, M_3 = abc$ are fixed then the value $M_2 = ab+bc+ca$ get its maximum value when $(a,b,c) = (x,x,y)$ or its permutation then $x \le y$, get its minimum value when $(a,b,c) = (z,z,t)$ or its permutation then $z \ge t$

***Proof***

In the proof for result 1, we have obtained this equality:

$$M_3 = \frac{2\left(3M_2 - M_1^2\right)x + M_1 M_2}{9} \Leftrightarrow M_2 = \frac{9M_3 + 2M_1^2 x}{6x + M_1} = \frac{M_1^2}{3} + \frac{27M_3 - M_1^3}{6x + M_1}$$

Notice that in $\mathbb{R}^+$ then $27M_3 - M_1^3 \le 0$ so $M_2$ get its maximum and minimum values when $x$ get its minimum and maximum values in order. Also notice that $x$ is a nonnegative solution of the equation: $2x^3 - M_1 x^2 + M_3 = 0$. Assume that this equation has three solutions: $a \le 0 \le b \le c$.

So $M_2$ get its maximum and minimum values when $x$ get values $b$ and $c$ in order.

We will prove that $b \le \frac{M_1}{3} \le c$ to deduce that $b \le y \le c$.

Indeed, we have: $\frac{M_1}{2} = a+b+c \Rightarrow a = \frac{M_1}{2} - b - c$. Replace this value $a$ into the expression: $ab+bc+ca = 0$ we will obtain that: $(b+c)\left(\frac{M_1}{2} - b - c\right) + bc = 0 \Leftrightarrow \frac{M_1}{3} = \frac{2\left(b^2+c^2+bc\right)}{3(b+c)}$.

From here we can come out with a conclusion $b \le \frac{M_1}{3} \le c$ and also obtain what we want to have.

*Result 3:* In $\mathbb{R}^+$, when two values $M_2 = ab+bc+ca, M_3 = abc$ are fixed then the value $M_1 = a+b+c$ get its minimum value when $(a,b,c) = (x,x,y)$ or its permutation then $x \ge y$.

***Proof***

We have $M_1 = 2x + \frac{M_3}{x^2}$ where $x$ is solution of the equation $x^3 - M_2 x + 2M_3 = 0$. This equation has two positive and one negative. Let call them $a \le 0 \le b \le c$

We have: $a+b+c = 0 \Rightarrow a = -b-c$ ; $abc = -(b+c)bc = -2M_3 \Rightarrow bc(b+c) = M_3$

From here we can find out that: $b \le M_3 \le c$ or $b \le y \le c$.

Now we will see $M_1(b)$ or $M_1(c)$ are minimum value of $M_1$. We have:

$$M_1(b) - M_1(c) = 2(b-c) + \frac{M_3(c-b)(c+b)}{b^2c^2} = (b-c)\frac{2b^2c^2 - M_3(b+c)}{b^2c^2} = (b-c)\frac{2b^2c^2 - bc(b+c)^2}{b^2c^2} \ge 0$$

So $M_1$ get its minimum when $x = c \ge y$.

From these three result, we can upgrade out ABC Theorem as:

## ABC Generalization-Extra

Consider a symmetric $n$ variables function $f(a_1, a_2, ..., a_n)$, where $f$ has minimum and $n \geq 3$.

We will consider $f(a_1, a_2, ..., a_n)$ as a three variables function $g(a_1, a_2, a_3)$ and $a_4, a_5, ..., a_n$ are considered as constant.

If $g$ is ABC-able with monotonic property then inequality $f(a_1, a_2, ..., a_n) \geq 0$ can be referred to two cases:

$i)$ $n-1$ variables are equal.

$ii)$ One variable is equal to 0 (This condition only occurs when apply ABC-able for variable $abc$)

The inequality is completely proved if it is correct in two cases given.

* ABC-able with monotonic property means $g''$ for one of three values $abc, ab+bc+ca, a+b+c$ is 0.

***Proof***

According to the old *ABC Generalization* then we can refer to the problem when one variable is equal to 0, or *m* variable are equal to *a* and the rest $n-m$ variable are equal to *b*.

In the first case, if there is one variable equal to 0 then there is nothing much to say.

In the second case, we need to prove that $m \leq 1$. Indeed assume that $m \geq 2$.

Notice that if $g$ is a $ABC$-able function with monotonic property then the minimum value *f* only can be obtained when one of three values $abc, ab+bc+ca, a+b+c$ get its maximum or minimum values (do not need to consider both maximum and minimum). Let assume that *f* get its minimum value when $abc$ get its minimum (the rest cases are very similar).

WLOG, assume that $a \geq b$ and $x_1 = a, x_2 = a, x_3 = b$. Clearly when we fix all rest variables $x_4, x_5, ..., x_n$ then $(a, a, b)$ can not be the point for the function $f(x_1, x_2, x_3)$ get its minimum value; because as we proved in the results this minimum can be obtained at the point $(x, x, y)$ where: $x \leq y$.

This contradiction prove that $m$ can not be greater than 1, and therefore there are $n-1$ variables are equal.

So we have completely the proof for the upgrade of $ABC-Generalization$.

We will return the problem 3 that we leaved before.

Notice that in this problem we have used ABC-able for the monotonic property of $ab+bc+ca$. Therefore, we only to consider one case when there are $n-1$ variables are equal, and refer to the problem (which is the case when $n-1$ are equal):

$$f(x) = n^2\left[(n-1)x^n+1\right]^2 - n^3x^{n-1}\left[(n-1)x^n+1\right] - (n-1)\left(x^n-1\right)^2 \geq 0$$

$$\Leftrightarrow n^2\left[(n-1)x^n+1\right]\left[(n-1)x^n+1-nx^{n-2}\right]-(n-1)\left(x^n-1\right)^2\geq 0 \ (*)$$

Notice two equality: $x^n-1=(x-1)\left(x^{n-1}+x^{n-2}+..+1\right)$

and $(n-1)x^n+1-nx^{n-2}=(x-1)^2\left[(n-1)x^{n-2}+(n-2)x^{n-3}+...+2x+1\right]$. Therefore:

$$(*)\Leftrightarrow (x-1)^2\left[n^2\left[(n-1)x^n+1\right]\left[(n-1)x^{n-2}+(n-2)x^{n-3}+...+2x+1\right]-(n-1)\left(x^{n-1}+x^{n-2}+...+1\right)^2\right]\geq 0$$

Notice that: $\left[(n-1)x^n+1\right]\left(\frac{1}{n-1}+1\right)\geq\left(x^{\frac{n}{2}}+1\right)^2$

$$\left[(n-1)x^{n-2}+(n-2)x^{n-3}+...+2x+1\right]\left(\frac{1}{n-1}+\frac{1}{n-2}+...+\frac{1}{2}+1\right)\geq\left(x^{\frac{n-2}{2}}+x^{\frac{n-3}{2}}+...+x^{\frac{1}{2}}+1\right)^2$$

Therefore: $n^2\left[(n-1)x^n+1\right]\left[(n-1)x^{n-2}+(n-2)x^{n-3}+...+2x+1\right]$

$$\geq\frac{n^2}{\left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{2}+...+\frac{1}{n-2}+\frac{1}{n-1}\right)}\left(x^{\frac{n}{2}}+1\right)^2\left(x^{\frac{n-2}{2}}+x^{\frac{n-3}{2}}+..+x^{\frac{1}{2}}+1\right)^2$$

$$\geq\frac{n(n-1)}{1+\frac{1}{2}+...+\frac{1}{n-1}}\left(1+...+x^{n-2}+x^{n-1}\right)^2$$

$$\geq\frac{n(n-1)}{n}\left(1+...+x^{n-2}+x^{n-1}\right)^2=(n-1)\left(1+...+x^{n-2}+x^{n-1}\right)^2$$

$$\Rightarrow\left[n^2\left[(n-1)x^n+1\right]\left[(n-1)x^{n-2}+(n-2)x^{n-3}+...+2x+1\right]-(n-1)\left(x^{n-1}+x^{n-2}+...+1\right)^2\right]\geq 0$$

So the proof was completed.

**Problem 4.** Given $a,b,c,d>0$ satisfy $abcd=1$. Prove that

$$2^8\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right)\left(1+d^2\right)\leq(a+b+c+d)^6$$

***Solution***

Let considering $f(a,b,c)=2^8\left(1+a^2\right)\left(1+b^2\right)\left(1+c^2\right)\left(1+d^2\right)-(a+b+c+d)^6$

$$=2^8\left[1+(a+b+c)^2+(ab+bc+ca)^2-2(ab+bc+ca)-2abc(a+b+c)-a^2b^2c^2\right]\left(1+d^2\right)-(a+b+c+d)^6$$

If we consider above function with the variable $ab+bc+ca$ then the inequality $f(a,b,c)\leq 0$ is $ABC$-able. Thus we just need to consider the following problem:

Given $x$, $y>0$ satisfy $x^3y=1$. Prove that $2^8\left(1+x^2\right)^3\left(1+y^2\right)\leq(3x+y)^6$

The inequality $\Leftrightarrow 2^8\left(1+x^2\right)^3\left(1+\frac{1}{x^6}\right)\leq\left(3x+\frac{1}{x^3}\right)^6$

**Problem 5.** Give positive numbers $a,b,c,d$ satisfying: $abcd=1$. Prove that

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}+\frac{1}{1+d^2}\ge 2 \qquad \textbf{\textit{(China MO)}}$$

***Solution***

Take $x=2-\frac{1}{1+d^2}>1$. We need to prove: $\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\ge x$

$$\Leftrightarrow (1+a^2)(1+b^2)+(1+b^2)(1+c^2)+(1+c^2)(1+a^2)\ge x(1+a^2)(1+b^2)(1+c^2)$$

$$\Leftrightarrow 1+2(a+b+c)^2+(ab+bc+ca)^2-4(ab+bc+ca)-2abc(a+b+c)$$

$$\ge x+x(a+b+c)^2+x(ab+bc+ca)^2-2x(ab+bc+ca)-2xabc(a+b+c)+xa^2b^2c^2$$

$$\Leftrightarrow (x-1)(ab+bc+ca)^2+(x-2)(a+b+c)^2+2(2-x)(ab+bc+ca)+2(1-x)abc(a+b+c)+x-1\le 0$$

The above inequality is $ABC-able$ with monotonic property when we consider the variable $ab+bc+ca$. We only need to consider the case:

Give positive numbers $x,y: x^3y=1$. Prove that: $\frac{3}{1+x^2}+\frac{1}{1+y^2}\ge 2$

$$\Leftrightarrow \frac{3}{1+x^2}+\frac{1}{1+\frac{1}{x^6}}\ge 2 \quad \Leftrightarrow \frac{3}{1+x^2}-\frac{3}{2}+\frac{x^6}{x^6+1}-\frac{1}{2}\ge 0$$

$$\Leftrightarrow \frac{3(1-x)(1+x)}{2(1+x^2)}+\frac{(x-1)(x^5+x^4+x^3+x^2+x+1)}{2(x^6+1)}\ge 0$$

$$\Leftrightarrow (x-1)\left[(x^2+1)(x^5+x^4+x^3+x^2+x+1)-3(x+1)(x^3+1)\right]\ge 0$$

$$\Leftrightarrow (x-1)^2(x+1)(x^2-x+1)(x^3+2x^2+4x+2)\ge 0$$

The equality occurs $\Leftrightarrow a=b=c=d=1$

**Problem 6 [Vasile MS 2005]** Give $n$ positive numbers satisfying: $a_1a_2...a_n=1$.

Prove that: $\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}+\frac{4n}{n+a_1+a_2+...+a_n}\ge n+2 \quad (1)$

***Solution***

$$(1)\Leftrightarrow \frac{a_1a_2+a_2a_3+a_3a_1}{a_1a_2a_3}+\frac{1}{a_4}+\frac{1}{a_5}+...+\frac{1}{a_n}+\frac{4n}{n+a_1+a_2+a_3+a_4+...+a_n}\ge n+2$$

Consider the variable $a_1a_2+a_2a_3+a_3a_1$ , clearly that it is $ABC-able$ with monotonous property, also we use the minimum value of $a_1a_2+a_2a_3+a_3a_1$ so the equality variables should be greater than the rest one. Therefore we only need to consider the inequality:

Give positive $x, y$ satisfying $x^{n-1}y = 1$ and $x \ge 1$, then:

$$\frac{n-1}{x} + \frac{1}{y} + \frac{4n}{n+(n-1)x+y} \ge n+2 \iff \frac{n-1}{x} + x^{n-1} + \frac{4n}{n+(n-1)x+\frac{1}{x^{n-1}}} \ge n+2$$

$$\iff \frac{x^n + n - 1 - nx}{x} \ge \frac{2\left[(n-1)x^n - nx^{n-1} + 1\right]}{nx^{n-1} + (n-1)x^n + 1}$$

$$\iff \frac{(x-1)^2\left[x^{n-2} + 2x^{n-3} + ... + n - 1\right]}{x} \ge \frac{2(x-1)^2\left[(n-1)x^{n-2} + (n-2)x^{n-3} + ... + 1\right]}{nx^{n-1} + (n-1)x^n + 1}$$

We need to prove that: $\dfrac{x^{n-2} + 2x^{n-3} + ... + n - 1}{x} \ge \dfrac{(n-1)x^{n-2} + (n-2)x^{n-3} + ... + 1}{nx^{n-1} + (n-1)x^n + 1}$

$$\iff \left(\frac{nx^{n-1} + (n-1)x^n + 1}{2x}\right)\left(x^{n-2} + 2x^{n-3} + ... + n - 1\right) \ge (n-1)x^{n-2} + (n-2)x^{n-3} + ... + 1$$

Notice that because $x \ge 1$ then $\dfrac{nx^{n-1} + (n-1)x^n + 1}{2x} \ge \dfrac{nx + (n-1)x + 1}{2x} \ge n - 1$. Hence:

LHS $\ge (n-1)\left(x^{n-2} + 2x^{n-3} + ... + n - 1\right) \ge (n-1)x^{n-2} + (n-2)x^{n-3} + ... + 1$

So we have the needed conclusion.

**Problem 7.** Given $x_1, x_2, ..., x_n > 0$ satisfying: $\dfrac{1}{x_1} + \dfrac{1}{x_2} + ... + \dfrac{1}{x_n} = n$. Prove that:

$$x_1 + x_2 + ... + x_n - n \le \mathrm{e}_{n-1}\left(x_1 x_2 ... x_n - 1\right) \text{ where } \mathrm{e}_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} < \mathrm{e}$$

***(Gabriel Dospinescu and Calin PoPa, MS, 1004)***

***Solution***

Notice that we can rewrite the problem as:

Give $x_1, x_2, ..., x_n > 0$ satisfying: $x_1x_2 + x_2x_3 + x_3x_1 = \left(n - \dfrac{1}{x_4} - .. - \dfrac{1}{x_n}\right)x_1x_2x_3$.

Prove that: $x_1 + x_2 + x_3 + ... + x_n - n \le \mathrm{e}_{n-1}\left(x_1x_2...x_n - 1\right)$ where $\mathrm{e}_{n-1} = \left(1 + \dfrac{1}{n-1}\right)^{n-1} < \mathrm{e}$

Hence if consider the inequality with variable $x_1 + x_2 + x_3$, the inequality is $ABC - able$.

Therefore our work is only this problem: Give positive numbers $a, b : \dfrac{n-1}{a} + \dfrac{1}{b} = n$, then :

$$g(a,b) = (n-1)a + b - n \le \mathrm{e}_{n-1}\left(a^{n-1}b - 1\right) \ (*) \text{ where } \mathrm{e}_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} < \mathrm{e}$$

Replace $b=\dfrac{a}{na-n+1}$ into (*) , we need to prove that:

$$f(a)=(n-1)a+\frac{a}{na-n+1}-n\le\left(1+\frac{1}{n-1}\right)^{n-1}\left(\frac{a^n}{na-n+1}-1\right) \text{ where } a\ge\frac{n-1}{n}$$

$$\Leftrightarrow \frac{n(n-1)a^2-(2n^2-2n)a+n^2-n}{a^n-na+n-1}\le\left(\frac{n}{n-1}\right)^{n-1}$$

$$\Leftrightarrow \frac{n(n-1)(a-1)^2}{(a-1)^2\left(a^{n-2}+2a^{n-3}+...+(k-1)a^{n-k}+...+(n-1)a^0\right)}\le\left(\frac{n}{n-1}\right)^{n-1}$$

$$\Leftrightarrow \frac{n(n-1)}{a^{n-2}+2a^{n-3}+...+(n-1)a^0}\le\left(\frac{n}{n-1}\right)^{n-1}$$. Notice that: $a\ge\dfrac{n-1}{n}$.

Therefore: LHS $\le\dfrac{n(n-1)}{\left(\frac{n-1}{n}\right)^{n-2}+2\left(\frac{n-1}{n}\right)^{n-3}+...+(n-1)}=f\left(\dfrac{n-1}{n}\right)=\left(\dfrac{n}{n-1}\right)^{n-1}$ (**)

At (**), notice that $f\left(\dfrac{n-1}{n}\right)$, which is also the value of $g(a,b)$ when $a\to\dfrac{n-1}{n},b\to\infty$, or exactly is: $\left(\dfrac{n}{n-1}\right)^{n-1}$ (**).

**4. Proposed Problem**

**Problem 4.1.** [Nguyen Anh Cuong] Give $a,b,c,d>0$. Prove that:

$$\frac{a^4+b^4+c^4+d^4}{abcd}+12\ge(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

**Problem 4.2.** Give positive numbers $a_1,a_2,...,a_n$ satisfying: $\sum\limits_{1\le i<j\le n}a_ia_j=1$.

Prove that: $a_1^2+...+a_n^2+ka_1a_2...a_n\ge\dfrac{2}{n-1}+\min\left\{2,k\left(\dfrac{2}{n(n-1)}\right)^{\frac{n}{2}}\right\}$

**Problem 4.3.** Give $x_1,x_2,..,x_n$ satisfying $x_1^2+x_2^2+...+x_n^2=1$. Prove that:

$$x_1^3+x_2^3+...+x_n^3+\frac{6}{(n-2)\left(\sqrt{n}+1\right)}\sum x_1x_2x_3\le 1$$

**Problem 4.4.** [Vasile Cirtoaje, MS, 2004] Given $x_1,x_2,...,x_n>0$. Prove that:

$$\left(x_1+x_2+...+x_n-n\right)\left(\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}-n\right)+x_1x_2...x_n+\frac{1}{x_1x_2...x_n}\ge 2$$

**Problem 4.5.** [Vasile Cirtoaje, MS, 2006]

Give $x_1,x_2,...,x_n\ge 0$ satisfying: $x_1+x_2+...+x_n=n$. Prove that:

$$\left(x_1x_2...x_n\right)^{\frac{1}{\sqrt{n-1}}}\left(x_1^2+x_2^2+...+x_n^2\right)\le n$$

## VII. ABC CYCLIC

The fourth weak point and also quite a hot problem in these days is cyclic problem. For now, there are no method which is really effectible for cyclic inequality.

### 1. Cyclic to symmetric with variable transformation

For some cyclic inequality, we only need a variable transformation step to refer to a symmetric problem. Notice that the variable transformation also should have the cyclic property, because if we still keep the symmetric property then we can never refer a cyclic to symmetric. Let consider some example:

**Problem 1 [Bodan-Mathlinks]**

Give real numbers $x,y,z>0$. Prove that: $1\ge \frac{x}{2x+y}+\frac{y}{2y+z}+\frac{z}{2z+x}>\frac{1}{2}$ (1)

***Solution***

(1) $\Leftrightarrow$ $1\ge \frac{1}{2+\frac{y}{x}}+\frac{1}{2+\frac{z}{y}}+\frac{1}{2+\frac{x}{z}}>\frac{1}{2}$. Take $a=\frac{y}{x},b=\frac{z}{y},c=\frac{x}{z}\Rightarrow abc=1$.

Then the inequality becomes: $1\ge \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}>\frac{1}{2}$

The readers can also see immediately that when we transform the above inequality to polynomial form then it is still 3-degrees with $a,b,c$, therefore linear polynomial with variable $ab+bc+ca$. Therefore fix $a+b+c$ and $abc$, and we need to consider the only case when $(a-b)(b-c)(c-a)=0$:

***The problem becomes:*** Give $a,b>0$ and $a^2b=1$. Prove that: $1\ge \frac{2}{2+a}+\frac{1}{2+b}>\frac{1}{2}$

Replace $b=\frac{1}{a^2}$ and refer to the problem: $1\ge \frac{2}{2+a}+\frac{a^2}{2a^2+1}>\frac{1}{2}$

The left inequality is equivalent to: $a(a-1)^2\ge 0$

The right inequality is equivalent to: $4a^2-\frac{a}{2}+1>0\Leftrightarrow (2a-1)^2+\frac{7a}{2}>0$

So we have completed the proof. The equality occurs $\Leftrightarrow$ $x=y=z>0$

**Problem 2.** Give $a,b,c>0$. Prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{6abc}{a^2b+b^2c+c^2a}\ge 5$ (1)

***Solution***

Take $x=\frac{a}{b};y=\frac{b}{c};z=\frac{c}{a}\Rightarrow xyz=1$. Then (1) $\Leftrightarrow$ $x+y+z+\frac{6}{xy+yz+zx}\ge 5$ (2)

This is a simple problem with $ABC-extra$.

We refer problem to the only case $(x-y)(y-z)(z-x)=0$, assume that $y=z$

$\Rightarrow\ xy^2=1\Rightarrow x=\dfrac{1}{y^2}$. Then (2) $\Leftrightarrow\ x+2y+\dfrac{6}{y^2+2xy}\geq 5\ \Leftrightarrow \dfrac{1}{y^2}+2y+\dfrac{6y}{y^3+2}-5\geq 0$

$\Leftrightarrow \dfrac{\left[y(2y+3)(y-1)^2+y+2\right](y-1)^2}{y^2(y^3+2)}\geq 0$ (is always true). The equality occurs $\Leftrightarrow\ a=b=c>0$

* In the above problems, we have introduced some technique to transform a cyclic to symmetric problems using variable transformation, and then use ABC to handle the rest. Now we will introduce another solution for cyclic case. With this idea, we can handle almost cyclic polynomial of which degrees is less than or equal to 5 in ¡ .

We will consider some example:

**Problem 4.** Prove that: $3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a)\geq 0,\ \forall a,b,c$

***Solution***

Here we will deal with a stronger problem: $197(a^4+b^4+c^4)+280(a^3b+b^3c+c^3a)\geq 0$ (1)

Take $a=2x+7y;b=2y+7z;c=2z+7x$. The inequality (1) is equivalent to:

$$\begin{aligned}f(x,y,z)=222743(x^4+y^4+z^4)+240296(x^3y+y^3x+y^3z+z^3y+z^3x+zx^3)+\\+92904\left(x^2y^2+y^2z^2+z^2x^2\right)+246960xyz(x+y+z)\geq 0\end{aligned}$$

$f(x,y,z)$ is a symmetric polynomial, more than that it also a homogenous polynomial of which degree is four. We only need to consider one case:

$f(x,1,1)\geq 0\Leftrightarrow 222743x^4+480592x^3+432768x^2+974512x+1018982\geq 0$

which is no longer difficult and will be remain for the readers.

This solution using ABC which is quite clear and nice, however the mystery inside it is that we have transform variable to refer to a symmetric problem $a=2x+7y;b=2y+7z;c=2z+7x$. It is not quite trivial to go to this transformation; we need some small testing and convention to find out. Our target is that we will make sure that all coefficients for cyclic expression will be equal after the transformation. More specify here we take: $a=x+ky,b=y+kz,c=z+kx$ and find $k$ so that two coefficient of two cyclic expression $x^3y+y^3z+z^3x,xy^3+yz^3+zx^3$ are equal. For the general problem: $x^4+y^4+z^4+m\left(x^3y+y^3z+z^3x\right)\geq 0$, we need to find $k$ such that: $mk^3-(m+4)k^2+(4-2m)k+m=0$.

For some case to find a good coefficient for the cyclic pair expression which as $x^4y+y^4z+z^4x$, $xy^4+yz^4+zx^4$ and $x^3y^2+y^3z^2+z^3x^2,x^2y^3+y^2z^3+z^2x^3$ we also need to take:

$a=x+my+nz,b=y+mz+nx,c=z+mx+ny$

So for now the readers might also guess that why this idea is only effectible for polynomial of which degrees are not greater than 5, from 6-degrees or above we have more than two cyclic expressions , for example with 6-degrees is $x^5y+y^5z+z^5x, xy^5+yz^5+zx^5$,

$x^4y^2+y^4z^2+z^4x^2, x^2y^4+y^2z^4+z^2x^4$ and $xyz(x^2y+y^2z+z^2x), xyz(xy^2+yz^2+zx^2)$

so we may not find a good system to find out $m,n$.

We will consider some another example to understand better this technique:

**Problem 5 [MnF].** Give positive numbers $a,b,c$. Prove that:

$$(13a^2-10ab-5b^2+9c^2)(a-b)^2+(13b^2-10bc-5c^2+9a^2)(b-c)^2+(13c^2-10ca-5a^2+9b^2)(c-a)^2 \geq 0$$

***Solution***

The inequality is equivalent to: $\left[(a-2b)(a-4b)-(b-2c)(b-4c)\right]^2+$

$$+\left[(b-2c)(b-4c)-(c-2a)(c-4a)\right]^2+\left[(c-2a)(c-4a)-(a-2b)(a-4b)\right]^2 \geq 0$$

This problem is a challenge problem more than a nice problem. Its author has tried for this inequality have two separated equality point $a=b=c$ and $a=2b=4c$. However if we have known about ABC-cyclic then this is a good problem to apply:

Take $a=2x-y, b=2y-z, c=2z-x$. The inequality is equivalent to:

$$f(x,y,z)=4\left(x^4+y^4+z^4\right)+21\left(x^2y^2+y^2z^2+z^2x^2\right)$$
$$-10\left(x^3y+xy^3+y^3z+yz^3+z^3x+zx^3\right)-5xyz(x+y+z)\geq 0$$

This is a fourth degrees polynomial and also a homogeneous one in $\Upsilon$, we only need to consider one case: $f(x,1,1)=4x^4-20x^3+37x^2-30x+9=(x-1)^2(2x-3)^2\geq 0$.

This problem is more special than the problem 3 in the point that it has two separate equality point . Sometime this case make it easer to find out the $k$ .

For example we assume that: $a=x+ky, b=y+kz, c=z+kx$. When the maximum or minimum occurs we should have two variable in $x,y,z$ equal; let assume that $y=z=1$ and then:

$a=x+k, b=k+1, c=kx+1$. From here, combine with this system:

$a=2b$ and $a=4c$ we can easily find out that $k=\frac{-1}{2}, x=\frac{3}{2}$ or $k=-1, x=1$.

From here, try in the inequality and we come out with the transformation given in the solution.

In above examples, all the problem has the domain is ¡ , however almost the problem will be in ¡ $^+$, and more specify is that if its degree is odd then we must consider in ¡ $^+$. So should we surrender in these cases? Actually it is not, let consider this example:

**Problem 6 [Nguyen Anh Cuong]:** Given $a,b,c \geq 0$. Prove that

$$23\left(a^3+b^3+c^3\right)+17\left(a^2b+b^2c+c^2a\right) \geq 37\left(ab^2+bc^2+ca^2\right)+9abc$$

***Solution***

We again use the idea taking $a=x+ky, b=y+kz, c=z+kx$. For this problem, the equality occurs in two separate points, and one of these can be specified in the case $a=-1, b=1, c=2$ and its permutation. In this case in three variables $x, y, z$ , there are two variable should be equal; let assume that $y=z$.

Then we will have the system: $x+ky=-1; (k+1)y=1; kx+y=2$. From here we can find out $k$ through this equation: $2k^2+3k+1=0$, solve it to obtain $k=-1$ or $k=-\frac{1}{2}$.

Choose $k=-\frac{1}{2}$, and take: $a=2x-y, b=2y-z, c=2z-x$ and the problem will become: $x^3+y^3+z^3+3xyz \geq xy(x+y)+yz(y+z)+zx(z+x)$ which is Shur inequality, can be solver by many ways or ABC. However the above inequality only valid for $x, y, z \geq 0$. Fortunately, $x=\frac{4a+2b+c}{7}, y=\frac{a+4b+2c}{7}, z=\frac{2a+b+4c}{7}$ are non-negative so the proof should be end here.

The variable transformation idea is really a good point to remember, however it is not a perfect solution anyway. Following we will introduce a more advance idea which should be perfect in some meanings.

**2. The connection between cyclic and transformation**

The readers may sometimes ask yourself if the symmetric is only "the sun" of cyclic inequality, because if the cyclic inequality is correct than we can refer to the symmetric one easily. This is correct in some meaning, for example this problem:

$\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}+\sqrt{3} \geq \sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}+\sqrt{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}$ is correct, then the following symmetric problem is also correct: $\sqrt{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}}+\sqrt{\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}}+2\sqrt{3} \geq 2\left(\sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}+\sqrt{\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\right)$

However the reverse question, if the symmetric one is correct then how about the cyclic one. A quick thinking come out with the answer is NO. For example:

$a^3+b^3+c^3+3abc \ge ab(a+b)+bc(b+c)+ca(c+a)$ is correct but its cyclic inequality is not correct: $a^3+b^3+c^3+3abc \ge 2(a^2b+b^2c+c^2a)$ is not valid for $(a,b,c)=(0,3,2)$.

However I claim that every cyclic inequality has its equal symmetric. And it is enough to solve the equal symmetric form to prove that the cyclic one is also correct. I will represent this technique and its application as following:

**SYMMETRIC TRANSFORMATION**

***Symmetric transformation technique for cyclic:***

Give a function with variable $a,b,c$ satisfying: $f(a,b,c)=f(b,c,a)=f(c,a,b)$.

If we take $g(a,b,c)=f(a,b,c)f(a,c,b)$; $h(a,b,c)=f(a,b,c)+f(a,c,b)$ then $g,h$ are symmetric inequality. Also the inequality $f(a,b,c)\ge 0$ are equivalent to: $\begin{cases} g(a,b,c)\ge 0 \\ h(a,b,c)\ge 0 \end{cases}$

The idea seem simple but let see its power when we combine it with $ABC$ :

**Problem 6.** Give $a,b,c\ge 0$ satisfying: $a+b+c=3$. Prove that: $P=a^2b+b^2c+c^2a\le 4$

***Solution***

Take: $f(a,b,c)=a^2b+b^2c+c^2a-4$

And: $g(a,b,c)=f(a,b,c)f(a,c,b)=9C^2+(39-18B)C+B^3-12B+16$

$$h(a,b,c)=f(a,b,c)+f(a,c,b)=ab(a+b)+bc(b+c)+ca(c+a)-8$$

Where: $B=ab+bc+ca; C=abc$. What we need to do is prove:

i) $g(a,b,c)\ge 0$ ii) $h(a,b,c)\le 0$

We will prove the second one first, since this is quite simple. The readers can also see that $h(a,b,c)\le 0$ very quickly using $ABC$.

However here we suggest a shorter solution using Schur:

$$a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$$

$$\Leftrightarrow a^3+b^3+c^3+6abc+3[ab(a+b)+bc(b+c)+ca(c+a)]\ge 3abc+4[ab(a+b)+bc(b+c)+ca(c+a)]$$

$$\Rightarrow (a+b+c)^3\ge 4[ab(a+b)+bc(b+c)+ca(c+a)]$$

$$\Leftrightarrow ab(a+b)+bc(b+c)+ca(c+a)\le \frac{(a+b+c)^3}{4}=\frac{27}{4}<8 \Rightarrow h(a,b,c)\le 0$$

Now the heavier part is the second part, in this part I have represented it in the form $ab+bc+ca, abc$ to apply $ABC$ for it. Here we will try to prove that $g$ is a monotonic function with variable $C$ by using derivative and prove that the derivative is positive or negative. Let consider it:

Take: $\omega(C)=9C^2+(39-18B)C+B^3-12B+16=9C^2+(39-18B)C+(B-2)^2(B+4)$

$$\omega'(C)=18C+39-18B$$

First notice that if $B\le\frac{39}{18}$ then everything is good for now, because $39-18B\ge0$ so $\omega(\mathrm{C})\ge0$. Hence just consider the case when $B>\frac{39}{18}$.

Notice that for every fixed value of $B$, $\omega'(C)=18C+39-18B=0$ has solution $C=\frac{18B-39}{18}$, or has no solution.

For the first case, we have: $\omega(\mathrm{C})\ge\omega\left(\frac{18B-39}{18}\right)=B^3-9B^2+27B-\frac{105}{4}$.

And now it is not so difficult to prove that $B^3-9B^2+27B-\frac{105}{4}\ge0$ for $B>\frac{39}{18}$

In the case $\omega'(C)=18C+39-18B=0$ has no solution, means $w(C)$ is monotonous, we refer to the original problem with the case $(a-b)(b-c)(c-a)=0$ or $abc=0$.

**i)** Given $a,b\ge0$ satisfying: $a+b=3$, prove that: $a^2b\le4$

This can be done quickly using AM – GM inequality: $a^2b=4\cdot\frac{a}{2}\cdot\frac{a}{2}\cdot b\le4\left(\frac{a+b}{3}\right)^3=4$

**ii)** Given $a,b\ge0$ satisfying: $2a+b=3$, prove that: $a^2b+b^2a+a^3\le4$.

Replace $b=3-2a$, the inequality $\Leftrightarrow$ $3a^3-9a^2+9a-4\le0$ (is true for $0\le a\le\frac{3}{2}$)

We have completely finished our proof. This proof is not the best solution for this problem, but it should be an effectible solution for these kinds of problems.

A question come out immediately is that is $a^2b+b^2c+c^2a$ get it maximum and minimum value when there are two equal variable or one variable is equal 0 when we fix $a+b+c, ab+bc+ca$ ? This is a very possible foresee, because for the above problem the equality occurs when there is a zero variable. Let consider this problem:

**Problem 7 [Nguyen Anh Cuong].** Give positive numbers $a,b,c$ satisfying:

$a^2+b^2+c^2=2(ab+bc+ca)$. Find the maximum value of: $P=\dfrac{8(a^2b+b^2c+c^2a)}{(a+b+c)^3}$.

***Solution***

The first remark is that this is a homogeneous cyclic inequality. We will use these properties step by step to come out with final solution:

Firstly, due to the homogeneous property, we can take $ab+bc+ca=1$.

*The problem becomes:* Give positive numbers $a,b,c$ satisfying: $a+b+c=2, ab+bc+ca=1$.

Find the maximum value of $P=a^2b+bc^2+ca^2$ .

Still the same idea to the first problem, we will take $Q=ab^2+bc^2+ca^2$. However here the problem ask us to find the maximum and minimum value ourselves so it is hard to define $g$ and $h$. However this can not make us surrender, we still go on with this idea.

Take $t=abc$, as what we proved for the first part of this article, we have $t\in\left[0,\dfrac{4}{27}\right]$.

We have: $P+Q=a^2b+b^2c+c^2a+ab^2+bc^2+ca^2=(a+b+c)(ab+bc+ca)-3abc=2-3t$

$$PQ=9(abc)^2+\left[(a+b+c)^3-6(a+b+c)(ab+bc+ca)\right]c+(ab+bc+ca)^3=9t^2-4t+1$$

$P,Q$ can obtain one of these two values $\dfrac{2-3t-\sqrt{-27t^2+4t}}{2};\dfrac{2-3t+\sqrt{-27t^2+4t}}{2}$.

To find out the maximum value, we should consider the case:

$P=f(t)=\dfrac{2-3t+\sqrt{-27t^2+4t}}{2}$ for $t\in\left[0,\dfrac{4}{27}\right]$. We have $f'(t)=0$ have the unique root $t=\dfrac{1}{27}$.

Therefore $f_{\max}=\max\left\{f(0),f\left(\dfrac{1}{27}\right),f\left(\dfrac{4}{27}\right)\right\}$. From here we have $f_{\max}=\dfrac{10}{9}$

when $abc=\dfrac{1}{27}$, when $a=0.04020492...;b=0.78243211...;c=1.17736297...$

Hence our speculation was wrong. How pity is it ! We can not find out a beautiful thing such as the symmetric world. Indeed the equality for cyclic inequality is always a troublesome but also very amazing and joyful. You can also think more about this article to find out that why almost cyclic inequalities get its maximum or minimum values when three variables are completely different. We will stop the research for now; following is some example and application problems using the symmetric transformation technique.

**Problem 8 [Nguyen Anh Cuong]** Give nonnegative $a,b,c$ satisfying $a+b+c=4$.

Prove that: $3(a^2b+b^2c+c^2a)+36\ge 5(ab^2+bc^2+ca^2)$

***Solution***

Take $f(a,b,c)=3\left(a^2b+b^2c+c^2a\right)-5\left(ab^2+bc^2+ca^2\right)+36$

And: $h(a,b,c)=f(a,b,c)+f(a,c,b)=72-2\left(ab(a+b)+bc(b+c)+ca(c+a)\right)$

$g(a,b,c)=f(a,b,c).f(a,c,b)=441C^2+(4312-1176B)C+64B^3-240B^2-288B+1296$

We will again use the same technique as the problem 7 to solve this problem.

Firstly prove that $h(a,b,c)\ge 0$. As the first problem,

$$ab(a+b)+bc(b+c)+ca(c+a)\le \frac{(a+b+c)^3}{4}=16<36 \text{ hence } h(a,b,c)\ge 0.$$

Secondly is proving $g(a,b,c)\ge 0$.

Rewrite $g(a,b,c)$ as $\omega(C)=441C^2+(4312-1176B)C+(B-3)^2(64B+144)$

So if $B\le \frac{4312}{1176}=\frac{11}{3}$ then we will have what we need immediately.

Consider $B>\frac{11}{3}$, we have: $\omega'(C)=882C+4312-1176B$, so using the same idea with the problem 7 we need to prove three things:

**i)** $\omega\left(\frac{12B-44}{9}\right)=64B^3-1024B^2+\frac{16384}{3}B-\frac{83200}{9}\ge 0$ when $B\ge\frac{11}{3}$:

this is quite easy and we will skip it (this is the case where there is no equality)

**ii)** Give $a,b\ge 0$ satisfying: $2a+b=4$. Prove that:

$$3\left(a^2b+b^2a+a^3\right)+36\ge 5\left(a^2b+b^2a+a^3\right)\Leftrightarrow 18\ge a^3+a^2b+b^2a$$

Replace $b=4-2a$ and we have: $3a^3-12a^2+16a-18\le 0$ where: $0\le a\le 2$, again remain for the readers (this is the case where there is no equality)

**iii)** Give $a,b\ge 0$ satisfying: $a+b=4$. Prove that: $3a^2b+36\ge 5ab^2$.

Replace $a=4-b$ and we have: $8b^3-44b^2+48b+36\ge 0\Leftrightarrow (b-3)^2(8b+4)\ge 0$

In conclusion, the inequality was proved completely.

The equality occurs when $(a,b,c)=(0,3,1)$ and its cyclic permutation.

**Problem 9.** [*Vasile Cirtoaje*]

Give real numbers: $a,b,c$ . Prove that: $\left(a^2+b^2+c^2\right)^2 \geq 3\left(a^3b+bc^3+c^3a\right)$

***Solution***

A very famous problem of Vasile Cirtoaje, a very amazing result in cyclic inequality world , a very strong result with the equality occurs at three separate points. And let see how the power of symmetric transformation versus this powerful problem.

Firstly, because of its homogeneous property we need to have this condition $a+b+c=1$.

Take $f(a,b,c)=\left(a^2+b^2+c^2\right)^2-3\left(a^3b+b^3c+c^3a\right)$ and

$$g(a,b,c)=f(a,b,c)+f(a,c,b)=2\left(a^2+b^2+c^2\right)^2-3\left[ab\left(a^2+b^2\right)+bc\left(b^2+c^2\right)+ca\left(c^2+a^2\right)\right]$$

$$h(a,b,c)=f(a,b,c)f(a,c,b)=63C^2+(21B^2-57B+12)C+49B^4-68B^3+42B^2-11B+1$$

Proving $g(a,b,c)\geq 0$ is in the border of $ABC$ and will be remained for the interested readers.

Here we will give the proof for $h(a,b,c)\geq 0$.

Take $\omega(c)=63c^2+\left(21b^2-57b+12\right)c+49b^4-68b^3+42b^2-11b+1$

We have: $\omega'(c)=126c+21b^2-57b+12$ . We will consider three cases:

**i)** $\omega\left(\dfrac{-21b^2+57b-12}{126}\right)=\dfrac{3969b^4-4914b^3+2277b^2-468b+36}{84}=\dfrac{\left(63b^2-39b+6\right)^2}{84}\geq 0$

**ii)** Two equal variables: Give the real numbers $x, y$ . Prove that: $\left(2x^2+y^2\right)^2\geq 3(x^4+x^3y+y^3x)$

Give $x=1$ and we refer to the problem: $y^4-3y^3+4y^2-3y+1\geq 0 \Leftrightarrow (y-1)^2\left(y^2-y+1\right)\geq 0$

**iii)** One variable is equal to 0: Give the real numbers $x, y$ . Prove that: $\left(x^2+y^2\right)^2\geq 3x^3y$

This can be done quickly using AM – GM inequality: $\left(\dfrac{1}{3}x^2+\dfrac{1}{3}x^2+\dfrac{1}{3}x^2+y^2\right)^2\geq \dfrac{16}{\sqrt{27}}x^3y>3x^3y$

Therefore the problem has been proved completely.

A remark about the power of this technique. Why this technique is powerful? The answer is that because the equivalent through every transformation. The only trouble point is that with which value of $b$ then $\omega'(c)$ will has solution or not. Actually this also can be solved by ABC, because the derivative expression is also a linear polynomial with abc . This remark is to see that, in some meanings, this technique is a possible approach to solve almost cyclic polynomial inequalities no matter what.

**Proposed Problems**

**Problem 5.1** *[VĐQ-MnF]* Give $a,b,c>0$. Find the maximum value of $k$ such that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{kabc}{(a+b)(b+c)(c+a)}\geq 3+\frac{k}{8}$$

**Problem 5.2**. Find the maximum value of $k$ such that:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+3k\geq k\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \text{ valid for } \forall a,b,c>0.$$

**Problem 5.3** Give positive $a,b,c$ and a positive real number p. Prove that :

$$\left(\frac{a}{a+pb}\right)^2+\left(\frac{b}{b+pc}\right)^2+\left(\frac{c}{c+pa}\right)^2\geq \min\left(1,\frac{3}{(1+p)^2}\right)$$

**Problem 5.4** *[Nguyen Anh Cuong]*

Give nonnegative real numbers $a,b,c$ such that: $a+b+c=5$. Prove that:

$$16\left(a^3b+b^3c+c^3a\right)+640\geq 11\left(ab^3+bc^3+ca^3\right)$$

**Problem 5.5** *[Vasile Cirtoaje]*

Prove that: $4(a+b+c)^3\geq 27\left(ab^2+bc^2+ca^2+abc\right)$ , $\forall a,b,c\geq 0$

**Problem 5.6.** Give $a,b,c>0$ satisfying: $abc=1$.

Prove that: $a^2b+b^2c+c^2a\geq abc$

**Problem 5.7** *[Nguyen Anh Cuong]*

i) Give nonnegative numbers $a,b,c$ satisfying: $a+b+c=3$.

Prove that: $(ab+bc+ca)\left(a^2b+b^2c+c^2a\right)\leq 9$

ii) Give nonnegative $a,b,c$ satisfying: $ab+bc+ca=3$.

Prove that: $(a+b+c)\left(a^2b+b^2c+c^2a\right)\geq 9$

**Problem 5.8** *[Nguyen Anh Cuong]*

Give nonnegative real numbers $x,y,z$. Prove that:

i) $2\left(x^5+y^5+z^5\right)+9\left(x^3y^2+y^3z^2+z^3x^2\right)\geq 11\left(x^2y^3+y^2z^3+z^2x^3\right)$

ii) $2\left(x^5+y^5+z^5\right)+x^4y+y^4z+z^4z\geq 3\left(xy^4+yz^4+zx^4\right)$

**3. Cyclic to Symmetric using auxiliary inequality**

In two previous parts, we have been familiar with cyclic inequalities and we solve it directly in many ways. That is truly the strongest problems. However in these days people do not always like it. We can use auxiliary inequalities to transform a cyclic inequality to a symmetric inequality to apply ABC for it.

To make this idea clearer, let consider this example:

**Problem 11 [Nguyen Anh Cuong]** Give positive numbers $a,b,c$ satisfying $a+b+c=3$.

Prove that: $\left(a^2b+b^2c+c^2a\right)(ab+bc+ca)\le 9\sqrt{\frac{a^2+b^2+c^2}{3}}$

***Solution***

Firstly let prove the auxiliary inequality:

Give positive numbers $a,b,c$ satisfying: $a+b+c=3$. Then: $a^2b+b^2c+c^2a\le\frac{9}{ab+bc+ca}$.

This is actually the problem given in the symmetric transformation part, the poof is remained for the readers. Here we will apply this inequality to transform the first cyclic problem to a symmetric problem. So $\left(a^2b+b^2a+c^2a\right)(ab+bc+ca)\le\frac{9}{ab+bc+ca}(ab+bc+ca)=9$

Our work is prove that: $9\sqrt{\frac{a^2+b^2+c^2}{3}}\ge 9$.

This one is trivial by using: $3\left(a^2+b^2+c^2\right)\ge(ab+bc+ca)^2$.

And we can end the proof here.

**Problem 12 [Nguyen Anh Cuong]** Give positive problems $a,b,c$ satisfying: $a+b+c=3$.

Prove that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\sqrt{\frac{a^2+b^2+c^2}{3}}+2$

***Solution***

This time we still use the same ideas, we need to have a good lower boundary for $ab^2+bc^2+ca^2$.

Also in the proposed problems in symmetric transformation part we have this result:

Give nonnegative $a,b,c$ satisfying: $ab+bc+ca=3$. Then $\left(a+b+c\right)(ab^2+bc^2+ca^2)\ge 9$.

Or rewrite as a homogeneous form is that: $(a+b+c)\left(ab^2+bc^2+ca^2\right)\ge(ab+bc+ca)^2$.

So with the condition $a+b+c=3$ we have this inequality: $ab^2+bc^2+ca^2\ge\frac{\left(ab+bc+ca\right)^2}{3}$

We only need to prove that: $\frac{(ab+bc+ca)^2}{3abc}\ge\sqrt{\frac{a^2+b^2+c^2}{3}}+2$

Consider the problem with variable $abc$, clearly this inequality is $ABC-able$.

We only need to consider two cases:

***Case 1:*** $c=0$, the inequality is trivially correct.

***Case 2:*** $a=b=x, c=y$, the inequality is equivalent to:

$\frac{(x^2+2xy)^2}{3x^2y} \ge \sqrt{\frac{2x^2+y^2}{3}}+2 \Leftrightarrow \frac{(x+2y)^2}{3y} \ge \sqrt{\frac{2x^2+y^2}{3}}+2$ where $x, y$ are positive real numbers satisfying $2x+y=3$. Replace $y=3-2x, x\in(0,1.5)$ we have:

$$\frac{(6-3x)^2}{3(3-2x)} \ge \sqrt{\frac{2x^2+(3-2x)^2}{3}}+2 \Leftrightarrow \frac{3(2-x)^2}{3-2x}-3 \ge \sqrt{2x^2-4x+3}-1$$

$$\Leftrightarrow \frac{3(x-1)^2}{3-2x} \ge \frac{2(x-1)^2}{\sqrt{2x^2-4x+3}+1} \Leftrightarrow \left(3\sqrt{2(x-1)^2+1}+4x-3\right)(x-1)^2 \ge 0$$

The proof can end here.

***Remarks:*** Through these two problems, the readers may also understand what that idea for this method is. We will try to find good upper and lower boundary for cyclic expression using (can use symmetric transformation) to transfer a cyclic problem to a weaker but nicer to handle symmetric problem.

Let review two evaluation we have used in above two problems:

$$\frac{(ab+bc+ca)^2}{a+b+c} \le a^2b+b^2c+c^2a \le \frac{(a+b+c)^5}{27(ab+bc+ca)} (*)$$

And another evaluation, which is exercise 5.02 and 5.03 in the symmetric transformation part.

$$(a+b+c)\sqrt[3]{a^2b^2c^2} \le a^2b+b^2c+c^2a \le \frac{4}{27}(a+b+c)^3-abc \quad (**)$$

Also notice that we refer to the symmetric problem simply because we want to use $ABC$ to refer to the case where one variable is 0 or two variables are equal.. Therefore let consider (*) and (**) in these two cases:

- When $c=0, b=1$, $(*) \Leftrightarrow \frac{a^2}{a+1} \le a^2 \le \frac{(a+1)^5}{27a}$ and $(**) \Leftrightarrow 0 \le a^2 \le \frac{4(a+1)^3}{27}$

Notice that $0 \le \frac{a^2}{a+1} \le a^2 \le \frac{4(a+1)^3}{27} \le \frac{(a+1)^5}{27a}$, therefore in the case our inequality very near to 0 when our variable reach 0 this evaluation should be:

$$\frac{(ab+bc+ca)^2}{a+b+c} \le a^2b+b^2c+c^2a \le \frac{4}{27}(a+b+c)^3-abc$$

• When $b=c=1$, $(*) \Leftrightarrow \frac{(2a+1)^2}{a+2} \le a^2+a+1 \le \frac{(a+2)^5}{27(2a+1)}$ and

$(**) \Leftrightarrow (a+2)\cdot\sqrt[3]{a^2} \le a^2+a+1 \le \frac{4}{27}(a+2)^3 - a$

Notice that $(a+2)\cdot\sqrt[3]{a^2} \le \frac{(2a+1)^2}{a+2} \le a^2+a+1 \le \frac{(a+2)^5}{27(2a+1)} \le \frac{4}{27}(a+2)^3 - a$ when $a \le 1$

And $\frac{(2a+1)^2}{a+2} \le (a+2)\sqrt[3]{a^2} \le a^2+a+1 \le \frac{4}{27}(a+2)^3 - a \le \frac{(a+2)^5}{27(2a+1)}$ when $a \ge 1$.

Therefore for a particular inequality, we may prefer the different evaluation to get the most effectible inequality. However remember that we should use in pair to have the best results.

**i)** For inequalities have equality occurs when three variable are equal and reach very near zero when there is a variable reach 0: $\frac{(ab+bc+ca)^2}{a+b+c} \le a^2b+b^2c+c^2a \le \frac{4}{27}(a+b+c)^3 - abc$

**ii)** For inequalities has equality occurs when three variable are equal and when apply ABC using the minimum value of $abc$ or $ab+bc+ca$ $(a \le b = c)$:

$$\frac{(ab+bc+ca)^2}{a+b+c} \le a^2b+b^2c+c^2a \le \frac{(a+b+c)^5}{27(ab+bc+ca)}$$

**iii)** For inequalities has equality occurs when three variable are equal and when apply ABC using the maximum value of $abc$ or $ab+bc+ca$ $(a \ge b = c)$

$$(a+b+c)\sqrt[3]{a^2b^2c^2} \le a^2b+b^2c+c^2a \le \frac{4}{27}(a+b+c)^3 - abc$$

The reasons above are also the way for the readers find out the best evaluation for yourself when you obtain a new evaluation.

Beside that we also have a very interesting evaluation based on Vasile inequality which was mentioned in the symmetric transformation part.

Given real numbers $a,b,c$. Then we have the inequality: $3(a^3b+b^3c+c^3a) \le (a^2+b^2+c^2)^2$

This inequality doesn't evaluate directly the expression $a^2b+b^2c+c^2a$, however we still can have its evaluation easily using the equality:

$$a^3b+b^3c+c^3a = (a+b+c)(a^2b+b^2c+c^2a) - abc(a+b+c) - a^2b^2 - b^2c^2 - c^2a^2$$

Therefore we have two good evaluations:

**iv)** $a^2b+b^2c+c^2a \le \frac{(a^2+b^2+c^2)^2 + 3abc(a+b+c) + 3(a^2b^2+b^2c^2+c^2a^2)}{3(a+b+c)}$

$$= \frac{(a+b+c) \ \ -4(a+b+c)^2(ab+bc+ca) + 7(ab+bc+ca)^2 - 3abc(a+b+c)}{3(a+b+c)}$$

These inequalities look really messy but are better than all the inequalities I have given in almost cases. Also despite of its shape, ABC can handle it quite easy because of there is only one $abc$ inside its form.

**Problem 12.** Give $a,b,c\ \ge 0$. Prove that: $\dfrac{12\left(a^2+b^2+c^2\right)}{(a+b+c)^2}+\dfrac{a^2b+b^2c+c^2a}{a^3+b^3+c^3}\ge 5$ (1)

***Solution***

$$(1)\Leftrightarrow \frac{12\left(a^2+b^2+c^2\right)}{(a+b+c)^2}+\frac{(a+b+c)(ab+bc+ca)-3abc-ab^2-bc^2-ca^2}{a^3+b^3+c^3}\ge 5$$

Apply the inequality: $a^2b+b^2c+c^2a\le\frac{4}{27}(a+b+c)^3-abc$ and we need to prove this inequality:

$$\frac{12\left(a^2+b^2+c^2\right)}{(a+b+c)^2}+\frac{(a+b+c)(ab+bc+ca)-2abc-\frac{4}{27}(a+b+c)^3}{a^3+b^3+c^3}\ge 5\quad (*)$$

The above problem can be handled using $ABC$ because it is actually a five degrees polynomial. We need to consider two cases:

**Case 1:** $b=1,c=0$, then: $(*)\Leftrightarrow\dfrac{12\left(a^2+1\right)}{(a+1)^2}+\dfrac{a(a+1)-\frac{4}{27}(a+1)^3}{a^3+1}\ge 5$

This is true since $\dfrac{2\left(a^2+1\right)}{(a+1)^2}\ge 1\Rightarrow\dfrac{12\left(a^2+1\right)}{(a+1)^2}\ge 6$ and $1+\dfrac{27a(a+1)-4(a+1)^3}{27(a^3+1)}\ge 0$

**Case 2:** $b=c=1$, then:

$$(*)\ \Leftrightarrow\frac{12\left(a^2+2\right)}{(a+2)^2}+\frac{(a+2)(2a+1)-2a-\frac{4}{27}(a+2)^3}{a^3+2}\ge 5$$

$$\Leftrightarrow\frac{12\left(a^2+2\right)}{(a+2)^2}-4\ge 1-\frac{-4a^3+30a^2+33a+22}{27\left(a^3+2\right)}$$

$$\Leftrightarrow\frac{8(a-1)^2}{(a+2)^2}\ge\frac{(31a+32)(a-1)^2}{27\left(a^3+2\right)}\Leftrightarrow\left(185a^3-156a^2-252a+304\right)(a-1)^2\ge 0$$

Therefore the problem has been solved.

***Remarks:*** in this problem when there is two equal variables, the most trouble case is when $a\ge 1$. Therefore apply the above inequality is reasonable. The value $12$ still can be reduced to smaller values, such as 10.5 … However for the following problem, we will need another convention.

**Problem 13 [Nguyen Anh Cuong]**

Give $a,b,c\ge 0$. Prove that: $\dfrac{7\sqrt{3\left(a^2+b^2+c^2\right)}}{(a+b+c)}+\dfrac{a^2b+b^2c+c^2a}{a^3+b^3+c^3}\ge 8$ (1)

***Proof***

$$(1) \Leftrightarrow \frac{7\sqrt{3\left(a^2+b^2+c^2\right)}}{a+b+c}+\frac{(a+b+c)(ab+bc+ca)-3abc-ab^2-bc^2-ca^2}{a^3+b^3+c^3} \ge 8$$

Using the inequality: $a^2b+b^2c+c^2a \le \frac{(a^2+b^2+c^2)^2+3abc(a+b+c)+3(a^2b^2+b^2c^2+c^2a^2)}{3(a+b+c)}$

And we need to prove that:

$$\frac{7\sqrt{3\left(a^2+b^2+c^2\right)}}{a+b+c}+\frac{3(a+b+c)^2(ab+bc+ca)-12abc(a+b+c)-\left(a^2+b^2+c^2\right)^2-3\left(a^2b^2+b^2c^2+c^2a^2\right)}{3\left(a^3+b^3+c^3\right)(a+b+c)} \ge 8$$

The inequality above is clearly $ABC-able$, we need to consider two cases :

**Case 1:** $c=0, b=1$, $a \le 1$ (because the inequality is still symmetric when $c=0$)

$$\frac{7\sqrt{3\left(a^2+1\right)}}{a+1}+\frac{3a(a+1)^2-\left(a^2+1\right)^2-3a^2}{3\left(a^3+1\right)(a+1)} \ge 8 \Leftrightarrow \frac{7\sqrt{3\left(a^2+1\right)}}{a+1}+\frac{3a^3+a^2+3a-a^4-1}{3\left(a^3+1\right)(a+1)} \ge 8$$

We have: $\frac{\sqrt{3\left(a^2+1\right)}}{a+1}=\sqrt{\frac{3}{2}}\cdot\frac{\sqrt{2\left(a^2+1\right)}}{a+1} \ge \sqrt{\frac{3}{2}} \Rightarrow \frac{7\sqrt{3\left(a^2+1\right)}}{a+1} \ge \frac{7\sqrt{6}}{2}$ and $\frac{3a^3+a^2+3a-a^4-1}{3\left(a^3+1\right)(a+1)} \ge 0$

Add two inequalities side by side we have: $\frac{7\sqrt{3\left(a^2+1\right)}}{a+1}+\frac{3a^3+a^2+3a-a^4-1}{3\left(a^3+1\right)(a+1)} \ge \frac{7\sqrt{6}}{2} > 8$

**Case 2:** $b=c=1$

$$\frac{7\sqrt{3\left(a^2+2\right)}}{a+2}+\frac{3(a+2)^2(2a+1)-12a(a+2)-\left(a^2+2\right)^2-3\left(2a^2+1\right)}{3\left(a^3+2\right)(a+2)} \ge 8$$

$$\Leftrightarrow \frac{7\sqrt{3\left(a^2+2\right)}}{a+2}-7 \ge 1-\frac{-a^4+6a^3+5a^2+12a+5}{3(a^4+2a^3+2a+4)}$$

$$\Leftrightarrow \frac{14(a-1)^2}{(a+2)\left(\sqrt{3(a^2+2)}+a+2\right)} \ge \frac{\left(4a^2+8a+7\right)(a-1)^2}{3\left(a^3+2\right)(a+2)}$$

$$\Leftrightarrow (a-1)^2\left[42\left(a^3+2\right)-\left(4a^2+8a+7\right)\left(\sqrt{3\left(a^2+2\right)}+a+2\right)\right] \ge 0$$

The rest work is to prove that: $f(a)=42\left(a^3+2\right)-\left(4a^2+8a+7\right)\left(\sqrt{3\left(a^2+2\right)}+a+2\right) \ge 0$ will be remained for the readers.

The proof end here.

***Remarks:*** In the above problems, we only find the evaluation for the expressions $a^2b+b^2c+c^2a$, $ab^2+bc^2+ca^2$. Actually that is enough to do already, because all the rest cyclic expressions can be represented through: $abc, ab+bc+ca, a+b+c, a^2b+b^2c+c^2a$.

We will not give a proof for the above result, however the interested readers may find out quickly using inductive. Instead of that, we will give some equality that the readers may use when solving problems. As normal, let assume that $a=x+y+z, b=xy+yz+zx, c=xyz$:

$$x^3y+y^3z+z^3x=a\left(x^2y+y^2z+z^2x\right)-b^2+ac$$

$$x^4y+y^4z+z^4x=\left(a^2-b\right)\left(x^2y+y^2z+z^2x\right)-ab^2+bc+a^2c$$

$$x^3y^2+y^3z^2+z^3x^2=b\left(x^2y+y^2z+z^2x\right)+bc-a^2c$$

$$x^5y+y^5z+z^5x=\left(a^3-2ab+c\right)\left(x^2y+y^2z+z^2x\right)-a^2b^2+a^3c+b^3$$

$$x^4y^2+y^4z^2+z^4x^2=(ab-c)\left(x^2y+y^2z+z^2x\right)+4abc-a^3c-b^3-3c^2$$

***Proposed Problems***

**Problem 14.** Prove that: $\dfrac{a^3+b^3+c^3}{a^2b+b^2c+c^2a}+\dfrac{8abc}{(a+b)(b+c)(c+a)}\geq 4,\ \forall a,b,c\geq 0$

**Problem 15. [Nguyen Anh Cuong]** Prove that:

$$\sqrt{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}+\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}\geq\sqrt{3}+1,\ \forall a,b,c\geq 0$$

**Problem 16. [Nguyen Anh Cuong]** Prove that:

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+4\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\geq 7,\ \forall a,b,c\geq 0$$

# §4.19. MIXING VARIABLES METHOD

**Main points:**

I. Introduction

II. Inequalities with three-variables

1. Unconditioned inequalities

2. MV method for conditioned inequalities

3. Trigonometric MV method in triangles

III. Using MV method with functions

IV. Three-variable inequalities involving boundary

V. SMV theorem - inequalities with four variables

VI. MV via convex function

VII. Undefined mixing variables – UMV

VIII. MV with mean value

IX. Inequality general induction (IGI)

X. Entirely mixing variables – EMV

1. EMV with the boundary at 0

2. EMV for inequalities with triangles

XI. Some special mixing variables techniques

XII. General mixing variables theorem (GMV)

XIII. Review & Proposed problems

## I. INTRODUCTION

• Dear reader, in our knowledge, for most inequalities, especially symmetric or permutation inequalities, equality occurs when variables are equal. A typical example is $AM-GM$ inequality, for example, for $n=3$:

**Example 1.1.** Let $x, y, z \geq 0$. Then $x+y+z \geq 3 \cdot \sqrt[3]{xyz}$.

Equality occurs if and only if $x=y=z \geq 0$.

• There are many such inequalities so we usually believe that equality occurs when variables are equal. This erroneous observation is understandable since it requires advanced mathematical ability in order to construct a symmetric or permutation inequality so that equality occurs at a state that all variables are not the same. Let us consider the following example.

**Example 1.2.** (VMO)

Let $x, y, z$ be real numbers such that $x^2+y^2+z^2=9$. Then $2(x+y+z)-xyz \leq 10$.

In this case, equality occurs for $(x, y, z) = (2; 2; -1)$ or any cyclic permutation.

• You might be surprised if you know that there are inequalities in which equality occurs when all variables have distinct values. For instance

**Example 1.3.** (Jackgarfukel) Given $a,b,c \geq 0$ . Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \leq \frac{5}{4}\sqrt{a+b+c}$$

In this case, equality occurs when $a = 3b > 0,\ c = 0$ or any cyclic permutation. We may ask why is $(3,\ 1,\ 0)$? Intuitively, we see there is something special about the fact that there is a variable which is 0. We observe that $a$, $b$, $c$ are non-negative, thus a variable which has value 0 is called variable which has value on the boundary (of the domain).

• Another interesting point is that for some special inequalities, equality occurs for more than once (excluding cyclic permutation). These inequalities are beautiful and difficult. Here, we introduce a well-known example:

**Example 1.4.** (Iran TST 1996) Given $a,b,c \geq 0$. Prove that

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geq \frac{9}{4}$$

Equality occurs when either $a = b = c > 0$ or $a = b > 0$, $c = 0$ (or any cyclic permutation). If you investigate on this example, you will see that this problem is actually consistent with the above examples.

• In summary, in the world of inequalities, the equalities usually occur in one of the following forms:

+ All variables are equal; we will call this “extreme value is at center”.

+ Some of variables are equal; we will call this “extreme value is symmetric.

+ One of variables is on the boundary; we will call this “extreme value is at the boundary.”

Mixing variables method (MV for short) is designed to solve such inequality by transforming the inequality to a simple inequality after deducing the number of variables. In some cases, after deducing the numbers of variables, we shall obtain a one-variable inequality which can be solved by investigating on its behavior as a function.

• We now present main techniques of MV method through some specific examples. This chapter is organized as following: in the first part, we shall consider 3-variable inequalities, then 4-variable inequalities and in last section, we will consider the general MV for $n$ variables. In the last section, we begin by introducing some “classical results”, and then we obtain some modifications and finally some general inequalities. We would like to transfer the natural idea of how to solve problems. Afterwards, readers can solve inequalities yourself.

**II. INEQUALITIES WITH THREE-VARIABLES**

• Assume that we need to prove $f(x,y,z)\geq 0$ for real numbers *x, y, z* satisfying some properties. We then process the following two steps:

***Step 1:*** (2 variables are equal)

Estimate $f(x, y, z) \geq f(t, t, z)$ for a suitable $t$ which depends on the relationship between *x, y, z*, for example, $t=\dfrac{x+y}{2}; t=\sqrt{xy}; t=\sqrt{\dfrac{x^2+y^2}{2}}$; …

***Step 2:*** See if $f(t, t, z) \geq 0$.

**Notice:** In some cases, it is much easier to normalize variables of the inequality before applying the above steps.

## 1. Unconditioned inequalities

For unconditioned inequalities, we often use the average quantities such as

$t=\dfrac{x+y}{2}; t=\sqrt{xy}; t=\sqrt{\dfrac{x^2+y^2}{2}}$. Let us consider following problems

**Problem 2.1.** Given $a,b,c\geq 0$ . Prove that $x+y+z\geq 3\cdot\sqrt[3]{xyz}$ (1)

***Proof***

***Method 1:*** The inequality (1) $\Leftrightarrow$ $f(x,y,z)=x+y+z-3\cdot\sqrt[3]{xyz}$ , $\forall$ *x, y, z* > 0

**Step 1:** Prove that: $f(x,y,z)\geq f(t,t,z)$ where $t=\dfrac{x+y}{2}$. From: $t^2\geq xy$ we have:

$$f(x,y,z)-f(t,t,z)=x+y+z-3\cdot\sqrt[3]{xyz}-\left[2t+z-3\cdot\sqrt[3]{t^2z}\right]=3\left(\sqrt[3]{t^2z}-\sqrt[3]{xyz}\right)\geq 0$$

**Step 2:** Prove that $f(t,t,z)=2t+z-3\cdot\sqrt[3]{t^2z}\geq 0$.

Indeed, we have: $f(t,t,z)\geq 0 \Leftrightarrow (2t+z)^3-27t^2z\geq 0 \Leftrightarrow (t-z)^2(8t+z)\geq 0$ (ans)

**Conclusion:** $f(x,y,z)\geq f(t,t,z)\geq 0$

***Method 2:*** The inequality (1) $\Leftrightarrow$ $f(x,y,z)=x+y+z-3\cdot\sqrt[3]{xyz}\geq 0$, $\forall$ *x, y, z* > 0.

**Step 1:** Prove that: $f(x,y,z)\geq f(t,t,z)$ where $t=\sqrt{xy}$. We have: $2t\leq x+y$ then:

$$f(x,y,z)-f(t,t,z)=x+y+z-3\cdot\sqrt[3]{xyz}-\left[2t+z-3\cdot\sqrt[3]{xyz}\right]=x+y-2t\geq 0$$

**Step 2:** Prove that $f(t,t,z)=2t+z-3\cdot\sqrt[3]{t^2z}\geq 0$.

Indeed, we have: $f(t,t,z)\geq 0 \Leftrightarrow (2t+z)^3-27t^2z\geq 0 \Leftrightarrow (t-z)^2(8t+z)\geq 0$ (ans)

**Conclusion:** $f(x,y,z)\geq f(t,t,z)\geq 0$

***Method 3:*** MV method for homogeneous inequality

**Step 1:** If $x+y+z=k>0$ then (1) $\Leftrightarrow (x+y+z)^3 \geq 27xyz \Leftrightarrow \left(\frac{x}{k}+\frac{y}{k}+\frac{z}{k}\right)^3 \geq 27\cdot\frac{x}{k}\cdot\frac{y}{k}\cdot\frac{z}{k}$

From that if we assume that $x+y+z=1$ (*), then (1) $\Leftrightarrow f(x,y,z)=1-27xyz\geq 0$

**Step 2:** We also notice that after replacing $x$ and $y$ by $t=\frac{x+y}{2}$ condition (*) still holds, i.e. $t+t+z=1$, thus, it is enough to consider the behavior of $xyz$.

According to $AM-GM$, $xy\leq\left(\frac{x+y}{2}\right)^2=t^2$, therefore $xyz\leq t^2z \Rightarrow f(x,y,z)\geq f(t,t,z)$

**Step 3:** Let $z=1-2t$ we have: $f(t,t,z)=1-27t^2(1-2t)=(1+6t)(1-3t)^2\geq 0$.

Under the condition (*), equality occurs if $\begin{cases} x=y \\ 3t=1 \end{cases} \Leftrightarrow x=y=\frac{1}{3} \Leftrightarrow x=y=z=\frac{1}{3}$.

Hence, in general, equality occurs if $x=y=z>0$.

***Method 4:***

**Step 1:** If $xyz=k^3>0$ then (1) $\Leftrightarrow \frac{x}{k}+\frac{y}{k}+\frac{z}{k}\geq 3\cdot\sqrt[3]{\frac{x}{k}\cdot\frac{y}{k}\cdot\frac{z}{k}}$

We then assume that $xyz=1$ (*), thus (1) $\Leftrightarrow f(x,y,z)=x+y+z-3\geq 0$

**Step 2:** We also notice that after replacing $x$ and $y$ by $t=\sqrt{xy}$ condition (*) still holds, i.e. $t.t.z=1$ thus, it is enough to consider the behavior of $x+y+z$. According to $AM-GM$, $x+y\geq 2\sqrt{xy}=2t$, therefore $x+y+z\geq 2t+z \Rightarrow f(x,y,z)\geq f(t,t,z)$.

**Step 3:** Let $z=\frac{1}{t^2}$ we have: $f(t,t,z)=2t+\frac{1}{t^2}-3=\frac{(t-1)^2(2t+1)}{t^2}\geq 0$.

Under the condition (*), equality occurs if $\begin{cases} x=y \\ t=1 \end{cases} \Leftrightarrow x=y=1 \Leftrightarrow x=y=z=1$.

Hence, in general, equality occurs if $x=y=z>0$.

• ***Remarks:***

**a)** Through the above example, we can prove inequality $f(x,y,z)\geq 0$ by proving the following two inequalities $f(x,y,z)\geq f(t,t,z)$ ; $f(t,t,z)\geq 0$.

**b)** We can easily apply MV method to an inequality which is in a normalized form. When considering an inequality which is not in normal form, we should choose a suitable MV method in order to transform the inequality to the simplest form as possible.

• Because $AM-GM$ ($n=3$) is very simple, MV method might not be impressive to some of readers. However, you will be convinced by the effectiveness of MV method through following examples.

**Problem 2.2.** Let $a,b,c\in\mathbb{R}$. Prove that $(a^2+2)(b^2+2)(c^2+2)\ge 9(ab+bc+ca)$

***(APMO 2004)***

***Proof***

Since the left hand side is even function with respect to $a$, $b$, $c$, it suffices to prove the inequality for non-negative real numbers $a$, $b$, $c$.

Letting $f(a,b,c)=(a^2+2)(b^2+2)(c^2+2)-9(ab+bc+ca)$ for $a,b,c\ge 0$.

We now consider the following difference:

$$d=f(a,b,c)-f\left(\sqrt{ab},\sqrt{ab},c\right)=\left(\sqrt{a}-\sqrt{b}\right)^2\left[2\left(\sqrt{a}+\sqrt{b}\right)^2c^2+4\left(\sqrt{a}+\sqrt{b}\right)^2-9c\right]$$

If we assume that $c=min\{a,b,c\}$ then it is obvious that $d\ge 0$ or $f(a,b,c)\ge f\left(\sqrt{ab},\sqrt{ab},c\right)$

It remains to prove that $f(t,t,c)\ge 0$. By representing $f(t,t,c)$ as a quadratic trinomial of $c$

$f(t,t,c)=(t^2+2)^2c^2-18tc+(2t^4-t^2+8)$, we have

$\Delta'=(9t)^2-(t^2+2)^2(2t^4-t^2+8)=-(t^2-1)^2(2t^4+11t^2+32)\le 0\Rightarrow f(t,t,c)\ge 0$

This completes our proof. Equality occurs if $a=b=c=1$.

• ***Remark:*** In symmetric inequalities, we can assume either $a\le b\le c$ or $a\ge b\ge c$, and in cyclic inequalities, we must assume either $a=min\{a,b,c\}$ or $a=max\{a,b,c\}$.

**Problem 2.3.** Given real numbers $a,b,c$. Find the minimum value of

$$f(a,b,c)=(a+b)^4+(b+c)^4+(c+a)^4-\frac{4}{7}(a^4+b^4+c^4)$$

***Solution***

WLOG, assume $a(a+b+c)\ge 0$. Consider the difference

$$f(a,b,c)-f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=\left[\frac{3}{28}(b^2+c^2)+3a(a+b+c)+\frac{15}{56}(b+c)^2\right](b-c)^2\ge 0$$

Thus $f(a,b,c)\ge f\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)$. If $b+c=0$ then $f(a,b,c)=\frac{3}{7}a^4\ge 0$

If $b+c\ne 0$, we standardize $b+c=2$. We now have $f(a,1,1)=2(a+1)^4+16-\frac{4}{7}(a^4+2)=g(a)$

It is easy to see that $g(a)\ge 0$ by investigating on the behavior of $g(a)$. Therefore $f(a,b,c)\ge 0$

Equality occurs if and only if $a=b=c=0$

**Problem 2.4 (Vasile)** Given $a,b,c\ge 0$ such that $ab+bc+ca=1$. Prove that:

$$\frac{1}{2a^2+bc}+\frac{1}{2b^2+ac}+\frac{1}{2c^2+ab}\ge\frac{8}{(a+b+c)^2}$$

***Proof***

Letting $f(a,b,c) = LHS - RHS$. We will prove that: $f(a,b,c) \ge f(a,t,t)$ , $t = \frac{b+c}{2}$.

First of all, we know $\frac{1}{2a^2+bc} \ge \frac{1}{2a^2+t^2}$, therefore we only need to prove

$$\frac{1}{2b^2+ac} + \frac{1}{2c^2+ab} \ge \frac{2}{2t^2+at} \Leftrightarrow (b-c)^2\left(2b^2+2c^2+8bc+a^2-5a(b+c)\right) \ge 0$$

Supposing $a = \min\{a,b,c\} \Rightarrow 2b^2+2c^2+8bc+a^2-5a(b+c) = 2b^2+2c^2+3bc-4a^2+5(b-a)(c-a) \ge 0$

$\Rightarrow f(a,b,c) \ge f(a,t,t)$. We now need to prove $f(a,t,t) \ge 0 \Leftrightarrow a^2(4a^2-7at+6t^2) \ge 0$ which is true

The problem is solved. Equality occurs if and only if $a = b = t, c = 0\,(t \ge 0)$ and its permutations

• ***Remark:*** The above problem is an example of inequalities whose optimal value is not at the centre. For MV method, the most important part is "mixing", it does not matter whether optimal value is at the centre or not. This is the distinct feature of MV method in comparison with traditional method. In general, it is not effective to apply classical inequalities for problems whose optimal value is not at the centre.

## 2. MV method for conditioned inequalities

MV method for conditioned inequalities is very different with the MV method we use in previous section. For example, with the condition $ab+bc+ca=1$, if we want to prove $f(a,b,c) \ge f(a,t,t)$, the variable t is not an "average quantity" of *b*, *c* but a quantity such that $2at+t^2=1$. In general:

To prove that $f(a,\ b,\ c) \ge 0$ where *a*, *b*, *c* satisfies $g(a,b,c)=0$ we need to prove $f(a,b,c) \ge f(a,t,t)$ where *t* satisfies $g(a,\ t,\ t)=0$. We present some examples below:

**Problem 2.5.** Given $a,b,c \ge 0$ such that $a+b+c=1$. Prove that

$$\sqrt{a+(b-c)^2} + \sqrt{b+(c-a)^2} + \sqrt{c+(a-b)^2} \ge \sqrt{3}$$

***Proof***

Let LHS = $f(a,b,c)$. Supposing $a = \min\{a,b,c\}$. As $a+b+c=1$, so $a \le \frac{1}{3}$.

We have $\sqrt{a+(b-c)^2} \ge \sqrt{a}$. Furthermore

$$\sqrt{b+(c-a)^2} + \sqrt{c+(a-b)^2} \ge \sqrt{2(b+c)+(2a-b-c)^2} \quad \Leftrightarrow (b-c)^2(3-8a) \ge 0$$

Therefore LHS $\ge \sqrt{a} + \sqrt{2(1-a)+(1-3a)^2} \ge \sqrt{3} \quad \Leftrightarrow a(3a-1)^2(4-3a) \ge 0$ which is true

The problem is solved completely. Equality occurs $\Leftrightarrow (a,b,c) = \left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\};\left\{\frac{1}{2},\frac{1}{2},0\right\}$.

**Problem 2.6.** (Phan Thanh Nam) Given $a+b+c\ge 3$. Determine all real numbers $k$ satisfying:

$$(a^2+k)(b^2+k)(c^2+k)\ge (1+k)^3 \quad (1)$$

***Solution***

***Step 1:*** First, we give some remarks to simplify the problem. Choose $c=0$, $a=b>\max\left\{\sqrt{|k|},\frac{3}{2}\right\}$ which implies $k>0$. Moreover, if $k>0$ satisfies the condition of the problem, so does every $k'>k$, since $(a^2+k')(b^2+k')(c^2+k')=\left[(a^2+k)+k'-k\right]\left[(b^2+k)+k'-k\right]\left[(c^2+k)+k'-k\right]$

$$\ge\left(\sqrt[3]{(a^2+k)(b^2+k)(c^2+k)}+k'-k\right)^3\ge\left(\sqrt[3]{(1+k)^3}+k'-k\right)^3=(1+k')^3$$

Therefore, it suffices to find the smallest real number $k$ satisfying (1).

We now show that the following statements are equivalent, for all $k>1$:

**(i)** (1) holds true for all real numbers $a$, $b$, $c$ satisfying $a+b+c\ge 3$.

**(ii)** (1) holds true for all nonnegative real numbers $a$, $b$, $c$ satisfying $a+b+c\ge 3$.

**(iii)** (1) holds true for all nonnegative real numbers $a$, $b$, $c$ satisfying $a+b+c=3$.

**(iv)** (1) holds true if $a=b=x, c=3-2x$, for all $x\in[0,1]$.

Indeed, the direction $(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(iv)$ is obvious, thus we only need to show the reverse one. We prove $(iv)\Rightarrow(iii)$

Assume that $a$, $b$, $c$ are non-negative real numbers such that $a+b+c=3$.

By symmetry, we may suppose that $c\ge a,b \Rightarrow 0\le a+b\le 2$.

Letting $f(a,b,c)=(a^2+k)(b^2+k)(c^2+k)-(1+k)^3$. Set $x=\frac{a+b}{2}$ then $x\in[0,1]$ and $c=3-2x$

We have: $f(a,b,c)-f(x,x,c)=(a-b)^2\left[2k-\left(\frac{a+b}{2}\right)^2-ab\right](c^2+k)\ge 0$

since $2k-\left(\frac{a+b}{2}\right)^2-ab\ge 2\left[k-\left(\frac{a+b}{2}\right)^2\right]\ge 0$

Besides, (iv) implies that $f(x,x,c)\ge 0$, thus $f(a,b,c)\ge 0$, which means that (iii) holds true. The direction $(iii)\Rightarrow(ii)$ is trivial for if $a$, $b$, $c$ is non-negative then the LHS of (1) increases as a function of $a$, $b$, $c$. Finally, to prove $(ii)\Rightarrow(i)$, we may replace ($a$, $b$, $c$) by $(|a|,|b|,|c|)$

***Step 2:*** Consider $k>0$, we will find all $k$ satisfying (iv)

We have: $f(x,x,3-2x)=(x-1)^2\left[6k^2+(9x^2-6x+3)k+4x^4-4x^3-3x^2-2x-1\right]$

Letting $g(k)=6k^2+(9x^2-6x+3)k+4x^4-4x^3-3x^2-2x-1$ then $g$ is a quadratic expression in $k$.

Since $x\in[0,1]$ , it follows that $4x^4-4x^3-3x^2-2x-1<0$, hence $g$ has 2 roots with opposite signs, where the positive one is $y=\frac{-3x^2+2x+1}{4}+\frac{1}{12}\sqrt{3(11-5x)(x+1)^3}$.

It is easily seen that $g(k)\ge 0\Leftrightarrow k\ge y$ (since $k>0$).

Therefore, we only need to find the maximum value of the function $y=y(x)$ on the domain $x\in[0,1]$

We have: $y'(x)=\frac{1}{2}\left[1-3x+\frac{(7-5x)\sqrt{1+x}}{\sqrt{3(11-5x)}}\right]$ ; $y'(x)=0\Rightarrow 10x^3-27x^2+12x+1=0$

This equation has a unique root in the interval [0;1] which is

$$x_0=\frac{\sqrt{41}}{5}\cos\left(\frac{2\pi-\arccos\left(\frac{139}{41\sqrt{41}}\right)}{3}\right)+\frac{9}{10}\approx 0,6631865028$$

Moreover, $y(x_0)>\max\{y(0),y(1)\}$ , hence $y$ attains its maximum value on [0,1] at the point $x_0$.

So all real numbers $k>0$ satisfying (iv) are $k\ge k_0=y(x_0)\approx 1,109926818$.

Because $k_0>1$, these are all values $k$ such that (i) holds true.

***Conclusion:*** All values of $k$ satisfying the condition of the problems are $k\ge k_0=y(x_0)\approx 1,109926818$.

**Problem 2.7. (VMO)** Let $x, y, z$ be real numbers such that $x^2+y^2+z^2=9$.

Prove that: $2(x+y+z)-xyz\le 10$.

***Proof***

Let $f(x,y,z)=2(x+y+z)-xyz$ and $t=\sqrt{\frac{y^2+z^2}{2}}$.

Consider $d=f(x,y,z)-f(x,t,t)=2(y+z-2t)-x(yz-t^2)$

Since $y+z-2t\le 0$ and $yz-t^2\le 0$, if $x\le 0$ then $d\le 0$. We now suppose that $x=\min\{x,y,z\}$.

• If $x\le 0$ then $f(x,y,z)\le f(x,t,t)$. We will to show that $f(x,t,t)\le 10$

Replace $t=\sqrt{\frac{9-x^2}{2}}\Rightarrow f(x,t,t)=2x+2\sqrt{2(9-x^2)}-\frac{1}{2}x(9-x^2)=g(x)$, $x\in[-3,0]$.

Considering function $g'(x)=\frac{3x^2}{2}-\frac{5}{2}-\frac{4x}{\sqrt{18-2x^2}}=0\Rightarrow x=-1\in[-3,0]$

Based on the table on the right hand side

$f(x,y,z)\le f(x,t,t)=g(x)\le 10,\ \forall x\in[-3,0]$

$f(x,y,z)=10\Leftrightarrow x=-1,\ y=z=2.$

| $x$ | $-3$ | | $-1$ | | $0$ |
|---|---|---|---|---|---|
| $g'$ | | $+$ | $0$ | $-$ | |
| $g$ | $-6$ | $\nearrow$ | $10$ | $\searrow$ | $6\sqrt{2}$ |

• If $x>0\Rightarrow y>0,\ z>0$.

Without using MV method, we consider the following two cases:

If $x\ge\frac{3}{4}$ then $f(x,y,z)=2(x+y+z)-xyz\le 2\sqrt{3(x^2+y^2+z^2)}-\left(\frac{3}{4}\right)^3=2\sqrt{27}-\frac{27}{64}<10$

If $x\le\frac{3}{4}$ then $f(x,y,z)=2(x+y+z)-xyz\le 2\left(\sqrt{2(y^2+z^2)}+\frac{3}{4}\right)\le 2\left(\sqrt{18}+\frac{3}{4}\right)<10$

The problem is solved completely. Equality occurs if $a=b=2,=c=-1$ or its permutation.

***Comment:*** The conditions of above problems are with average quantities; therefore MV method is pretty much similar to Section 1. Now, we will consider problems with special conditions

**Problem 2.8.** Given $\begin{cases} a,b,c \ge 0 \\ ab+bc+ca=1 \end{cases}$. Prove that $\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2} \ge \frac{9}{4}$

***(Iran TST 1996)***

*Proof*

Let LHS = $f(a, b, c)$ . Supposing $t \ge 0$ is a variable such that $t^2+2at=1$. Consider:

$$d = f(a,b,c) - f(a,t,t) = \frac{1}{(b+c)^2} - \frac{1}{(2t)^2} + \frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} - \frac{2}{(a+t)^2}$$

We will prove $d \ge 0$ by transforming $d$ into the form of $(b-c)^2 A$ where A > 0

We have: $t^2+2at = ab+bc+ca \Leftrightarrow (a+t)^2 = (a+b)(a+c)$. Therefore

$$b+c-2t = (a+b)+(a+c)-2(a+t) = (a+b)+(a+c)-2\sqrt{(a+b)(a+c)}$$

$$= (\sqrt{a+b}-\sqrt{a+c})^2 = \frac{(b-c)^2}{(\sqrt{a+b}+\sqrt{a+c})^2}$$

So $d = \dfrac{(2t-b-c)(2t+b+c)}{(b+c)^2(2t)^2} + \dfrac{1}{(a+b)^2} + \dfrac{1}{(a+c)^2} - \dfrac{2}{(a+b)(a+c)}$

$$= \frac{-(b-c)^2(2t+b+c)}{(\sqrt{a+b}+\sqrt{a+c})^2(b+c)^2(2t)^2} + \frac{(b-c)^2}{(a+b)^2(a+c)^2}$$

$$= (b-c)^2\left(\frac{1}{(a+b)^2(a+c)^2} - \frac{2t+b+c}{(\sqrt{a+b}+\sqrt{a+c})^2(b+c)^2(2t)^2}\right)$$

Now, let us assume $a = \min\{a,b,c\}$, thus $a \le t$ and $2t+b+c \le (\sqrt{a+b}+\sqrt{a+c})^2$,

$(a+b)^2(a+c)^2 \le (b+c)^2(2t)^2$. Therefore $d \ge 0$ or equivalently $f(a,b,c) \ge f(a,t,t)$

On the other hand, replace $a = \dfrac{1-t^2}{2t}$ into $f(a,t,t)$ and after transformation, we get

$f(t,t,c) = \dfrac{(1-t^2)(1-3t^2)^2}{4t^2(1+t^2)} + \dfrac{9}{4} \ge \dfrac{9}{4}$. The problem is solved.

Equality occurs if and only if $a=b=c=\dfrac{1}{\sqrt{3}}$ or $a=0, b=c=1$ and its permutation

**Problem 2.9 [Le Trung Kien, Vo Quoc Ba Can]**

Given $a,b,c \ge 0$ such that $ab+bc+ca+6abc=9$. Prove that $a+b+c+3abc \ge 6$

***Proof***

Letting $f(a,b,c) = a+b+c+3abc$. Supposing $2at+t^2+6at^2=9$ $(0 \le t \le 3)$

Consider the difference: $d = f(a,b,c) - f(a,t,t) = (b+c-2t) + 3a(bc-t^2)$

With the given conditions, to transform $(b+c-2t)$, $(bc-t^2)$ into $A(b-c)^2$ is really complicated.

We will deal with this in a more "sophisticated" way

The condition $ab+bc+ca+6abc = 2at+t^2+6at^2 \Leftrightarrow \dfrac{a}{6a+1}(b+c-2t) = t^2 - bc$

Notice that if $\begin{cases} b+c-2t<0 \\ t^2-bc<0 \end{cases} \Leftrightarrow \dfrac{b+c}{2} < t < \sqrt{bc}$. This can never happen. Therefore $\begin{cases} b+c-2t \ge 0 \\ t^2-bc \ge 0 \end{cases}$

So $d = (b+c-2t) + 3a(bc-t^2) = (b+c-2t)\left(1 - \dfrac{3a^2}{6a+1}\right)$. Now, let us assume $a = \min\{a,b,c\}$,

then $a \le 1$ which implies $1 - \dfrac{3a^2}{6a+1} \ge 0$. Thus $d \ge 0$ or equivalently $f(a,b,c) \ge f(a,t,t)$

Replace $a = \dfrac{9-t^2}{2t+6t^2}$ and after transformation, we have $f(a,t,t) = \dfrac{3(3-t)(t+1)(t-1)^2}{2t+6t^2} + 6 \ge 6$

The problem is solved completely.

Equality occurs if $a=b=c=1$ or $a=0, b=c=3$ or its permutation

## 3. Trigonometric MV method in triangles

Trigonometric inequalities related to triangles are also 3-variable inequalities. We present few examples to illustrate the power of MV method for this kind of inequalities:

**Problem 2.10.** Given ΔABC. Find the minimum value of $(1+\cos^2 A)(1+\cos^2 B)(1+\cos^2 C)$

***Proof***

Supposing $C = \min\{A,B,C\} \Rightarrow 0 \le C \le \dfrac{\pi}{3}$ (*). Letting $f(A,B,C) = (1+\cos^2 A)(1+\cos^2 B)(1+\cos^2 C)$

We will prove that: $f(A,B,C) \ge f\left(\dfrac{A+B}{2}, \dfrac{A+B}{2}, C\right)$ (1). Indeed, we have:

(1) $\Leftrightarrow (1+\cos^2 A)(1+\cos^2 B) \ge \left(1+\cos^2 \dfrac{A+B}{2}\right)^2 \Leftrightarrow \sin^2 \dfrac{A-B}{2}[6\cos C - \cos(A-B) - 1] \ge 0$ (2)

Because of (*) we have: $6\cos C - \cos(A-B) - 1 \ge 3-1-1 > 0 \Rightarrow$ (2) is true $\Rightarrow$ (1) is also true,

or equivalently $f(A,B,C) \ge f\left(\dfrac{A+B}{2}, \dfrac{A+B}{2}, C\right) = g(C) = \dfrac{1}{4}(3-\cos C)^2(1+\cos^2 C)$

It is easy to see that $g(C) \ge \dfrac{125}{64}$. From (1) $\Rightarrow f(A,B,C) \ge \dfrac{125}{64}$

Equality occurs $\Leftrightarrow \Delta ABC$ is an equilateral triangle. Thus $\min f = \dfrac{125}{64}$

**Problem 2.11.** Given a non-obtuse triangle $ABC$.

Find the minimum value of $P=\dfrac{\sin A+\sin B+\sin C}{\cos A+\cos B+\cos C}$

***Solution***

Supposing $A=max\{A,B,C\}$, we have $\frac{\pi}{2}\ge A\ge\frac{\pi}{3}$. Letting $x=\cos\frac{B-C}{2};x\in[0,1]$

$$P=f(x)=\frac{\sin A+2\cos\frac{A}{2}x}{\cos A+2\sin\frac{A}{2}x}\Rightarrow f'(x)=\frac{\cos\frac{3A}{2}}{\left(\cos A+2\sin\frac{A}{2}x\right)^2}\le 0\Rightarrow f(x)\text{is decreasing in }[0,1]$$

$$\Rightarrow f(x)\ge\frac{\sin A+2\cos\frac{A}{2}}{\cos A+2\sin\frac{A}{2}}=g(A)\text{. We also have } g'(A)=\frac{\sin\frac{3A}{2}-1}{\left(\cos A+2\sin\frac{A}{2}\right)^2}\le 0$$

$$\Rightarrow g(A)\text{ is decreasing }\left[\frac{\pi}{3},\frac{\pi}{2}\right)\Rightarrow g(A)\ge g\left(\frac{\pi}{2}\right)=1+\frac{\sqrt{2}}{2}\Rightarrow P\ge 1+\frac{\sqrt{2}}{2}$$

Equality occurs if and only if: $A=\frac{\pi}{2};B=C=\frac{\pi}{4}$ or its permutations. Thus $\min P=1+\frac{\sqrt{2}}{2}$

**Problem 2.12.** Given a non-obtuse triangle $ABC$. Prove that:

$$\left(\frac{\sin A\sin B}{\sin C}\right)^2+\left(\frac{\sin B\sin C}{\sin A}\right)^2+\left(\frac{\sin A\sin C}{\sin B}\right)^2\ge\frac{9}{4}$$

***Proof***

WLOG, we assume that $\frac{\pi}{2}\ge A\ge\frac{\pi}{3}$. We can rewrite the above inequality in the form

$$f^2(A,B,C)\ge\frac{9}{4}+2\left(\sin^2 A+\sin^2 B+\sin^2 C\right)$$

where $f(A,B,C)=\dfrac{\sin A\sin B}{\sin C}+\dfrac{\sin A\sin C}{\sin B}+\dfrac{\sin B\sin C}{\sin A}$

Consider the difference: $d=f(A,B,C)-f\left(A,\frac{B+C}{2},\frac{B+C}{2}\right)=\dfrac{\sin^2\frac{B-C}{2}}{\sin A}\left(\dfrac{4\sin^2 A\sin^2\frac{A}{2}}{\sin B\sin C}-\dfrac{1}{2}\right)$

Since $\frac{\pi}{2}>A\ge\frac{\pi}{3}$, we have $\dfrac{4\sin^2 A\sin^2\frac{A}{2}}{\sin B\sin C}\ge 16s\text{in}^4\frac{A}{2}\ge 1$. Hence $d\ge 0$

We also notice that $\sin^2 B+\sin^2 C\le 2\cos^2\frac{A}{2}$ , so we only need to prove:

$$f^2\left(A,\frac{B+C}{2},\frac{B+C}{2}\right)\ge\frac{9}{4}+2\left(\sin^2 A+\sin^2 B+\sin^2 C\right)\Leftrightarrow\cos A(\cos A+1)(2\cos A-1)^2\ge 0$$

This is true because $\frac{\pi}{2}\ge A\ge\frac{\pi}{3}$. The inequality is then solved.

Equality occurs $\Leftrightarrow$ $\Delta$ABC is an equilateral triangle or an isosceles right triangle.

***Comment:*** If we use the formula $\dfrac{\sin B\sin C}{\sin A}=\dfrac{1}{\cot B+\cot C}$ and $\sum\cot B.\cot C=1$ then the above inequality is equivalent to (Iran 1996): $(ab+bc+ca)\left[\dfrac{1}{(a+b)^2}+\dfrac{1}{(b+c)^2}+\dfrac{1}{(c+a)^2}\right]\ge\dfrac{9}{4}$

**III. USING MV METHOD WITH FUNCTIONS**

In this section, we will study MV method with functions; this is an important technique in MV method. In section II. in order to prove $f(x, y, z) \geq f(t, t, z)$ we consider the expression $d = f(x, y, z) - f(t, t, z)$ then we show that $d \geq 0$. It is a suitable if $f$ is a simple polynomial or polynomial fraction. However with more complicated function, such as $f(x, y, z) = x^k + y^k + z^k$ (for $k > 0$) we need to evaluate intermediate quantity by investigating the behavior of the function. In details, assume that we need to prove $f(x, y, z) \geq f(x, t, t)$ where $t = \frac{y+z}{2}$, let us consider function $g(s) = f(x, t+s, t-s)$ for $s \geq 0$. We then prove that $g$ is increasing for $\forall s \geq 0$, thus $g(s) \geq g(0)$.

We would like to emphasize that this is a difficult and sophisticated technique in MV method. The following examples illustrates the beauty as well as the power of MV method

**Problem 3.1.** Given $k \geq 0$ and $a, b, c \geq 0$. Prove that:

$$\left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \min\left\{2, \frac{3}{2^k}\right\} \quad (*)$$

***Proof***

**Step 1:** It is enough to show the inequality under the case $2 = \frac{3}{2^k} \Leftrightarrow k = \frac{\ln 3}{\ln 2} - 1$

We leave it as a food for thought for readers to find out the reason.

***Notice:*** For $k = \frac{\ln 3}{\ln 2} - 1$ equality occurs for $a = b = c > 0$ or $\left[\begin{array}{l} a = b, c = 0 \\ b = c, a = 0 \\ c = a, b = 0 \end{array}\right.$

**Step 2:** Without lost of generality, we assume $a + b + c = 1$ and $b \geq c \geq a$.

Let $t = \frac{b+c}{2}$ and $m = \frac{b-c}{2}$, it follows that $b = t + m$, $c = t - m$, $a = 1 - 2t$. Therefore:

$$(*) \Leftrightarrow f(m) = \left(\frac{1-2t}{2t}\right)^k + \left(\frac{t+m}{1-t-m}\right)^k + \left(\frac{t-m}{1+m-t}\right)^k \geq 2 \text{ where } k = \frac{\ln 3}{\ln 2} - 1$$

Since $c \geq a$, we have $3t - 1 \geq m \geq 0$, and $1 \geq b + c = 2t$ thus $\frac{1}{2} \geq t \geq \frac{1}{3}$.

Let us consider function $f(m)$ where $m \in [0, 3t-1]$ and $t \in \left[\frac{1}{3}, \frac{1}{2}\right]$ is a constant.

We have $f'(m) = \frac{k(t+m)^{k-1}}{(1-t-m)^{k+1}} - \frac{k(t-m)^{k-1}}{(1+m-t)^{k+1}} \geq 0 \Leftrightarrow \frac{k(t+m)^{k-1}}{(1-t-m)^{k+1}} \geq \frac{k(t-m)^{k-1}}{(1+m-t)^{k+1}}$

$$\Leftrightarrow g(m) = [\ln(t-m) - \ln(t+m)] - \frac{1+k}{1-k}[\ln(1-t-m) - \ln(1-t+m)] \geq 0$$

Then: $g'(m)=\left(\frac{1}{t-m}+\frac{1}{t+m}\right)+\frac{1+k}{1-k}\left(\frac{1}{1-t-m}+\frac{1}{1-t+m}\right)\geq 0 \Leftrightarrow$

$$\frac{-2t}{(t-m)(t+m)}+\frac{1+k}{1-k}\cdot\frac{2(1-t)}{(1-t-m)(1-t+m)}\geq 0 \Leftrightarrow \frac{-t}{t^2-m^2}+\frac{1+k}{1-k}\cdot\frac{1-t}{(1-t)^2-m^2}\geq 0 \quad (1)$$

Since $k=\frac{\ln 3}{\ln 2}-1$, $\frac{1+k}{1-k}\geq 2$. Therefore, it is enough to prove that

$$\frac{-t}{t^2-m^2}+\frac{2(1-t)}{(1-t)^2-m^2}\geq 0 \Leftrightarrow u(m)=-t+4t^2-3t^3+3tm^2-2m^2\geq 0$$

Now $u'(m)=2(3t-2)m<0\ \ \forall t\in\left[\frac{1}{3},\frac{1}{2}\right] \Rightarrow u(m)\geq u(3t-1)=2(3t-1)(2t-1)^2\geq 0.$

$\Rightarrow$ (1) holds $\Rightarrow g'(m)\geq 0 \Rightarrow g(m)$ is increasing $\Rightarrow g(m)\geq g(0)=0 \Rightarrow f'(m)\geq 0$

$\Rightarrow f(m)$ is increasing $\Rightarrow f(m)\geq f(0)=\left(\frac{1-2t}{2t}\right)^k+2\left(\frac{t}{1-t}\right)^k.$

**Step 3:** We shall prove that $f(0)=h(t)=\left(\frac{1-2t}{2t}\right)^k+2\left(\frac{t}{1-t}\right)^k\geq 2$, $\forall t\in\left[0,\frac{1}{3}\right]$

$$h'(t)=\frac{2kt^{k-1}}{(1-t)^{k+1}}-\frac{k}{2^k}\cdot\frac{(1-2t)^{k-1}}{t^{k+1}}\leq 0 \Leftrightarrow 2^{k+1}t^{2k}\leq[(1-t)(1-2t)]^{k-1} \quad (2)$$

In the last inequality, the left hand side is an increasing function with respect to $t$ and the right hand side is a decreasing function with respect to $t$, and since $t\leq\frac{1}{3}$ then it is sufficient to show that: $2^{k+1}\left(\frac{1}{3}\right)^{2k}\leq\left[\left(1-\frac{1}{3}\right)\left(1-\frac{2}{3}\right)\right]^{k-1}$.

It is easy to see that the above inequality holds, thus $h(t)$ is decreasing.

Therefore $h(t)\geq h\left(\frac{1}{3}\right)=2$ which completes our proof. Comparing $t\in\left[\frac{1}{3},\frac{1}{2}\right]$ and $t\in\left[0,\frac{1}{3}\right]$

***Remarks:*** To see the beauty of above inequality, we consider some special cases

**a)** For $k = 1$, we get Nesbit inequality: $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$

There are 12 ways to prove the inequality, e.g.

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+3=\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}$$
$$=(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq(a+b+c)\frac{9}{(b+c)+(c+a)+(a+b)}=\frac{9}{2}$$

**b)** For $k=\frac{1}{2}$, we get: $\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2$

Here is a solution using AM – GM inequality:

$$\sum_{cyc}\sqrt{\frac{a}{b+c}}=\sum_{cyc}\frac{2a}{2\sqrt{a(b+c)}}\geq\sum_{cyc}\frac{2a}{a+b+c}=\frac{2(a+b+c)}{a+b+c}=2$$

**c)** For $k \geq \frac{2}{3}$, we have: $\left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \frac{3}{2^k}$

This is a nice inequality although constant $k = \frac{2}{3}$ is not the best. Here is a solution

$$a+b+c = a + \frac{b+c}{2} + \frac{b+c}{2} \geq 3 \cdot \sqrt[3]{a\left(\frac{b+c}{2}\right)^2} \Rightarrow \left(\frac{2a}{b+c}\right)^{\frac{2}{3}} \geq \frac{3a}{a+b+c} \Rightarrow \sum_{cyc}\left(\frac{a}{b+c}\right)^{\frac{2}{3}} \geq \frac{3}{2^{\frac{2}{3}}}$$

**Problem 3.2.** Let $k > 0$, $a, b, c \geq 0$ and $a + b + c = 3$. Prove that:

$$(ab)^k + (bc)^k + (ca)^k \leq \max\left\{3, \left(\frac{3}{2}\right)^{2k}\right\} \quad (1)$$

***Proof***

Without lost of generality we may assume that $b \geq c$ (we shall choose $a = \min\{a, b, c\}$ or $a = \max\{a, b, c\}$ in a suitable way).

Let $t = \frac{b+c}{2}$ and $m = \frac{b-c}{2}$ it follows that $b = t + m$, $c = t - m$. Then we can rewrite (1) as following:

$$f(m) = a^k\left[(t+m)^k + (t-m)^k\right] + \left(t^2 - m^2\right)^k \leq \max\left\{3, \left(\frac{3}{2}\right)^{2k}\right\}$$

Now consider $f(m)$ on $m \in [0, t]$. After some manipulations, we get:

$$f'(m) = ka^k\left[(t+m)^{k-1} - (t-m)^{k-1}\right] - 2km\left(t^2 - m^2\right)^{k-1}$$

$$f'(m) \geq 0 \Leftrightarrow g(m) = a^k\left[(t-m)^{1-k} - (t+m)^{1-k}\right] - 2m \geq 0$$

It is enough to consider that case $k > 1$ (for $k \leq 1$ the inequality is trivial).

Since $g''(m) = a^k k(k-1)\left[(t-m)^{-k-1} - (t+m)^{-k}\right] > 0$, $g'(m)$ is increasing, it follows that $g'(m) = 0$ has at most one solution in $(0, t)$. Because $g(0) = 0$, $g(t) = +\infty$ there are two possible cases $g(m) > 0$, $\forall m \in (0; t]$ or $g(m) = -0+$, $\forall m \in [0, t]$.

$\Leftrightarrow f'(m) > 0$, $\forall m \in (0; t]$ or $f'(m) = -0+$, $\forall m \in [0, t]$

$\Leftrightarrow f(m)$ goes up or $f(m)$ goes down and then goes up.

In both cases, the maximum value occurs in the boundary, this means $f(m) \leq \max\{f(0), f(t)\}$

For $m = t \Rightarrow c = 0 \Rightarrow f(t) = (ab)^k \leq \left(\frac{a+b}{2}\right)^{2k} = \left(\frac{3}{2}\right)^{2k}$

For $m = 0 \Rightarrow b = c = t \Rightarrow f(0) = 2t^k a^k + t^{2k} = 2t^k(3-2t)^k + t^{2k} = h(t)$

We have: $h'(t) = -4k(3-2t)^{k-1}t^k + 2k(3-2t)^k t^{k-1} + 2kt^{2k-1}$

$$h'(t) \geq 0 \Leftrightarrow -2\left(\frac{3-2t}{t}\right)^{k-1} + \left(\frac{3-2t}{t}\right)^k + 1 \geq 0 \Leftrightarrow u(x) = x^k - 2x^{k-1} + 1 \geq 0 \text{ for } x = \frac{3-2t}{t}$$

We get: $u'(x) = [kx - 2(k-1)]x^{k-2}$. It follows from that fact that $u'(x)$ has at most one solution in $\mathbb{R}^+$ that $u(x)$ has at most two solutions in $\mathbb{R}^+$, one of which is $x = 1$. By the way, we now suppose that $a = \min\{a, b, c\}$. It is enough to consider the case $t \geq 1$ or equivalently $x \leq 1$. Since $u(x)$ has at most one solution in (0, 1) then $h'(t)$ has at most one solution in $\left(1, \frac{3}{2}\right)$.

Notice that $h'(1) = 0$, $h'\left(\frac{3}{2}\right) > 0$. Therefore $h(t)$ is increasing in $\left[1, \frac{3}{2}\right]$ or $h(t)$ has a form (−0+) for $t \in \left[1, \frac{3}{2}\right]$. In both cases, the maximal value of $h(t)$ occurs in the boundary, this means that

$$h(t) \leq \max\left\{f(1), f\left(\frac{3}{2}\right)\right\} = \max\left\{3, \left(\frac{3}{2}\right)^{2k}\right\}$$

• ***Remarks:*** We do not assume $a = min\{a, b, c\}$ from beginning to point out that by applying mix variables method, we do not need to assume any orders between the variables.

Furthermore, after proving

$$\begin{cases} f(a,b,c) \leq \max\left\{\left(\frac{3}{2}\right)^{2k}, f(a,t,t)\right\} \text{ for } t = \dfrac{b+c}{2} \\ f(a,b,c) \leq \max\left\{\left(\frac{3}{2}\right)^{2k}, f(t,t,c)\right\} \text{ for } t = \dfrac{a+b}{2} \end{cases} \qquad (*)$$

we can continue in a different way as follows.

For $a, b, c$ fixed, consider the following sequence defined by:

$$(a_0, b_0, c_0) = (a,b,c);\ (a_{2n-1}, b_{2n-1}, c_{2n-1}) = \left(a_{2n-2}, \frac{b_{2n-2}+c_{2n-2}}{2}, \frac{b_{2n-2}+c_{2n-2}}{2}\right)\ \forall n \in \mathbb{Z}^+$$

and $(a_{2n}, b_{2n}, c_{2n}) = \left(\frac{a_{2n-1}+b_{2n-1}}{2}, \frac{a_{2n-1}+b_{2n-1}}{2}, c_{2n-1}\right)$ $\forall n \in \mathbb{Z}^+$. Then we have:

$$f(a,b,c) \leq \max\left\{\left(\frac{3}{2}\right)^{2k}, f(a_n, b_n, c_n)\right\},\ \forall n \in \mathbb{Z}^+$$

Since $a + b + c = 3$ then sequences $\{a_n\}, \{b_n\}, \{c_n\}$ converge to 1, it follows that:

$$f(a,b,c) \leq \max\left\{\left(\frac{3}{2}\right)^{2k}, f(1,1,1)\right\} = \max\left\{\left(\frac{3}{2}\right)^{2k}, 3\right\} \text{ (ans)}$$

• ***Comments:*** We shall generalize the above technique to obtain some well-known MV methods, namely, strongly mixing variables (SMV for short) and undefined mixing variables (UMV for short) which will be introduced in the next section. By using the continuity of function, we obtain more generalized methods than SMV and UMV.

On the other hand, after having (*), we have some special ways to obtain the solution; this will be introduced in the section on 4-variable problems. In the section on 3-variable inequality, we use only simple approach.

Here is another example, in which we use geometric mean.

**Problem 3.3.** (Pham Kim Hung) Given $a, b, c > 0$ such that $abc = 1$. Prove that:

**a)** $81(1+a^2)(1+b^2)(1+c^2) \le 8(a+b+c)^4$

**b)** $64(1+a^3)(1+b^3)(1+c^3) \le (a+b+c)^6$

***Proof***

**a)** Letting $f(a,b,c)=8(a+b+c)^4-81(1+a^2)(1+b^2)(1+c^2)$. We then assume that $a \ge b$.

Consider function $g(t)=f\left(ta,\frac{b}{t},c\right)$ where $t\in\left[\sqrt{\frac{b}{a}},1\right]$. We have:

$$g'(t)=32\left(a-\frac{b}{t^2}\right)\left(ta+\frac{b}{t}+c\right)^3-81\left(a-\frac{b}{t^2}\right)\left(ta+\frac{b}{t}\right)(1+c^2)$$

Since $t\in\left[\sqrt{\frac{b}{a}},1\right]$ it follows that $g'(t) \ge 0$ provided that: $32(d+c)^3 \ge 81d(1+c^2)$ for $d=ta+\frac{b}{t}$

Indeed: $32(d+c)^3 \ge 32d(d^2+2dc+3c^2) \ge 32d\left(3\sqrt[3]{d^4c^2}+3c^2\right) > 81d(1+c^2)$ since $d^2c \ge 4$.

Thus $g'(t) \ge 0$ for $t\in\left[\sqrt{\frac{b}{a}},1\right]$ therefore $g(t)$ is increasing in $\left[\sqrt{\frac{b}{a}},1\right] \Rightarrow g(1) \ge g\left(\sqrt{\frac{b}{a}}\right)$,

this means that $f(a,b,c) \ge f(s,s,c)$ where $s=\sqrt{ab}$.

Finally, we prove that $f(s, s, c) \ge 0$ for $s^2c = 1$. Let $s=\frac{1}{\sqrt{c}}$, we get:

$$f(s,s,c)=f\left(\frac{1}{\sqrt{c}},\frac{1}{\sqrt{c}},c\right)=8\left(\frac{2}{\sqrt{c}}+c\right)^4-81\left(1+\frac{1}{c}\right)^2(1+c^2)$$

$$=\left(\frac{\sqrt{c}-1}{c}\right)^2\left(8c^5+16c^{\frac{9}{2}}+24c^4+96c^{\frac{9}{2}}+87c^3+78c^{\frac{5}{2}}+99c^2+120c^{\frac{3}{2}}-21c+94\sqrt{c}+47\right) \ge 0 \text{ (ans)}$$

This completes the proof. Equality occurs if $a=b=c=1$.

**b)** Define $f(a,b,c)=(a+b+c)^6-64(1+a^3)(1+b^3)(1+c^3)$

Suppose $a \ge b \ge c$. Consider the function $g(t)=f(ta,\frac{b}{t},c)$ where $t\in\left[\sqrt{\frac{b}{a}},1\right]$. We have:

$$g'(t)=6\left(a-\frac{b}{t^2}\right)\left(ta+\frac{b}{t}+c\right)^5-192\left(a-\frac{b}{t^2}\right)\left(t^2a^2+ab+\frac{b^2}{t^2}\right)(1+c^3)$$

Since $t\in\left[\sqrt{\frac{b}{a}},1\right]$, $g'(t) \ge 0$ if the inequality $\left(ta+\frac{b}{t}+c\right)^5 \ge 32\left(t^2a^2+ab+\frac{b^2}{t^2}\right)(1+c^3)$ holds true.

Setting $d = t^2a^2 + ab + \dfrac{b^2}{t^2}$, we have: $\left(ta + \dfrac{b}{t} + c\right)^5 \geq 3\left(ta + \dfrac{b}{t} + c\right)^4$

$$= 3\left(d + 2ab + 2tac + 2\cdot\frac{bc}{t}\right)^2 \geq 3(d+6)^2 \geq 72d > 32d\left(1+c^3\right) \text{ (since } c \leq 1\text{)}$$

Hence $g'(t) \geq 0$ for $t \in \left[\sqrt{\dfrac{b}{a}}, 1\right]$, which implies that g is an increasing function on $\left[\sqrt{\dfrac{b}{a}}, 1\right]$.

In particular, $g(1) \geq g\left(\sqrt{\dfrac{b}{a}}\right)$, which means $f(a,b,c) \geq f(s,s,c)$ where $s = \sqrt{ab}$.

Finally, we show that $f(s, s, c) \geq 0$ where $s^2c = 1$. Substituting $s = \dfrac{1}{\sqrt{c}}$, we obtain:

$$f(s,s,c) = f\left(\frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}, c\right) = \left(\frac{2}{\sqrt{c}} + c\right)^6 - 64\left(1 + \frac{1}{c\sqrt{c}}\right)^2 (1+c^3)$$

$$= c^6 + 12c^4\sqrt{c} - 4c^3 + 32c\sqrt{c} + 112 + \frac{64}{c\sqrt{c}} > 0 \text{ since } c \leq 1.$$ The proof is completed.

**Problem 3.4. (Phan Thanh Viet)** Given $a, b, c \geq 0$, $a + b + c = 1$. Determine the greatest positive real number $k$ satisfying: $\sqrt{a + k(b-c)^2} + \sqrt{b + k(c-a)^2} + \sqrt{c + k(a-b)^2} \leq \sqrt{3}$ (1)

***Solution***

• ***Step 1:*** Let $a = 1, b = c = 0$. Then (1) implies that $k \leq 1 - \dfrac{\sqrt{3}}{2}$.

We now show that inequality (1) holds true for $k = 1 - \dfrac{\sqrt{3}}{2}$.

Define $f(a,b,c) = \sqrt{a + k(b-c)^2} + \sqrt{b + k(c-a)^2} + \sqrt{c + k(a-b)^2}$

Without loss of generality, we may suppose that $a \geq b \geq c$.

We prove $f(a,b,c) \leq f(a,t,t)$ (2) where $t = \dfrac{b+c}{2}$. Let $b = t + s$, $c = t - s$.

Since $a + b + c = 1$, $a = 1 - 2t$. Then the conditions $a \geq b \geq c \geq 0$ implies $1 \geq 3t + s \geq 4s \geq 0$.

Consider $f(a,b,c) = g(s) = \sqrt{1 - 2t + 4ks^2} + \sqrt{t + s + k(3t - s - 1)^2} + \sqrt{t - s + k(3t + s - 1)^2}$,

where $s \in \left[0; \dfrac{1}{4}\right]$ and $t \in \left[0; \dfrac{1}{3}\right]$. Note that (2) is true if we have $g(s) \leq g(0)$, so it suffices to show that g(s) is an decreasing function. We have:

$$g'(s) = \frac{4ks}{\sqrt{1-2t+4ks^2}} + \frac{1+2k-6kt+2ks}{2\sqrt{t+s+k(3t-s-1)^2}} + \frac{-1-2k+6kt+2ks}{2\sqrt{t-s+k(3t+s-1)^2}}$$

Since $1-2t+4ks^2 \ge t-s+k(3t+s-1)^2 \Leftrightarrow (1+s-3t)(3kt+3ks+1-k) \ge 0$ , to obtain $g'(s) \le 0$

we only need to show that $\dfrac{1+2k-6kt+2ks}{2\sqrt{t+s+k(3t-s-1)^2}} + \dfrac{-1-2k+6kt+10ks}{2\sqrt{t-s+k(3t+s-1)^2}} \le 0$

$$\Leftrightarrow (1+2k-6kt-10ks)^2(t+s+k(3t-s-1)^2) \ge (1+2k-6kt+2ks)^2(t-s+k(3t+s-1)^2)$$

$$\Leftrightarrow h(s) = 1-24k^2t-30kt+72k^2t^2+144k^2ts+144k^3t-432k^3t^2+6k-32k^2s-8ks+40k^2s^2$$

$$+16k^3s-16k^3+80k^3s^2+48k^3s^3-96k^3ts+432k^3t^3+144k^3t^2s-240k^3ts^2 \ge 0$$

We have: $h'(s) = 144k^2t-32k^2-8k+80k^2s+16k^3+160k^3s+144k^3s^2-96k^3t+144k^3t^2-480k^3ts$

$h''(s) = 80k^2+160k^3+288k^3s-480k^3t \ge 0 \quad \forall s \in \left[0;\frac{1}{4}\right]$ (since $t \in \left[0;\frac{1}{3}\right]$)

Hence $h'(s) \le h'\left(\frac{1}{4}\right) < 0$, which implies $h(s) \ge h\left(\frac{1}{4}\right) > 0$. We may now conclude that $g'(s) \le 0$

It follows that $f(a,b,c) = g(s) \le g(0) = f(a,t,t)$ where $t = \dfrac{b+c}{2}$.

• ***Step 2:*** It remains to prove the inequality for the case $b = c$:

$$\sqrt{a} + \sqrt{2(1-a)+k(1-3a)^2} \le \sqrt{3} \Leftrightarrow 2(1-a)+k(1-3a)^2 \le \left(\sqrt{3}-\sqrt{a}\right)^2$$

$$\Leftrightarrow k(1-3a)^2 \le (1-\sqrt{3a})^2 \Leftrightarrow (1-\sqrt{3a})^2\left[k(1+\sqrt{3a})^2-1\right] \le 0$$

The last inequality is obviously true since $(1+\sqrt{3a})^2 \le (1+\sqrt{3})^2 = \dfrac{1}{k}$.

Therefore, the greatest real number k satisfying the condition of the problem is $k = 1-\dfrac{\sqrt{3}}{2}$.

***Comment:*** From understanding the method to applying the method skillfully is a long way. The most important thing is that you need to be willing to deal with the problem to the very end, do not stop when you confront complicated calculations. Successes will make you more confident. We now present a problem in which the solution might frighten some of you, however we hope you will be calm to see hidden beauty of the problem

**Problem 3.5.** Given $\begin{cases} a,b,c \ge 0 \\ a+b+c=3 \end{cases}$. Find the maximum value of $S = \dfrac{ab}{3+c^2} + \dfrac{bc}{3+a^2} + \dfrac{ca}{3+b^2}$

***Solution***

WLOG, supposing $a \ge b \ge c$.

Put $a = s + t$, $b = s - t$ then we can rewrite S as: $f(t)=\dfrac{c(s-t)}{3+(s+t)^2}+\dfrac{c(s+t)}{3+(s-t)^2}+\dfrac{s^2-t^2}{3+c^2}$

We now investigate $f(t)$ for $t \in [0, s-c]$. We have:

$$f'(t)=\frac{-c}{3+(s+t)^2}-\frac{2c(s^2-t^2)}{\left[3+(s+t)^2\right]^2}+\frac{c}{3+(s-t)^2}+\frac{2c(s^2-t^2)}{\left[3+(s-t)^2\right]^2}-\frac{2t}{3+c^2}$$

$$=\frac{4cst}{uv}+\frac{8cst(s^2-t^2)(u+v)}{u^2v^2}-\frac{2t}{3+c^2}, \forall t\in(0,s-c) \quad \text{where } u=3+(s+t)^2, v=3+(s-t)^2.$$

If $f'(t) < 0$, $\forall t \in (0, s-c)$ (we shall prove it later), then:

$$f(t)\le f(0)=\frac{2cs}{3+s^2}+\frac{s^2}{3+c^2}=\frac{2s(3-2s)}{3+s^2}+\frac{s^2}{3+(3-2s)^2}=g(s) \quad (1)$$

Consider $g(s)$ for $s\in\left[1;\frac{3}{2}\right]$. We see that:

$$g'(s)=\frac{24s-12s^2}{\left[3+(3-2s)^2\right]^2}+\frac{18-24s-6s^2}{(3+s^2)^2}=\frac{108(s^2-3s+4)(s-1)^2(-s^2-3s+6)}{\left[3+(3-2s)^2\right]^2(3+s^2)^2}$$

Obviously $s^2-3s+4>0$ and $-s^2-3s+6=\left(\frac{\sqrt{33}-3}{2}-s\right)\left(s+\frac{\sqrt{33}+3}{2}\right)$

thus $g'(s)$ is positive in $(1, s_0)$ and negative in $\left(s_0;\frac{3}{2}\right)$ for $s_0=\frac{\sqrt{33}-3}{2}=1,372281323...$

Hence for every $s\in\left[1;\frac{3}{2}\right]$ we always have: $g(s)\le g(s_0)=\frac{11\sqrt{33}-45}{24}$ (2)

In (1) and (2), equality occurs if $t = 0$ and $s=s_0$, or equivalently if $a=b=s_0$ and $c=3-2s_0$.

So, the maximum value is $\frac{11\sqrt{33}-45}{24}=0.757924546...$ when $a=b=\frac{\sqrt{33}-3}{2}=1,372281323...$,

$c=6-\sqrt{33}=0.255437353...$

Finally we prove $f'(t) < 0$, $\forall t \in (0, s-c)$. We shall prove

for $t \in (0, s-c)$ that: $\frac{4cs}{uv}<\frac{1}{3+c^2}$ (3) and $\frac{8cs(s^2-t^2)(u+v)}{u^2v^2}\le\frac{1}{3+c^2}$ (4)

***Prove (3):*** It follows from $c+2s=1$ and $s > 1$ that $cs<1$. Moreover:

$u=3+(s+t)^2>4, v=3+(s-t)^2>3+c^2$ which yields (3).

***Prove (4):*** Using AM – GM inequality we have:

$$u^2v^2 = \left[3+(s+t)^2\right]^2\left[3+(s-t)^2\right]^2 \geq 16\left(s^2-t^2\right)$$

and $2cs(u+v)\left(3+c^2\right) = 4cs\left(3+s^2+t^2\right)\left(3+c^2\right) \leq \left(\dfrac{4cs+3+s^2+t^2+3+c^2}{3}\right)^3$

Using $c = 3-2s$, together with $t \leq s-c = 3s-3$, we yield:

$$4cs+3+s^2+t^2+3+c^2 \leq 4(3-2s)s+6+s^2+(3s-3)^2+(3-2s)^2 = 12+6(s-1)(s-2) \leq 12$$

then $2cs(u+v)(3+c^2) < 4^3$.

Thus: $\dfrac{8cs\left(s^2-t^2\right)(u+v)}{u^2v^2} = 4\cdot\dfrac{s^2-t^2}{u^2v^2}\cdot\dfrac{2cs(u+v)\left(3+c^2\right)}{3+c^2} \leq 4\cdot\dfrac{1}{4^4}\cdot\dfrac{4^3}{3+c^2} = \dfrac{1}{3+c^2}$

It follows from (3) and (4) that $f'(t) < 0$, $\forall t \in (0, s-c)$.

Hence $\operatorname{Max} S = \dfrac{11\sqrt{33}-45}{24} = 0.757924546...$

• ***Remark:*** We can also apply the above technique to the following problem:

**Problem 3.6.** Let $a$, $b$, $c > 0$ and $a+b+c = 3$. Find the maximum value of:

$$S(a,b,c) = \frac{bc}{a^2+k}+\frac{ca}{b^2+k}+\frac{ab}{c^2+k} \text{ where } k \geq 3 \text{ is a constant.}$$

**Problem 3.7.** Let $a$, $b$, $c > 0$; $a+b+c = 3$ and $S(a,b,c) = \dfrac{bc}{a^2+k}+\dfrac{ca}{b^2+k}+\dfrac{ab}{c^2+k}$

Find all $k$ such that $S(a,b,c) \leq S(1,1,1) = \dfrac{3}{1+k}$

Firstly, since $S(1,1,1) \geq S\left(\dfrac{3}{2},\dfrac{3}{2},0\right)$ then $k \geq 3$. Using similar approach as in 3.4 we then need to consider the case $a = b \geq c$ and $S_{\max} = \max\left\{\dfrac{3}{1+k}, S\left(s_0,s_0,3-s_0\right)\right\}$

where $s_0$ is the greatest solution of the equation $6s^3+(7k-9)s^2-18ks+k^2+9k$.

Thus we must find all $k \geq 3$ satisfying: $S\left(s_0,s_0,3-s_0\right) \leq \dfrac{3}{1+k}$

The possible value of $k$ is $[3, k_0]$ where $k_0 = 3{,}2690313...$ (solution of an equation with high degree). We also notice that in the most of inequalities concerning to find the best value of a constant (which is complicated), we must use MV method.

**IV. THREE-VARIABLE INEQUALITIES INVOLVING BOUNDARY**

In the previous section, "mixing variables" means "two variables are equal", and now, in this section, "mixing variables" should be understood as "moving one variable to the boundary".

For example, if we want to prove $f(x, y, z) \geq 0$ for $x, y, z \geq 0$, we hope that $f(x, y, z) \geq f(0, s, t)$, for suitable $s, t$ corresponding to $a, b, c$. Finally, it is enough to show that $f(0, s, t) \geq 0$.

We now begin with Schur inequality.

**Problem 4.1.** Let $a, b, c \geq 0$. Prove that $a^3 + b^3 + c^3 + 3abc \geq a^2(b+c) + b^2(c+a) + c^2(a+b)$

***(Schur inequality)***

*Proof*

In II., we have already proved it by using MV method when two variables are equal. We also observe that equality occurs either $a = b = c$ or $a = b, c = 0$ (and any cyclic permutation).

Let $f(a,b,c) = a^3 + b^3 + c^3 + 3abc - a^2(b+c) - b^2(c+a) - c^2(a+b)$

We hope that $f(a, b, c) \geq f(0, a + b, c)$. Now we consider:

$$d = f(a, b, c) - f(0, a + b, c) = ab(5c - 4a - 4b)$$

We then find that it is impossible to get $d \geq 0$ for every $a, b, c$.

Unfortunately it is not true! But why not $d \geq 0$. We now observe that $f(a, b, c)$ is decreasing if two variables are touching (which occurred in II.), now if we replace $(a, b, c)$ by $(0, a+b, c)$ variables are moving far from each other. We now replace $(a, b, c)$ by $\left(0, b+\frac{a}{2}, c+\frac{a}{2}\right)$ and consider: $d_a = f(a,b,c) - f\left(0, b+\frac{a}{2}, c+\frac{a}{2}\right) = a(a+b-2c)(a+c-2b)$

We now can assume that $d_a \geq 0$. Indeed, by symmetry, we may assume that $d_a = \max\{d_a, d_b, d_c\}$

So, if $d_a < 0$ then $0 > d_a d_b d_c = abc(b+c-2a)^2(c+a-2b)^2(a+b-2c)^2$, a contradiction!

Thus $d_a \geq 0$ then $f(a, b, c) \geq f(0, s, t)$ where $s = b + \frac{a}{2}, t = c + \frac{a}{2}$. Finally, we see that

$f(0,s,t) = t^3 + s^3 - t^2 s - ts^2 = (t+s)(t-s)^2 \geq 0$ this completes our proof.

**Problem 4.2. (Hojoo Lee)** Given $\begin{cases} a,b,c\ge 0 \\ ab+bc+ca=1 \end{cases}$ (*). Prove that $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge\frac{5}{2}$

***Proof***

We notice that equality occurs for $a = b = 1$, $c = 0$ and any cyclic permutation.

If $c = 0$, then we must prove: A = $\frac{1}{a}+\frac{1}{b}+\frac{1}{a+b}\ge\frac{5}{2}$, for $ab = 1$.

Let $s = a + b \Rightarrow s\ge 2\sqrt{ab}=2$. Then A = $s+\frac{1}{s}=\left(\frac{s}{4}+\frac{1}{s}\right)+\frac{3s}{4}\ge 2\sqrt{\frac{s}{4}\cdot\frac{1}{s}}+\frac{3\cdot 2}{4}=\frac{5}{2}$

Let $f(a,\ b,\ c)=\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}$. We hope that $f\left(a,b,c\right)\ge f\left(a+b,\frac{1}{a+b},0\right)$

(due to $ab + bc + ca = 1$). Consider: $f\left(a,b,c\right)-f\left(a+b,\frac{1}{a+b},0\right)$

$$=\left(\frac{1}{a+b}+\frac{1}{a+\frac{1-ab}{a+b}}+\frac{1}{b+\frac{1-ab}{a+b}}\right)-\left(\frac{1}{a+b}+a+b+\frac{1}{a+b+\frac{1}{a+b}}\right)$$

$$=\frac{1}{1+a^2}+\frac{1}{1+b^2}-1-\frac{1}{1+(a+b)^2}=\frac{2+a^2+b^2}{(1+a^2)(1+b^2)}-\frac{2+(a+b)^2}{1+(a+b)^2}$$

$$=\frac{\left[2+a^2+b^2\right]\left[1+(a+b)^2\right]-(1+a^2)(1+b^2)\left[2+(a+b)^2\right]}{(1+a^2)(1+b^2)\left[1+(a+b)^2\right]}$$

$$=\frac{2ab-a^2b^2\left[2+(a+b)^2\right]}{(1+a^2)(1+b^2)\left[1+(a+b)^2\right]}=\frac{ab\left[2(1-ab)-ab(a+b)^2\right]}{(1+a^2)(1+b^2)\left[1+(a+b)^2\right]}$$

Then $d\ge 0$ provided $2(1-ab)\ge ab(a+b)^2$. Therefore, assume that $c = max\{a,\ b,\ c\}$,

we get: $2(1-ab) = 2c(a+b)\ge(a+b)^2\ge ab(a+b)^2\ \Rightarrow d\ge 0\ \Rightarrow$ (ans)

• ***Remark:*** Problem 4.2 is interesting but it is a corollary of a well-know inequality, namely Iran TST 1996. Indeed, since $ab + bc + ca = 1$ then it follows from Iran TST 1996 that

$$\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)^2=\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{4(a+b+c)}{(a+b)(b+c)(c+a)}\ge\frac{9}{4}+4=\frac{25}{4}$$

due to $a + b + c = (a + b + c)(ab + bc + ca)\ge(a + b)(b + c)(c + a)$.

It is natural to raise a question that whether Inequality Iran TST 1996 can be solved by forcing one variable to the boundary or not? This can be answered by considering the following problem.

**Problem 4.3. (Le Trung Kien)** Given $a, b, c \geq 0$, $ab + bc + ca = 1$ (*). Prove that

$$\frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}} \geq 2+\frac{1}{\sqrt{2}}$$

***Proof***

First of all, consider the case $c = 0$, we need to prove

$$A = \frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}} \geq 2+\frac{1}{\sqrt{2}}, \text{ where } ab = 1$$

We leave this simple exercise for readers.

Now, we will prove $f(a,b,c) \geq f\left(0, a+b, \frac{1}{a+b}\right)$

Note that $f(a, b, c)$ in 4.3 is quite complicated therefore it is difficult to use MV method by considering the difference. Fortunately, $f(a, b, c)$ can be expressed in terms of $x = a+b$ and $c$ as following:

$$f(a,b,c) = \frac{1}{\sqrt{a+b}}+\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}} = \frac{1}{\sqrt{a+b}}+\frac{\sqrt{(c+a)}+\sqrt{(b+c)}}{\sqrt{(b+c)(c+a)}}$$

$$= \frac{1}{\sqrt{a+b}}+\frac{\sqrt{2c+a+b+2\sqrt{c^2+1}}}{\sqrt{c^2+1}} = \frac{1}{\sqrt{x}}+\frac{\sqrt{2c+x+2\sqrt{c^2+1}}}{\sqrt{c^2+1}} = g(c)$$

We have: $g'(c) = \dfrac{1-cx-c^2-c\sqrt{c^2+1}}{\sqrt{(c^2+1)^3(2c+x+2\sqrt{c^2+1})}} \leq 0$

Therefore, $g(c)$ is decreasing. Note that: $cx = ca+cb \leq ca+cb+ab = 1 \Leftrightarrow c \leq \dfrac{1}{x}$.

Thus: $g(c) \geq g\left(\dfrac{1}{x}\right) = \sqrt{x}+\dfrac{1}{\sqrt{x}}+\sqrt{\dfrac{x}{x^2+1}} \Rightarrow f(a,b,c) \geq f\left(0, a+b, \dfrac{1}{a+b}\right)$

The problem is then solved. Equality occurs $\Leftrightarrow$ $a = 0, b = c = 1$ or its permutation

***Remark:***

• The technique of transforming expressions of $a$, $b$, $c$ into expression of $a+b, c$ if sophisticated. Note that a conditioned inequality of three variables is equivalent to an unconditioned inequality of two variable. The above technique helps us to utilize the given condition when transforming it to inequality of 2-variables. Based on this idea, we have a "surprised" proof for Iran TST 96 inequality.

With the condition $ab+bc+ca=1$. Let $x=a+b$, we have

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}=\frac{1}{x^2}+\frac{x^2+2c^2+4cx-2}{\left(c^2+1\right)^2}=g(c)$$

This is a function of two variables.

Differentiate $g(c)$ we have $g'(c)=-\dfrac{4\left[cx^2+(3c^2-1)x+c^3-3c\right]}{\left(c^2+1\right)^3}$

Consider following cases:

***Case 1:*** $c\ge 1$, then: $c\ge x\ge 1$

We have $1=ab+bc+ca\le\dfrac{x^2}{4}+cx\Leftrightarrow x^2\ge 4-4cx$, therefore

$cx^2+(3c^2-1)x+c^3-3c\ge c(4-cx)+(3c^2-1)x+c^3-3c=c^3+c-c^2x-x\ge 0$

$\Leftrightarrow g'(c)\le 0$. So: $g(c)\le g\left(\dfrac{1}{x}\right)=f\left(0,a+b,\dfrac{1}{a+b}\right)$.

***Case 2:*** $c\le 1$. Again, we have: $x^2\ge 4-4cx\Leftrightarrow c\ge\dfrac{4-x^2}{4x}$.

Consider the second derivative of *g(c)*:

$$g''(c)=\frac{4\left[\left(5c^2-1\right)x^2+12c(c^2-1)x+3c^4-18c^2+3\right]}{\left(c^2+1\right)^3}$$

We will prove: $g''(c)\le 0\Leftrightarrow h(x)=\left(5c^2-1\right)x^2+12c(c^2-1)x+3c^4-18c^2+3\le 0$

Since $h(x)$ is a quadratic function with positive coefficient, we have $h(x)\le\max\left\{h(0),h\left(\dfrac{1}{c}\right)\right\}$.

Also $h(0)=3c^4-18c^2+3\le 0$; $h\left(\dfrac{1}{c}\right)=3c^4-6c^2-\dfrac{1}{c^2}-4\le 0$

Therefore $h(x)\le 0\Leftrightarrow g''(c)\le 0$, so:

$$g(c)\le\max\left\{g\left(\frac{1}{x}\right),g\left(\frac{4-x^2}{4x}\right)\right\}\Leftrightarrow f(a,b,c)\le\max\left\{f\left(0,a+b,\frac{1}{a+b}\right),f(t,t,c)\right\}$$

We now only to prove the problem when there are two variables with the same value or one variable is 0. The first case is proved above. In the second case, we assume $c=0$, thus the condition becomes $ab=1$. We have

$$\frac{1}{(a+b)^2}+\frac{1}{b^2}+\frac{1}{a^2}\ge\frac{9}{4}\Leftrightarrow ab(a-b)^2(8a^2+8b^2+15)\ge 0$$

The problem is now solved completely.

• We will now show examples that MV method where variables are equal cannot be applied. Applying MV method where one variable moves to the boundary is suitable.

**Problem 4.4. (Jackgarfukel)**

Given $a, b, c \geq 0$. Prove that $\dfrac{a}{\sqrt{a+b}}+\dfrac{b}{\sqrt{b+c}}+\dfrac{c}{\sqrt{c+a}} \leq \dfrac{5}{4}\sqrt{a+b+c}$ (*)

*Proof*

We now consider when the equality occurs. It is easy to see that if $a = b = c$ then the equality does not occur. We may assume that $c = 0$, then (*) becomes $\dfrac{a}{\sqrt{a+b}}+\sqrt{b} \leq \dfrac{5}{4}\sqrt{a+b}$ (1)

Let $a + b = 1$. We get (1) $\Leftrightarrow 1-b+\sqrt{b} \leq \dfrac{5}{4} \Leftrightarrow \left(\sqrt{b}-\dfrac{1}{2}\right)^2 \geq 0$ (ans)

Thus equality occurs for $a = 3b > 0$, $c = 0$ (and any cyclic permutation).

Without lost of generality, assume that $a = max\{a, b, c\}$.

Normalize so that $a+b+c=1$.

Let $t=\dfrac{a+c}{2}$ and $s=\dfrac{a-c}{2}$, thus $a = t + s$, $c = t - s$, $b = 1 - 2t$.

Then (*) $\Leftrightarrow$ $f(s)=\dfrac{t+s}{\sqrt{s+1-t}}+\dfrac{1-2t}{\sqrt{1-t-s}}+\dfrac{t-s}{\sqrt{2t}} \leq \dfrac{5}{4}$ (2)

We shall prove that $f(s) \leq \max\{f(0), f(t)\}$ for $s \in [0, t]$. We have:

$$f'(s)=\frac{1}{\sqrt{s+1-t}}-\frac{t+s}{2(s+1-t)^{\frac{3}{2}}}+\frac{1-2t}{2(1-t-s)^{\frac{3}{2}}}-\frac{1}{\sqrt{2t}}$$ and

$$f''(s)=-\frac{1}{(s+1-t)^{\frac{3}{2}}}+\frac{3(t+s)}{4(s+1-t)^{\frac{5}{2}}}+\frac{3(1-2t)}{4(1-t-s)^{\frac{5}{2}}}$$. By using $b=1-2t \geq 0$ we obtain

$$f'''(s)=\frac{9}{4(s+1-t)^{\frac{5}{2}}}-\frac{15(t+s)}{8(s+1-t)^{\frac{7}{2}}}+\frac{15(1-2t)}{8(1-t-s)^{\frac{7}{2}}}=\frac{18+3s-33t}{8(s+1-t)^{\frac{7}{2}}}+\frac{15(1-2t)}{8(1-t-s)^{\frac{7}{2}}}>0$$

Because $f'''(s)>0$ $\forall s \in [0, t]$ then it follows from Rolle theorem that equation $f'(s) = 0$ has at most two solutions in $[0, t]$. On the other hand, it is easy to prove that $f'(0) \leq 0$ and $f'(t) \geq 0$, therefore $f'(s)$ may change its sign at most once $(0, t)$, moreover $f'(s)$ is as the following possible cases: $f'(s) > 0$, $\forall s \in (0, t)$ or $f'(s) < 0$, $\forall s \in (0, t)$ or $f'(s)$ with form –0+ on $(0, t)$. We then obtain $f(s) \leq \max\{f(0), f(t)\}$ $\forall s \in [0, t]$.

Now from $f(0) \leq \dfrac{5}{4}$ and $f(t) \leq \dfrac{5}{4}$ we have $f(s) \leq \max\{f(0), f(t)\} \leq \dfrac{5}{4}$

• ***Remark:*** Similarly, we can prove the generalization of above problem:

"Let $a, b, c \geq 0$ and $k \notin (0, 1)$. We have:

$\dfrac{a}{(a+b)^k}+\dfrac{b}{(b+c)^k}+\dfrac{c}{(c+a)^k} \leq C_k\,(a+b+c)^{1-k}$ where $C_k = 1+k-k^2$".

**Problem 4.5. (Phan Thanh Nam)** Given $a, b, c \geq 0$. Prove that:

$$\left(a^2+b^2+c^2\right)^2 \geq 4(a+b+c)(a-b)(b-c)(c-a)$$

***Solution***

Define $f(a,b,c)=\left(a^2+b^2+c^2\right)^2-4(a+b+c)(a-b)(b-c)(c-a)$.

Without loss of generality we may suppose that $c=\min\{a,b,c\}$. If $a\geq b\geq c$ then the inequality is trivial since $f(a,b,c)\geq 0$, hence we only need to consider the case $b\geq a\geq c$.

We then have:

$$f(a,b,c)-f(a,b,0)=\left(a^2+b^2+c^2\right)-(a^2+b^2)^2+4(b-a)c(a^2+ab+b^2-3c^2)\geq 0$$

Finally, $f(a,b,0)=(a^2+b^2)^2-4(b^2-a^2)ab=(a^2+2ab-b^2)^2\geq 0$, hence the proof is completed

Equality holds if $(a,b,c)=\left((\sqrt{3}-1)t,t,0\right)$ where $t\geq 0$ (and its permutations).

• Applying MV method when one variable moves to the boundary is essential when considering cyclic inequalities. But what's about symmetric inequalities?

**Problem 4.6. (Pham Kim Hung)** Given $a, b, c \geq 0$, $a+b+c=3$. Prove that:

$$\left(a^3+b^3+c^3\right)\left(a^3b^3+b^3c^3+c^3a^3\right)\leq 36(ab+bc+ca)$$

*Proof*

Without lost of generality, we may assume $a\geq b\geq c$. Put

$$f(a,b,c)=36(ab+bc+ca)-\left(a^3+b^3+c^3\right)\left(a^3b^3+b^3c^3+c^3a^3\right)$$

$$\Rightarrow f(a,b+c,0)=36a(b+c)-\left[a^3+(b+c)^3\right]a^3(b+c)^3$$

We shall prove that: $f(a, b, c)\geq f(a, b+c, 0)$. Indeed, we have:

$$\begin{cases}36(ab+bc+ca)=36a(b+c)+36bc\geq 36a(b+c)\\ \left(a^3+b^3+c^3\right)\left(a^3b^3+b^3c^3+c^3a^3\right)\leq\left[a^3+(b+c)^3\right]a^3(b+c)^3\end{cases}\Rightarrow f(a,b,c)\geq f(a,b+c,0)$$

Thus it is enough to show the case $c = 0$, or equivalently

$$36ab\geq a^3b^3\left(a^3+b^3\right)\Leftrightarrow 36\geq a^2b^2\left(a^3+b^3\right)$$

Letting $t = ab$ we rewrite the inequality as: $t^2(27-9t)\leq 36\Leftrightarrow t^3+4\geq 3t^2$

Using AM – GM inequality, we have: $t^3+4=\frac{t^3}{2}+\frac{t^3}{2}+4\geq 3\cdot\sqrt[3]{\frac{t^3}{2}\cdot\frac{t^3}{2}\cdot 4}=3t^2$

Equality occurs when $c = 0$ and $a + b = 3$, $ab = 2 \Leftrightarrow a = 2$, $b = 1$, $c = 0$ or its permutations.

## V. SMV THEOREM - INEQUALITIES WITH FOUR VARIABLES

We begin with a well-known problem:

**Problem 5.1. (IMO SL, Vietnam)** Let $a, b, c, d \ge 0$, $a + b + c + d = 1$.

Prove that: $abc + bcd + cda + dab \le \frac{1}{27} + \frac{176}{27}abcd$ (1)

***Proof***

Since equality occurs for $a = b = c = d = \frac{1}{4}$ or $a = b = c = \frac{1}{3}, d = 0$ (or any cyclic permutation), normal evaluation is not suitable for this problem.

Letting $f(a, b, c, d) = abc + bcd + cda + dab - kabcd$ where $k = \frac{176}{27}$. We have:

$$f(a, b, c, d) = ab(c + d - kcd) + cd(a + b)$$

Hence, we hope that $f(a, b, c, d) \le f(t, t, c, d)$ where $t = \frac{a+b}{2}$. Since $0 \le ab \le t^2$, we need to have $c + d - kcd \ge 0$. Otherwise, if $c + d - kcd < 0$ then we have:

$$f(a, b, c, d) = ab(c + d - kcd) + cd(a + b) \le cd(a + b) \le \left[\frac{c+d+(a+b)}{3}\right]^3 = \frac{1}{27}$$

Thus, we may assume that $f(a,b,c,d) \le f\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)$. This means that we can use our method without any additional assumptions for $a$, $b$. By the symmetry, if $s = \frac{c+d}{2}$ then we have:

$$f(a,b,c,d) \le f(t,t,c,d) \le f(t,t,s,s) = f(t,s,t,s) \le$$

$$\le f\left(\frac{t+s}{2}, \frac{t+s}{2}, t, s\right) \le f\left(\frac{t+s}{2}, \frac{t+s}{2}, \frac{t+s}{2}, \frac{t+s}{2}\right) = f\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{27} \Rightarrow \text{(ans)}$$

• ***Remark:***

**a)** In this proof, we split our inequality into two cases: one solved by MV method, one is obviously true. We also make the proof more clear by supposing the existence of $(a_0, b_0, c_0, d_0)$ such that $f(a_0, b_0, c_0, d_0) > \frac{1}{27}$. Moreover, we also suppose that:

$$f(a,b,c,d) \le \max\left\{\frac{1}{27}, f\left(\frac{a+b}{2}, \frac{a+b}{2}, c, d\right)\right\} \text{ (1)}$$

**Problem 5.2.** Given $a_1, a_2, ..., a_n \ge 0$ such that $a_1 + a_2 + ... + a_n = n$. Prove that

$$(n-1)\left(a_1^2 + a_2^2 + ... + a_n^2\right) + na_1a_2...a_n \ge n^2$$

It is simple when $n = 2$ or $n = 3$. We now prove the inequality when $n = 4$:

Given $a, b, c, d \ge 0$ such that $a + b + c + d = 4$ then we have:

$$3\left(a^2 + b^2 + c^2 + d^2\right) + 4abcd \ge 16 \text{ (1)}$$

***Proof***

WLOG, assume $a \le b \le c \le d$ . Let $f(a,b,c,d) = 3\left(a^2+b^2+c^2+d^2\right) + 4abcd - 16$ .

Consider the difference: $f(a,b,c,d) - f(a,b,\frac{c+d}{2},\frac{c+d}{2}) = \left(\frac{3}{2} - ab\right)(c-d)^2$

Since $a \le b \le c \le d$ we have $ab \le \sqrt{abcd} \le \left(\frac{a+b+c+d}{4}\right)^2 = 1$.

Therefore: $f(a,b,c,d) \ge f(a,b,\frac{c+d}{2},\frac{c+d}{2})$. Replace $c+d = 4-a-b$ , we need to prove:

$$f(a,b,\frac{c+d}{2},\frac{c+d}{2}) = 3\left(a^2+b^2+\frac{(4-a-b)^2}{2}\right) + (4-a-b)^2 ab - 16 \ge 0$$

Let $x = \frac{a+b}{2}$, $y = ab$ the above inequality is equivalent to $h(y) = (2x^2-8x+5)y + (3x-2)^2 \ge 0$

If $2x^2-8x+5 \ge 0$ then this is true. If $2x^2-8x+5 \le 0$ then because $y \le x^2$,

$h(y) = (2x^2-8x+5)x^2 + (3x-2)^2 = 2(x-1)^2(x^2-2x+2) \ge 0$ . Thus (1) is true

Equality occurs $\Leftrightarrow$ $a=b=c=d=1$ or $a=0, b=c=d=\frac{4}{3}$ or its permutations.

***Remark:***

Solutions of problem 5.1 and 5.2 with $n = 4$ are simple and beautiful. However, we cannot see how the method used in these solutions can be applied to other inequalities with 4 or more variables. We now present a general method:

## 1. SMV [Strongly Mixing Variables]

### Theorem 1 [SMV]

Let $D \in \mathbb{R}^n, D = \left\{x = (x_1, x_2, \ldots x_n) \mid x_i \ge \alpha, \sum x_i \ge \alpha, \sum x_i = ns = const\right\}$ and $s_0 = (s, s, \ldots, s) \in D$

- Consider the mapping $T$: $D \to D$ as following

Given $a = (a_1, a_2, \ldots a_n) \in D, a \ne s_0$, we select a pair of $i \ne j$ (depends on function $f$ below) such that $a_i \ne a_j$, then replace $a_i, a_j$ by its average.

- $f: D \to \mathbb{R}$ is a continuous function such that: $f(a) \ge f(T(a)), \forall a \in D$.

Then: $f(a) \ge f(s_0), \forall a \in D$

(The proof of this theorem will be presented in the next section).

Based on SMV theorem, consider problem 5.2 with $n \ge 4$

Let $f(a_1, a_2, \ldots, a_n) = (n-1)\left(a_1^2 + a_2^2 + \ldots + a_n^2\right) + n a_1 a_2 \ldots a_n - n^2$ and assume that $a_1 \le a_2 \le \ldots \le a_n$.

Consider: $f(a_1, a_2, \ldots, a_n) - f\left(a_1, \frac{a_2+a_n}{2}, a_3, \ldots, a_{n-1}, \frac{a_2+a_n}{2}\right) = (a_2 - a_n)^2 \left(\frac{n-1}{2} - \frac{n}{4} a_1 a_3 \ldots a_{n-1}\right)$

If $\frac{2(n-1)}{n} \ge a_1 a_3 \ldots a_{n-1}$ then $f(a_1, a_2, \ldots, a_n) \ge f\left(a_1, \frac{a_2+a_n}{2}, a_3, \ldots, a_{n-1}, \frac{a_2+a_n}{2}\right)$.

We also have $n=a_1+a_2+...+a_n\geq 2a_1+\frac{n-2}{n-3}(a_3+...+a_{n-1})\geq(n-2)\cdot\sqrt[n-2]{2a_1a_3...a_{n-1}\left(\frac{n-2}{n-3}\right)^{n-3}}$

$\Leftrightarrow \frac{n^{n-2}(n-3)^{n-3}}{2(n-2)^{2n-5}}\geq a_1a_3...a_{n-1}$

We will prove that: $\frac{2(n-1)}{n}\geq\frac{n^{n-2}(n-3)^{n-3}}{2(n-2)^{2n-5}}\Leftrightarrow 4(n-1)(n-2)^{2n-5}\geq n^{n-1}(n-3)^{n-3}$

Using AM-GM: $(n-2)^{2n-4}=\left(\frac{n-3+n-1}{2}\right)^{2(n-2)}\geq(n-3)^{n-2}(n-1)^{n-2}$

Thus, we only need to prove $4(n-3)(n-1)^{n-1}\geq n^{n-1}(n-2)\Leftrightarrow 4\left(1-\frac{1}{n}\right)^{n-1}-1+\frac{1}{n-3}\geq 0$

This is true because $4\left(1-\frac{1}{n}\right)^{n-1}\geq 4\left(1-\frac{1}{4}\right)^{4-1}=\frac{27}{16}>1$

Therefore $f(a_1,a_2,...,a_n)\geq f\left(a_1,\frac{a_2+a_n}{2},a_3,...a_{n-1},\frac{a_2+a_n}{2}\right)$. By SMV theorem, we only need to consider the case $a_1=n-(n-1)x, a_2=a_3=....a_n=x$ where $0\leq x\leq\frac{n}{n-1}$.

The inequality becomes: $(n-1)\left[(n-(n-1)x)^2+(n-1)x^2\right]+nx^{n-1}(n-(n-1)x)-n^2\geq 0$

$$\Leftrightarrow (n-1)x^n-nx^{n-1}-(n-1)^2x^2+2(n-1)^2x+2n-n^2\leq 0$$

Let $g(x)=(n-1)x^n-nx^{n-1}-(n-1)^2x^2+2(n-1)^2x+2n-n^2\leq 0$. We have

$g'(x)=n(n-1)x^{n-1}-n(n-1)x^{n-2}-2(n-1)^2x+2(n-1)^2=(n-1)(x-1)\left[nx^{n-2}-2(n-1)\right]$Hence

$g'(x)$ has two real solutions $x=1$ and $x=\sqrt[n-2]{\frac{2(n-1)}{n}}\geq 1$.

It is easy to see that $g(x)\leq\max\left\{g(1),g\left(\frac{n}{n-1}\right)\right\}$

On the other hand $g(0)=g\left(\frac{n}{n-1}\right)=0$ so $g(x)\leq 0$.

The problem is solved completely.

Equality occurs $\Leftrightarrow$ $a_1=a_2=...=a_n=1$ or $a_1=0, a_2=...=a_n=\frac{n}{n-1}$.

Through the above examples, you might see the power and fineness of SVM. Now we will consider problem 5.1 using SMV theorem.

WLOG, assume that $a\leq b\leq c\leq d$.

Consider $f(a,b,c,d)=abc+bcd+cda+dab-\frac{176}{27}abcd=ac(b+d)+bd\left(a+c-\frac{176}{27}ac\right)$

From the given conditions, we have: $a+c\leq\frac{1}{2}(a+b+c+d)=\frac{1}{2}$,

therefore $\frac{1}{a}+\frac{1}{c}\geq\frac{4}{a+c}\geq 8\geq\frac{176}{27}\Rightarrow f(a,b,c,d)\leq f\left(a,\frac{b+d}{2},c,\frac{b+d}{2}\right)$

By SMV theorem, we only need to prove with the case $a=1-3t, b=c=d=t$

Replace $a=1-3t$, we have $3at^2+t^3 \le \frac{1}{27}+\frac{176}{27}at^3 \Leftrightarrow (1-3t)(4t-1)^2(11t+1)\ge 0$

Equality occurs $\Leftrightarrow$ $a=b=c=d=\frac{1}{4}$ or $a=b=c=\frac{1}{3}, d=0$ or its permutations.

Through the application of SMV with two problems, one with $n$ variable and one with 4 variables, you can see that it is simple with 4 variables while MV method with $n$ variables is really complicated. Therefore, SMV is a perfect choice for inequalities with 4 variables (given that MV method is applicable), and on the other hand 4-variable inequality is the one use SMV the most. Let reconsider the problem 2.2.1 to affirm this:

Based on the solution, we see that the condition guarantees $f(a,b,c,d)\ge f\left(a,b,\frac{c+d}{2},\frac{c+d}{2}\right)$ is that $ab\le\frac{3}{2}$.

And if $a\le b\le c\le d$ then we have $ab\le 1$. However, we also notice that $ac\ge ab$ and $ac\le 1$.

Thus we can use SMV and then just prove the problem where three variables are equal.

In general, for all symmetric 4-variable problems, if we have $f(a,b,c,d)\ge f(a,b,\frac{c+d}{2},\frac{c+d}{2})$ for $a\le b\le c\le d$ then we also have $f(a,b,c,d)\ge f(a,\frac{b+d}{2},c,\frac{b+d}{2})$, therefore do we only need to prove the case where three variable are equal. This property is not always true for inequalities with n variables, even it is true then the proof is not simple. In the following, we thus only introduce applications of SMV in inequalities with 4 variables.

**Problem 5.3. (Phan Thanh Nam)** Given *a, b, c, d* $\ge 0$ and $a+b+c+d=4$.

Prove that: $abc+bcd+cda+dab+(abc)^2+(bcd)^2+(cda)^2+(dab)^2\le 8$

***Proof***

Letting $f(a, b, c, d) = abc+bcd+cda+dab+(abc)^2+(bcd)^2+(cda)^2+(dab)^2$

Assume $a\ge b\ge c\ge d$. Consider the difference: $f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)-f(a,b,c,d)$

$$=\left(\frac{a-c}{2}\right)^2\left\{(b+d)+\left[\left(\frac{a+c}{2}\right)^2+ac\right](b^2+d^2)-2b^2d^2\right\}$$

$$\ge\left(\frac{a-b}{2}\right)^2(b+d+4abcd-2b^2d^2)\ge 0 \text{ (since } abcd\ge b^2d^2)$$

Therefore: $f(a,b,c,d)\le f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)$.

By SMV theorem, we only need to prove with the case $a=b=c=x, d=4-3x$, $0\le x\le\frac{4}{3}$:

$$x^3+3x^2(3-3x)+3x^4(4-3x)^2+x^6\le 8\Leftrightarrow (x-1)^2(28x^4-16x^3-12x^2-8)\le 0$$

It is easy to prove: $28x^4-16x^3-12x^2-8\le 0$ with $0\le x\le \frac{4}{3}$.

Thus, the problem is solved. Equality occurs if and only if $a=b=c=d=1$.

**Problem 5.4.** Given $a,b,c,d\ge 0$ such that $a+b+c+d=1$. Prove that:

$$a^4+b^4+c^4+d^4+\frac{148}{27}abcd\ge \frac{1}{27}$$

***Proof***

Assume $a\ge b\ge c\ge d$. and let $f(a,b,c,d)=a^4+b^4+c^4+d^4+\frac{148}{27}abcd-\frac{1}{27}$

Consider the difference $d=f(a,b,c,d)-f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)=\left[\frac{7}{8}(a-c)^2+3ac-\frac{37}{27}bd\right](a-b)^2$

Since $ac\ge bd\Rightarrow d\ge 0\Rightarrow f(a,b,c,d)\ge f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)$.

By SMV theorem, we only need to consider the case $a=b=c=\frac{1-d}{3}$. We have:

$$f(a,b,c,d)=\frac{(1-d)^4}{27}+d^4+\frac{148d(1-d)^3}{729}-\frac{1}{27}=\frac{2d(4d-1)^2(19d+20)}{729}\ge 0$$

Equality occurs $\Leftrightarrow$ $a=b=c=d=\frac{1}{4}$ or $a=b=c=\frac{1}{3},d=0$ or its permutations.

**Problem 5.5.** Given $a, b, c, d\ge 0$ and $a+b+c+d=4$. Prove that

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2)\ge(1+a)(1+b)(1+c)(1+d)$$

*Proof*

Let $f(a,b,c,d)=(1+a^2)(1+b^2)(1+c^2)(1+d^2)-(1+a)(1+b)(1+c)(1+d)$

And assume that $a\le b\le c\le d$. We will prove: $f(a,b,c,d)\ge f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)$.

Indeed, since $a+c\le 2$ we have

$$(1+a^2)(1+c^2)-\left[1+\left(\frac{a+c}{2}\right)^2\right]^2=(a-c)^2\left(\frac{1}{2}-\frac{(a+c)^2+4ac}{16}\right)\ge 0$$

Using AM – GM, $(1+a)(1+c)\le\left(1+\frac{a+c}{2}\right)^2$.

Therefore $f(a,b,c,d)\ge f\left(\frac{a+c}{2},b,\frac{a+c}{2},d\right)$

Now we only need to consider the case $a=b=c=x,c=4-3x$:

$$f(x,x,x,4-3x)=(1+x^2)^3\left[1+(4-3x)^2\right]-(1+x)^3(5-3x)$$

$$=(x^6+3x^4+3x^2+1)(9x^2-24x+17)-(x^3+3x^2+3x+1)(5-3x)$$

$$=9x^8-24x^7+44x^6-72x^5+81x^4-68x^3+54x^2-36x+12$$

$=\left(9x^6-6x^5+23x^4-20x^3+18x^2-12x+12\right)(x-1)^2$

$=\left[x^4(3x-1)^2+2x^4+5x^2(2x-1)^2+10x^2+3(x-2)^2\right](x-1)^2\geq 0$

The problem is thus solved. Equality occurs if and only if $a=b=c=d=1$

**Problem 5.6 [Tukervic inequality]** Given $a,b,c,d\geq 0$. Prove that

$$a^4+b^4+c^4+2abcd\geq a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2 \quad (1)$$

***Proof***

Assume that $a\geq b\geq c\geq d$. Let $f(a,b,c,d)=LS(1)-RS(1)$

$=a^4+b^4+c^4+d^4+2abcd-a^2c^2-b^2d^2-\left(a^2+c^2\right)\left(b^2+d^2\right)$

$\Rightarrow f(a,b,c,d)-f\left(\sqrt{ac},b,\sqrt{ac},d\right)=(a-c)^2\left((a+c)^2-b^2-d^2\right)\geq 0$

By S.M.V, we only need to consider the case $a=b=c=t$.

The inequality $\Leftrightarrow 3t^4+d^4+2t^3d\geq 3t^4+3t^2d^2 \Leftrightarrow d^4+t^3d+t^3d\geq 3t^2d^2$

This is true due to AM – GM.

Equality occurs $\Leftrightarrow a=b=c=d$ or $a=b=c, d=0$ or its permutations

**Problem 5.7 (Pham Kim Hung)** Given $x,y,z,t\geq 0$ such that $x+y+z+t=4$.

Prove that: $(1+3x)(1+3y)(1+3z)(1+3t)\leq 125+131xyzt$

***Proof***

Let $f(x,y,z,t)=(1+3x)(1+3y)(1+3z)(1+3t)-131xyzt$

WLOG, we assume $x\geq y\geq z\geq t$. Consider the difference:

$$f(x,y,z,t)-f\left(\frac{x+z}{2},y,\frac{x+z}{2},t\right)=\frac{(x-z)^2}{4}\left(131yt-9(1+3y)(1+3t)\right)$$

Since $x\geq y\geq z\geq t$ we have $y+t\leq 2$, this leads to $9(1+3y)(1+3t)\geq 131yt$

Therefore $f(x,y,z,t)\leq f\left(\frac{x+z}{2},y,\frac{x+z}{2},t\right)$.

By S.M.V, we only need to consider the case $x=y=z=a\geq 1\geq t=4-3a$:

$(1+3a)^3(1+3(4-3a))\leq 125+131a^3(4-3a) \Leftrightarrow (a-1)^2(3a-4)(50a+28)\leq 0$

This is obviously true.

Equality occurs if $x=y=z=t=1$ or $x=y=z=\frac{4}{3}, t=0$ or its permutations.

• Now, we would like to conclude the section on MV method for "specific" inequalities (inequalities with 3 or 4 variables) to move on a new section with n-variable inequalities. You will see that, this is much more difficult. However, the main methods are developed from inequalities with 3 or 4 variables.

## VI. MV VIA CONVEX FUNCTION

Convex function plays an important role in the theory of inequalities. We now recall some basic definitions:

**1. Definition:** Function $f$: $[a, b] \to \mathbb{R}$ is called convex if:

$$f\left(tx+(1-ty)\right)\leq tf(x)+(1-t)f(y),\ \forall x,y\in[a,b],\forall t\in[0,1]$$

**2. Properties:**

***2.1***. Assume that $f$ has second derivative in interval (a, b), then $f$ is convex on [a, b] if $f''(x)\geq 0,\ \forall x\in(a,b)$.

***2.2.*** If $f$ is convex on [a, b] then $f$ is continuous on [a, b]. On the other hand, if $f$ is continuous on [a, b] then $f$ is convex on [a, b] if $f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}$, $\forall$ *x, y* $\in$ *(a, b).*

***2.3.*** Jensen inequality: Assume that $f$ is a convex function in $[a, b]$. Then we have

(i) $f\left(\frac{x_1+x_2+...+x_n}{n}\right)\leq\frac{f(x_1)+f(x_2)+...+f(x_n)}{n}$ $\forall x_1, x_2,\dots, x_n \in [a, b]$.

(ii) Let $x_i$ be $n$ numbers in [a, b ] and $\lambda_i$ be $n$ non-negative numbers with $\sum_{i=1}^{n}\lambda_i=1$, we have:

$$f\left(\lambda_1x_1+\lambda_2x_2+...+\lambda_nx_n\right)\leq\lambda_1f(x_1)+\lambda_2f(x_2)+...+\lambda_nf(x_n)$$

We now show you how to apply convex function, property 2.2 and Jensen inequality, in MV method.

**Problem 6.1.** Let *x, y, z* be real numbers such that $x+y+z=1$. Prove that

$$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\leq\frac{9}{10}\ (1)\qquad \textbf{\textit{(Poland 1992)}}$$

***Proof***

Consider $f(t)=\frac{t}{1+t^2}$ then (1) $\Leftrightarrow$ $f(x)+f(y)+f(z)\leq 3f\left(\frac{x+y+z}{3}\right)$.

We now prove that $f$ concave or $-f$ is convex. We have: $-f''(t)=\frac{2t(3-t^2)}{(1+t^2)^3}$ then $-f''(t)\geq 0$, $\forall t\in\left[0,\sqrt{3}\right]$. Thus if $x,y,z\in\left[0,\sqrt{3}\right]$ then done. In the remaining cases, for the simplicity, we can assume $x \geq y \geq z$. Since $x + y + z = 1$ and $x, y, z \notin [0, 1]$ then $z < 0$ which yields $f(z) < 0$.

If $y<\frac{1}{2}$ then $\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}<\frac{1}{2}+\frac{2}{5}+0=\frac{9}{10}$ thus it is sufficient to prove that $y\geq\frac{1}{2}$

If $0 \ge z \ge -\frac{1}{2}$ together with $y \ge \frac{1}{2}$ then: $\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2} \le \frac{x}{1+\left(\frac{1}{2}\right)^2}+\frac{y}{1+\left(\frac{1}{2}\right)^2}+\frac{4}{5}z=\frac{4}{5}<\frac{9}{10}$

Hence, it is enough to consider $z<-\frac{1}{2}$

If $-\frac{1}{2}>z\ge -3$ then: $\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}\le \frac{1}{2}+\frac{1}{2}-\frac{3}{10}=\frac{7}{10}<\frac{9}{10}$

If $z < -3$ then: $2x \ge x + y = 1 - z \ge 4$ so $x \ge 2$ and therefore:

$\frac{x}{1+x^2}+\frac{y}{1+y^2}+\frac{z}{1+z^2}<\frac{2}{5}+\frac{1}{2}+0=\frac{9}{10}$ (ans).

***Remark:*** If two of $x, y, z$ are in $\left[0,\sqrt{3}\right]$ then using MV method, they are equal. Using convex function in applying MV method is a good choice but it is suitable when the inequality has form $f\left(x_1\right)+f\left(x_2\right)+\ldots+f\left(x_n\right)$). Otherwise, we need more trick. Let us explorer more details.

**Theorem:** Let $f: [a, b] \to \mathbb{R}$ be a convex function. Then we have:

$$f(x)\le \text{Max}\left\{f(a), f(b)\right\}, \forall x\in[a,b]$$

**Proof:** Since $f$ is continuous then $f$ attaints its maximum value at $x_0\in[a,b]$.

Consider $\left|x_0-a\right|\le\left|x_0-b\right|\Rightarrow x_1=2x_0-a\in[a,b]$

From the definition, we have: $f(a)+f\left(x_1\right)\ge 2f\left(\frac{a+x_1}{2}\right)=2f\left(x_0\right)$

it follows that $f(a)=f\left(x_0\right)$. The case $\left|x_0-b\right|\le\left|x_0-a\right|$ is similar.

**Problem 6.2.** Given $a\ge b\ge c\ge 0$. Prove that: $2\left(\sqrt{a}-\sqrt{c}\right)^2\ge a+b+c-3\cdot\sqrt[3]{abc}$

***(USA TST 2004)***

***Proof***

Let $f(b)=a+b+c-3\cdot\sqrt[3]{abc}-2\left(\sqrt{a}-\sqrt{c}\right)^2$

Since $f''(b)=\frac{2\cdot\sqrt[3]{abc}}{3b^2}>0$ , we have $f(b)\le\max\left\{f(a), f(c)\right\}$. In addition:

$f(a)=2a+c-3\cdot\sqrt[3]{a^2c}-2\left(\sqrt{a}-\sqrt{c}\right)^2=-\left(c+3\cdot\sqrt[3]{a^2c}-4\sqrt{ac}\right)\le 0$ (AM – GM)

$f(c)=a+2c-3\cdot\sqrt[3]{ac^2}-2\left(\sqrt{a}-\sqrt{c}\right)^2=-\left(a+3\cdot\sqrt[3]{ac^2}-4\sqrt{ac}\right)\le 0$ (AM – GM)

The problem is thus solved. Equality occurs $\Leftrightarrow$ $a=b=c$

The generalization with $n$ variables of the above problem can be stated as following:

**Problem 6.3.** Given $a_1\ge a_2\ge\ldots\ge a_n\ge 0$. Prove that

$$(n-1)\left(\sqrt{a_1}-\sqrt{a_n}\right)^2\ge\ a_1+a_2+\ldots+a_n-n\cdot\sqrt[n]{a_1a_2\ldots a_n}$$

We now introduce two problems which are quite difficult to solve.

**Problem 6.4.** Let $0 < p < q$, and $x_i \in [p,q], i = 1,2,..,n$ be real numbers. Prove that:

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \le n^2 + \left[\frac{n^2}{4}\right]\frac{(p-q)^2}{pq}$$

Here we denote by $[x]$ the greatest integer number lest than or equal to $x$

***Proof***

Since $x_i \in [p,q]$, it is easy to see that the maximum value attains if $x_i \in \{p,q\}$ for every $i$.

We now assume that there are $k$ numbers $x_i$ which equals to $p$ and there are $n - k$ numbers which equals to $q$. Then the left hand side is equal to: $[kp + (n-k)q]\left(\frac{k}{p} + \frac{n-k}{q}\right)$

$$= k^2 + (n-k)^2 + k(n-k)\left(\frac{p}{q} + \frac{q}{p}\right) = n^2 + k(n-k)\frac{(p-q)^2}{pq} = n^2 + \frac{1}{4}\left[n^2 - (n-2k)^2\right]$$

Since $k$ is a integer number then $n^2 - (n-2k)^2 \le n^2$ ($n$ is even) and $n^2 - (n-2k)^2 \le n^2 - 1$ ($n$ is odd)

This completes our present proof.

For every $i$, we denote the left hand side by $f(x_i)$ a function of $x_i$, we now prove that:

$f(x_i) \le \max\{f(p), f(q)\}$. And equality holds if $x_i \in \{p,q\}$.

We have: $f(x) = Ax + \frac{B}{x} + C$. We can use derivation tool and easily to point out that equality occurs for $x_i \in \{p,q\}$. Here is a different approach.

Note that: $f(x_i) - f(p) = (x_i - p)\left(A - \frac{B}{x_i p}\right)$; $f(x_i) - f(q) = (x_i - q)\left(A - \frac{B}{x_i q}\right)$

Since then if $f(x_i) > \max\{f(p), f(q)\}$ then $x_i \notin \{p,q\}$ and $A - \frac{B}{x_i p} > 0, A - \frac{B}{x_i q} < 0$

$\Rightarrow \frac{B}{x_i p} < A < \frac{B}{x_i q}$ a contradiction $p < q$. Thus $f(x_i) \le \max\{f(p), f(q)\}$.

Let us consider the case of equality: Assume that $f(x_i) = \max\{f(p), f(q)\}$ for $x_i \notin \{p,q\}$.

If $f(x_i) = f(p)$ then $A = \frac{B}{x_i p} > \frac{B}{x_i q}$, therefore $f(x_i) - f(q) < 0$ (a contradiction).

If $f(x_i) = f(q)$ then $A = \frac{B}{x_i q} < \frac{B}{x_i p}$, therefore $f(x_i) - f(p) < 0$ (a contradiction).

Thus $f(x_i) = \max\{f(p), f(q)\}$ which is equivalent to $x_i \in \{p,q\}$

***Remarks:*** (Generalized problem): Let $a_i \in [a, A], b_i \in [b, B]$ for $0 < a \le A$ and $0 < b \le B$.

Find the maximum value of $T = \dfrac{\left(a_1^2 + ... + a_n^2\right)\left(b_1^2 + ... + b_n^2\right)}{a_1 b_1 + ... + a_n b_n}$

Ones can easy see that an upper bound of T is $\dfrac{1}{4}\left(\sqrt{\dfrac{AB}{ab}} + \sqrt{\dfrac{ab}{AB}}\right)^2$ by Polya.

It is natural to ask what the upper bound of AM-GM inequality is?

**Problem 6.5.** (Phan Thanh Nam) Let $0 < p < q$, and $n$ numbers $x_i \in [p, q]$.

Prove that: $T = \dfrac{x_1}{x_2} + \dfrac{x_2}{x_3} + ... + \dfrac{x_n}{x_1} \le n + \left[\dfrac{n}{2}\right]\dfrac{(p-q)^2}{pq}$

***Proof***

If follows from the above properties of function $f(x) = Ax + \dfrac{B}{x} + C$ that for every $i$, by replacing $x_i$ by $p$ or $x_i$ by $q$ $T$ is increasing. When $T$ is constant, $x_1, x_2, ..., x_n \in \{p, q\}$.

After finite replacing, we get $x_1, x_2, ..., x_n \in \{p, q\}$. If $x_1 = x_2 = ... = x_n = q$ then $T = n$ is the minimum value. In order to obtain the maximum value, we just assume the existence of $x_i = p$. Without lost of generality, we can suppose that $x_1 = p$. After replacing $x_2 = q$ we substitute $x_3 = p$. At the end of this process, we obtain

$T = \dfrac{n}{2}\left(\dfrac{p}{q} + \dfrac{q}{p}\right)$ (if n is even) and $T = \dfrac{n-1}{2}\left(\dfrac{p}{q} + \dfrac{q}{p}\right) + 1$ (if n is odd)

or equivalently $T = n + \left[\dfrac{n}{2}\right]\dfrac{(p-q)^2}{pq}$, $\forall n$

Equality occurs if $x_{2i} = q$, $x_{2i+1} = p$.

***Comment:*** You can see that the idea of "mixing variables" turns up early even with classical approach. Although classical inequalities are not powerful tools, however we can "stand on the shoulder of the giants". We have seen two important MV methods in previous sections (centre and boundary). In special cases when optima is at the centre, convex functions give us another interesting MV method which will be introduced in the next section.

### VII. UNDEFINED MIXING VARIABLES – UMV

MV method with convex functions is useful in many problems. However, its disadvantage is that it can not be used when functions of variable are not explicit or differentiation if too complicated. Undefined mixing variables method is designed to deal with this issue

***UMV – Undefined Mixing Variables Theorem:*** Given

- $D \subset \left\{x = (x_1, x_2, ...x_n) \in \mathbb{R}^n \mid x_i \geq 0, \forall i = 1, ..., n\right\}$, $D$ is closed and bounded. Let $\Lambda$ be the set of elements in $D$ such that $t$ coordinates are 0 and the rest are equal $(t \geq 0)$.
- 2 mappings $T_1, T_2 : D \to D$ such that: For each element $a = (a_1, a_2, ...a_n) \in D \setminus \Lambda$, select 2 indices $i \neq j$ such that $a_i = \min\{a_t > 0, t = 1, ..., n\}$ and $a_j = \max\{a_1, a_2, ...a_n\}$, then replace $a_i, a_j$ by $\alpha, \beta \in (a_i, a_j)$ (corresponding to $T_1$) and $\alpha' < a_i < a_j < \beta'$ (corresponding to $T_2$).
- $f : D \to \mathbb{R}$ is continuous such that: $f(a) \geq \min\left\{f(T_1(a)), f(T_1(a))\right\}, \forall a \in D$

then $f(x) \geq \min\limits_{y \in \Lambda}\{f(y)\}, \forall x \in D$

The theorem seems to be very abstract; however the essence is quite simple.

We begin to see applications of UMV with the following problem.

**Problem 7.1. [Le Trung Kien]** Given $a, b, c \geq 0$. Prove that:

$$a^3 + b^3 + c^3 + 9abc + 4(a + b + c) \geq 8(ab + bc + ca) \quad (1)$$

***Proof***

Let $f(a,b,c) = a^3 + b^3 + c^3 + 9abc + 4(a + b + c) - 8(ab + bc + ca)$

First of all, we consider the difference (i.e. see if two variables are equal):

$$f(a,b,c) - f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) = (3b + 3c - 9a + 8)\frac{(b-c)^2}{4}$$

We can not conclude anything. We now try to see if a variable is zero:

$$f(a,b,c) - f(a, b+c, 0) = -(3b + 3c - 9a + 8)bc$$

Since $(3b + 3c - 9a + 8)$ is either non-negative or non-positive, therefore

$$f(a,b,c) \geq \min\left\{f\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right); f(a, b+c, 0)\right\}$$

Using U.M.V: $f(a,b,c) \geq \min\left\{f(x,x,x); f(y,y,0); f(z,0,0)\right\}$

where $x = \dfrac{a+b+c}{3}; y = \dfrac{a+b+c}{2}; z = a + b + c$

Now: $f(z,0,0)=z^3+4z\ge 0$ ; $f(x,x,x)=12x^3-24x^2+12x=12x(x-1)^2\ge 0$

$f(y,y,0)=2y^3-8y^2+8y=2y(y-2)^2\ge 0 \Rightarrow f(a,b,c)\ge 0$.

Equality occurs $\Leftrightarrow (a,b,c)=(0,0,0);(1,1,1);(2,2,0)$.

***Comment:*** You can see the power of UMN variables through above example. When normal MV techniques does not work, combination of these techniques actually solves the problem.

**Problem 7.2 [Le Trung Kien]** Given $a,b,c,d\ge 0$ such that $a+b+c+d=4$.

Prove that: $2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)\ge abc+bcd+cda+dab+4$

***Proof***

Let $2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\right)-abc-bcd-cda-dab-4$. Consider the difference:

- $f(a,b,c,d)-f\left(\frac{a+b}{2},\frac{a+b}{2},c,d\right)=2\left(\sqrt{a}+\sqrt{b}-\sqrt{2(a+b)}\right)+(c+d)\left(\frac{(a+b)^2}{4}-ab\right)$

$$=\frac{(a-b)^2}{4}\left[c+d-\frac{8}{\left(\sqrt{a}+\sqrt{b}\right)^2\left(\sqrt{a}+\sqrt{b}+\sqrt{2(a+b)}\right)}\right]=\frac{(a-b)^2}{4}(c+d-X)$$

- $f(a,b,c,d)-f(a+b,0,c,d)$

$$=2\left(\sqrt{a}+\sqrt{b}-\sqrt{a+b}\right)-(c+d)=ab\left[\frac{4}{\sqrt{ab}\left(\sqrt{a}+\sqrt{b}+\sqrt{a+b}\right)}-c-d\right]=ab(Y-c-d)$$

It is easy to see that: $\dfrac{4}{\sqrt{ab}\left(\sqrt{a}+\sqrt{b}+\sqrt{a+b}\right)}\ge\dfrac{8}{\left(\sqrt{a}+\sqrt{b}\right)^2\left(\sqrt{a}+\sqrt{b}+\sqrt{2(a+b)}\right)}$

or equivalently $Y\ge X$ thus $\left[\begin{array}{l}c+d\le Y\\ c+d\ge X\end{array}\right.\Rightarrow f(a,b,c)\ge\min\left\{f\left(\frac{a+b}{2},\frac{a+b}{2},c,d\right);f(a+b,0,c,d)\right\}$

By U.M.V theorem we have

$$f(a,b,c,d)\ge\min\left\{f(1,1,1,1);f\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0\right);f(2,2,0,0),f(4,0,0,0)\right\}=0$$

The problem is then solved. Equality occurs $\Leftrightarrow (a,b,c,d)=(1,1,1,1);(4,0,0,0)$.

Now we turn back to Problem 5.2

**Problem 7.3.** Given $a_1,a_2,...,a_n\ge 0$ such that $a_1+a_2+...+a_n=n$. Prove that

$$(n-1)\left(a_1^2+a_2^2+...+a_n^2\right)+na_1a_2...a_n\ge n^2$$

***Proof***

Let $f\left(a_1,a_2,\ldots,a_n\right)=(n-1)\left(a_1^2+a_2^2+\ldots+a_n^2\right)+na_1a_2\ldots a_n$. We have

$$f\left(a_1,a_2,\ldots,a_n\right)-f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\ldots,a_n\right)=\frac{n\left(a_1-a_2\right)^2}{4}\left(\frac{2(n-1)}{n}-a_3a_4\ldots a_n\right)$$

$$f\left(a_1,a_2,\ldots,a_n\right)-f\left(0,a_1+a_2,a_3,\ldots,a_n\right)=-na_1a_2\left(\frac{2(n-1)}{n}-a_3a_4\ldots a_n\right)$$

$$\Rightarrow\ f\left(a_1,a_2,\ldots,a_n\right)\geq\min\left\{f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,\ldots,a_n\right),f\left(0,a_1+a_2,a_3,\ldots,a_n\right)\right\}$$

By UMV, we only need to prove when there is at least one number in $a_1,a_2,\ldots,a_n$ is 0 and the rest are 1. These cases are simple enough to prove.

***Remark:*** Examples of using UMV in inequalities with 3 variables, 4 variables and n variables, we would like to state that UMV is independent of the number of variables. This is because UMV uses arbitrary pair of variables. This feature of UMV helps us to solve many complicated problems.

**Problem 7.4 (Pham Kim Hung)** Given $a_1,a_2,\ldots,a_n\geq 0$ such that $a_1+a_2+...+a_n=n$.

Find the minimal value of $S=a_1^2+a_2^2+\ldots+a_n^2+a_1a_2...a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+......+\frac{1}{a_n}\right)$

***Solution***

Let $f\left(a_1,a_2,....a_n\right)=a_1^2+a_2^2+....+a_n^2+a_1a_2...a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+......+\frac{1}{a_n}\right)$

Consider the differences

- $f\left(a_1,a_2,....a_n\right)-f\left(0,a_1+a_2,a_3,....,a_n\right)=-a_1a_2\left[2-a_3a_4.....a_n\left(\frac{1}{a_3}+\frac{1}{a_4}+......+\frac{1}{a_n}\right)\right]$ (*)
- $f\left(a_1,a_2,....a_n\right)-f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},.a_3,....,a_n\right)$

$$=\frac{\left(a_1-a_2\right)^2}{4}\left[2-a_3a_4.....a_n\left(\frac{1}{a_3}+\frac{1}{a_4}+......+\frac{1}{a_n}\right)\right]\ (**)$$

$$\Rightarrow\ f\left(a_1,a_2,....a_n\right)\geq\min\left\{f\left(0,a_1+a_2,a_3,....,a_n\right),f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},.a_3,....,a_n\right)\right\}$$

Thus, by UMV we have: $f\left(a_1,a_2,....a_n\right)\geq\underset{k=1,n}{\mathrm{Min}}\,f\left(\frac{n}{k},\frac{n}{k},\ldots,\frac{n}{k},0,0...,0\right)$

Therefore, the minimum value is $\min\left\{2n,\frac{n^2}{n-1}+\left(\frac{n}{n-1}\right)^{n-1},\frac{n^2}{n-2}\right\}$

## VIII. MV WITH MEAN VALUE

Here is another technique in applying convex function by using MV method.

**Theorem 1:** Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then:

$$f(a)+f(b)\geq f(x)+f(a+b-x), \forall x\in[a,b]$$

***Proof***

Since $x\in[a,b]$ then $x=ta+(1-t)b$ for $t\in[0,1]$. Then: $a+b-x=(1-t)a+tb$

Using the definition of convex function, we have:

$$f(x)+f(a+b-x)=f[ta+(1-t)b]+f[(1-t)a+tb]$$

$$\leq[tf(a)+(1-t)f(b)]+[(1-t)f(a)+tf(b)]=f(a)+f(b)$$

**Theorem 2:** (Jensen inequality) Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then

$\forall\ x_1,x_2,...,x_n\in[a,b]$ , we have: $f(x_1)+f(x_2)+...+f(x_n)\geq nf(\frac{x_1+x_2+...+x_n}{n})=nf(T)$

***Proof***

• ***Step 1:*** If $x_1=x_2=...=x_n=T=\frac{x_1+x_2+...+x_n}{n}$ (*) then the inequality holds.

• ***Step 2:*** If (*) does not hold, without lost of generality, we can assume that $x_1>T>x_2$.

By replacing $(x_1,x_2,...x_n)$ by $(T,x_1+x_2-T,...,x_n)$ function $f$ is increasing. Moreover, after processing Step 2, the number of variables which is equal to T is increasing, therefore, (*) holds after at most $(n-1)$ steps.

*Comments:* In $n$-variable inequalities ($n\geq 4$), mixing variables to the arithmetic mean value is better than mixing variables to the center. Indeed, let us consider more examples.

**Problem 8.1.** Let $a_1,a_2,...a_n>0$ and $a_1a_2...a_n=1$. Prove that:

For $k = 4(n-1)$ we always have: $\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}+\frac{k}{a_1+a_2+...+a_n}\geq n+\frac{k}{n}$ (1)

***Proof***

For $n = 1$, $n = 2$ the assertion is trivial, we now need to prove under the case $n\geq 3$.

**Proposition 1:** Let $f(a_1,a_2,...a_n)=\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}+\frac{k}{a_1+a_2+...+a_n}$.Then we have:

($i$) If $a_1\leq x\leq a_2$ and $a_1a_2\leq 1$ then $f(a_1,a_2,...a_n)\geq f\left(x,\frac{a_1a_2}{x},a_3,...,a_n\right)$

(*ii*) If $(1-a_1)(1-a_2)\left[ka_1a_2-\sum_{i=1}^{n}a_i\left(\sum_{i=3}^{n}a_i+a_1a_2+1\right)\right]\ge 0$ then $f(a_1,a_2,...a_n)\ge f(1,a_1a_2,a_3,...,a_n)$

(*iii*) If $a_1,a_2\ge 1\ge a_3$ then $f(a_1,a_2,...a_n)\ge \min\{f(1,a_1a_2,a_3,...,a_n),f(1,a_2,a_1a_3,a_1a_i,...,a_n)\}$

***Proof:***

(***i***) We see that

$$f(a_1,a_2,...a_n)-f\left(x,\frac{a_1a_2}{x},a_3,...,a_n\right)=\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{x}-\frac{x}{a_1a_2}+\frac{k}{A+a_1+a_2}-\frac{k}{A+x+\frac{a_1a_2}{x}}$$

$$=\frac{(x-a_1)(x-a_2)\left[(A+a_1+a_2)\left(A+x+\frac{a_1a_2}{x}\right)-ka_1a_2\right]}{xa_1a_2(A+a_1+a_2)\left(A+x+\frac{a_1a_2}{x}\right)} \text{ where } A=\sum_{i=3}^{n}a_i$$

According to AM – GM: $(A+a_1+a_2)\left(A+x+\frac{a_1a_2}{x}\right)\ge n^2\ge 4(n-1)=k\ge ka_1a_2 \Rightarrow$ (ans).

(***ii***) From the above inequalities, we put $x$ = 1 then: $f(a_1,a_2,...a_n)-f(1,a_1a_2,a_3,...,a_n)=$

$$=\frac{(1-a_1)(1-a_2)\left[ka_1a_2-(A+a_1+a_2)(A+a_1a_2+1)\right]}{a_1a_2(A+a_1+a_2)(A+a_1a_2+1)} \text{ which yields the assertion.}$$

(***iii***) We consider two following cases: $(1-a_1)(1-a_2)\left[ka_1a_2-\sum_{i=1}^{n}a_i\left(\sum_{i=3}^{n}a_i+a_1a_2+1\right)\right]\ge 0$

***Case 1:*** If $ka_1a_2\ge\sum_{i=1}^{n}a_i\left(\sum_{i=3}^{n}a_i+a_1a_2+1\right)$ then from (*ii*) we get $f(a_1,a_2,...a_n)\ge f(1,a_1a_2,a_3,...,a_n)$.

***Case 2:*** If $ka_1a_2\le\sum_{i=1}^{n}a_i\left(\sum_{i=3}^{n}a_i+a_1a_2+1\right)$ since $a_3\le 1\le a_2$ then we have:

$$\frac{\sum_{i\ne 1,3}a_i+a_1a_3+1}{a_1a_3}=\frac{\sum_{i=1}^{n}a_i-a_1-a_3+1}{a_1a_3}+1\ge\frac{\sum_{i=1}^{n}a_i-a_1-a_2+1}{a_1a_2}+1=\frac{\sum_{i=3}^{n}a_i+a_1a_2+1}{a_1a_2}$$

$\Rightarrow ka_1a_3\le\sum_{i=1}^{n}a_i\left(\sum_{i\ne 1,3}a_i+a_1a_3+1\right)$. Again by using (*ii*) we have: $f(a_1,a_2,...a_n)\ge f(1,a_2,...,a_1a_i,...,a_n)$.

***Remark:*** Proposition 1 helps us to reduce the number of variables to one.

**Proposition 2:** Inequality (1) with variables $a_1, a_2, \dots a_n > 0$ and $a_1 a_2 \dots a_n = 1$ can be deduced to the case where there are $n - 1$ variables which are equal and $\leq 1$.

***Proof:***

• ***Step 1:*** Reduce to the case where there are $n - 1$ variables which are less than or equal to 1.

Assume that at least two variables are greater than 1, without lost of generality, we may assume $a_1, a_2$. Using Proposition 1 (*iii*) we can replace $(a_1, a_2, \dots a_n)$ by another such that $f$ is unchanged, and moreover, the number of variables which are equal to 1 is increasing at least 1. Hence, we obtain our conclusion after at most $(n - 1)$ steps.

• ***Step 2:*** Deduce these variables are equal.

Assume that there are $a_1 \leq a_2 \leq \dots \leq a_{n-1} \leq 1$ such that its geometric mean value equals to $x$.

If they are not equal, then $a_1 < x < a_{n-1}$. Using Proposition 1 (*i*) we can replace $(a_1, a_2, \dots a_{n-1}, a_n)$ by $\left(x, a_2, \dots, \frac{a_1 a_{n-1}}{x}, a_n\right)$. Then $f$ is not decreasing and then number of variables which are equal to $x$ is increasing at least 1. On the other hand, since $\frac{a_1 a_{n-1}}{x} \leq \frac{a_1}{x} \leq 1$ then $a_1 \leq x$, this means that we can apply this step again and again.

Now we prove the one-variable problem: $f\left(x, x, \dots, x, \frac{1}{x^{n-1}}\right) \geq f(1, 1, \dots, 1)$ for $x \leq 1$

Letting $g(x) = f\left(x, x, \dots, x, \frac{1}{x^{n-1}}\right) = \frac{n-1}{x} + x^{n-1} + \frac{k}{(n-1)x + \frac{1}{x^{n-1}}}$ for $x \in (0, 1]$

After some calculations, we get:

$$g'(x) = -\frac{n-1}{x^2} + (n-1)x^{n-2} - \frac{k\left[n-1-\frac{n-1}{x^n}\right]}{\left[(n-1)x + \frac{1}{x^{n-1}}\right]^2} = (n-1)\frac{x^n - 1}{x^2}\left[\frac{(n-1)x^n - 1}{(n-1)x^n + 1}\right]^2$$

Since $k = 4(n - 1)$ then $g'(x) \leq 0$ for $x \in (0, 1]$, it follows that $g(x) \geq g(1)$ which completes our proof.

**Problem 8.2.** Let $a_1, a_2, \dots a_n$ be $n$ real numbers such that $a_1 . a_2 \dots a_n = 1$. Prove that:

$$\left(1 + a_1^2\right)\left(1 + a_2^2\right)\dots\left(1 + a_n^2\right) \leq \frac{2^n}{n^{2n-2}}\left(a_1 + a_2 + \dots + a_n\right)^{2n-2}$$

***Proof***

The cases $n = 1$, $n = 2$ are trivial. We now consider the case $n \geq 3$.

It is sufficient to prove $f(a_1, a_2, \dots a_n) \geq 0$ or prove $g(a_1, a_2, \dots a_n) \geq 0$,

where $f(a_1, a_2, \dots a_n) = k(a_1 + a_2 + \dots + a_n)^{2n-2} - \left(1 + a_1^2\right)\left(1 + a_2^2\right)\dots\left(1 + a_n^2\right)$

$$g(a_1, a_2, \dots a_n) = \ln k + (2n-2)\ln(a_1 + a_2 + \dots + a_n) - \ln\left(1 + a_1^2\right) - \ln\left(1 + a_2^2\right) - \dots - \ln\left(1 + a_n^2\right)$$

**Proposition 1:**

($i$) If $a_1 \ge 1 \ge a_2, a_3$ then: $f(a_1,a_2,...a_n) \ge \min\{f(1,a_1a_2,a_3,...,a_n), f(a_1,1,a_2a_3,...,a_n)\}$

($ii$) If $a_1 = \max\{a_i\}_{i=1}^n$ and $a_1 \ge x \ge a_2 \ge 1$ then: $g(a_1,a_2,...a_n) \ge g\left(x, \dfrac{a_1a_2}{x}, a_3,...,a_n\right)$

***Proof***

($i$) Consider $f(a_1,a_2,...a_n) - f(1,a_1a_2,a_3,...,a_n) =$

$$= ks^{2n-2} - ku^{2n-2} + \left[2(1+a_1^2a_2^2) - (1+a_1^2)(1+a_2^2)\right](1+a_3^2)...(1+a_n^2)$$

(where $s = a_1 + a_2 + ... + a_n, u = 1 + a_1a_2 + ... + a_n$)

$$= k(a_1 + a_2 - 1 - a_1a_2)(s^{2n-3} + s^{2n-4}u + ... + u^{2n-3}) + (1-a_1^2)(1-a_2^2)(1+a_3^2)...(1+a_n^2)$$

$$= -(1-a_1)(1-a_2)\left[k(s^{2n-3} + s^{2n-4}u + ... + u^{2n-3}) - (1+a_1)(1+a_2)(1+a_3^2)...(1+a_n^2)\right]$$

Similarly, we have: $f(a_1,a_2,...a_n) - f(a_1,1,a_2a_3,...,a_n) =$

$$= -(1-a_2)(1-a_3)\left[k(s^{2n-3} + s^{2n-4}v + ... + v^{2n-3}) - (1+a_2)(1+a_3)(1+a_1^2)(1+a_4^2)...(1+a_n^2)\right]$$

(where $v = 1 + a_1 + a_2a_3 + a_4 + ... + a_n$)

From the above equalities, we see that:

- If $k(s^{2n-3} + s^{2n-4}u + ... + u^{2n-3}) - (1+a_1)(1+a_2)(1+a_3^2)...(1+a_n^2) \ge 0$ (2)

  then $f(a_1,a_2,...a_n) \ge f(1,a_1a_2,a_3,...,a_n)$.

- If $k(s^{2n-3} + s^{2n-4}v + ... + v^{2n-3}) - (1+a_2)(1+a_3)(1+a_1^2)(1+a_4^2)...(1+a_n^2) \le 0$ (3)

  then $f(a_1,a_2,...a_n) \ge f(a_1,1,a_2a_3,...,a_n)$.

Therefore, it is necessary to prove either (2) or (3).

For example, we assume that (2) not hold, we shall prove that (3) hold.

It is enough to show that $u \ge v$ and $(1+a_1)(1+a_3^2) \le (1+a_3)(1+a_1^2)$

This is followed form some simple calculation

$$u - v = a_3 + a_1a_2 - a_1 - a_2a_3 = (1-a_2)(a_1-a_3) \ge 0$$

$$(1+a_1)(1+a_3^2) - (1+a_3)(1+a_1^2) = (a_3-a_1)(a_1a_3 + a_1 + a_3 - 1) \le 0$$

This proves Proposition ($i$).

(***ii***) Because of the presence of function *ln* we must use derivation tool instead of applying the above method. Consider: $g(t) = \ln k + 2(n-1)\ln\left(ta_1 + \dfrac{a_2}{t} + a_3 + ... + a_n\right)$

$$-\ln(1+t^2a_1^2) - \ln\left(1+\frac{a_2^2}{t^2}\right) - \ln(1+a_3^2)... - \ln(1+a_n^2) \text{ where } t \in \left[\sqrt{\frac{a_2}{a_1}}, 1\right]$$

We have: $g'(t)=\dfrac{2(n-1)\left(a_1-\dfrac{a_2}{t^2}\right)}{ta_1+\dfrac{a_2}{t}+a_3+...+a_n}-\dfrac{2ta_1^2-\dfrac{2a_2^2}{t^3}}{\left(1+t^2a_1^2\right)\left(1+\dfrac{a_2^2}{t^2}\right)}=$

$$=2\left(a_1-\frac{a_2}{t^2}\right)\left[\frac{n-1}{ta_1+\dfrac{a_2}{t}+a_3+...+a_n}-\frac{ta_1+\dfrac{a_2}{t}}{\left(1+t^2a_1^2\right)\left(1+\dfrac{a_2^2}{t^2}\right)}\right]$$

Since $t\in\left[\sqrt{\dfrac{a_2}{a_1}},1\right]$ then $a_1-\dfrac{a_2}{t^2}\ge 0$. Therefore, denote by $T$ the last factor, it is enough to show that $T\ge 0$ in order to show that $g$ is increasing (in $\left[\sqrt{\dfrac{a_2}{a_1}},1\right]$)

Denote $c=\sqrt{\left(1+t^2a_1^2\right)\left(1+\dfrac{a_2^2}{t^2}\right)}, d=ta_1+\dfrac{a_2}{t}$

We have: $T\ge 0\Leftrightarrow\dfrac{n-1}{d+a_3+...+a_n}\ge\dfrac{d}{c^2}\Leftrightarrow(n-1)c^2\ge d^2+d\left(a_3+...+a_n\right)$

Since $c\ge d$ (Cauchy-Schwarz inequality) then it suffices to prove that $(n-2)c\ge a_3+...+a_n$

And it is true since $c>a_1a_2\ge a_1\ge\max\left\{a_3,...,a_n\right\}$

Choose $t_0=\max\left\{x,\dfrac{a_1a_2}{x}\right\}\cdot\dfrac{1}{a_1}$ , we get $t_0\in\left[\sqrt{\dfrac{a_2}{a_1}},1\right], t_0a_1=\max\left\{x,\dfrac{a_1a_2}{x}\right\},\dfrac{a_2}{t_0}=\min\left\{x,\dfrac{a_1a_2}{x}\right\}$

Since $g$ is increasing in $\left[\sqrt{\dfrac{a_2}{a_1}},1\right]$ then $g(1)\ge g(t_0)$ (ans).

This completes the proof of Proposition (*ii*).

Turn back to our problem. We say that $\left(a_1,a_2,...a_n\right)$ is substituted by $\left(b_1,b_2,...b_n\right)$ if $f\left(a_1,a_2,...a_n\right)\ge f\left(b_1,b_2,...b_n\right)$ or $g\left(a_1,a_2,...a_n\right)\ge g\left(b_1,b_2,...b_n\right)$.

**Proposition 2:** We always deduce to the case where there are $(n-1)$ variables which are equal and $\ge 1$.

***Proof***

• ***Step 1:*** Deduce to the case where there are $n-1$ variables which are equal and $\ge 1$.

Assume that there exist $a_2,a_3<1$. It is easy to see that at least one of which is greater than 1, without lost of generality, we can assume that it is $a_1$. By using Proposition 1 (*i*), we can substitute $\left(a_1,a_2,...a_n\right)$ by $\left(1,a_1a_2,a_3,...a_n\right)$ or by $\left(a_1,1,a_2a_3,...a_n\right)$. We observe that the number of variables which are equal to 1 is increasing, thus, we cannot substitute over $n-1$ times.

• ***Step 2:*** We now claim that we can substitute $n-1$ variables which are greater than or equal to 1 by their GM. Indeed, assume that $a_1 \ge a_2 \ge ... \ge a_{n-1} \ge 1 \ge a_n$ and put $x = \sqrt[n-1]{a_1 a_2 ... a_{n-1}} \ge 1$. If there exists at least one variable which is different from the first $n-1$ variables then $a_1 > x > a_{n-1}$. By using Proposition (*ii*), we can substitute $(a_1, a_2, ... a_{n-1}, a_n)$ by $\left(x, a_3, ..., \frac{a_1 a_2}{x}, a_n\right)$. Since $\frac{a_1 a_{n-1}}{x} \ge a_{n-1} \ge 1$ (because $a_1$ is the greatest number in $\{a_i\}_{i=1}^{n-1}$, this means that $a_1 > x$) then our substitute is valid. After at most $n$ – 1 times, the first $n$ – 1 variable are equal to $x$. We now turn to the one-variable problem.

Consider the following function $h(x) = g\left(x, x, ..., x, \frac{1}{x^{n-1}}\right)$

$= \ln k + 2(n-1)\ln\left[(n-1)x + \frac{1}{x^{n-1}}\right] - (n-1)\ln(1+x^2) - \ln\left(1 + \frac{1}{x^{2n-2}}\right)$ for $x \ge 1$.

We have: $$h'(x) = 2(n-1)\frac{n-1-\frac{n-1}{x^n}}{(n-1)x + \frac{1}{x^{n-1}}} - \frac{2(n-1)x}{1+x^2} - \frac{-\frac{2(n-1)}{x^{2n-1}}}{1+\frac{1}{x^{2n-2}}}$$

$$= \frac{2(n-1)}{x}\left[\frac{(n-1)(x^n-1)}{(n-1)x^n+1} - \frac{x^2}{1+x^2} + \frac{1}{1+x^{2n-2}}\right] = \frac{2(n-1)}{x}\left[\frac{(n-1)(x^n-1)}{(n-1)x^n+1} - \frac{x^{2n}-1}{(1+x^2)(1+x^{2n-2})}\right]$$

Since $x \ge 1$ then $h'(x) \ge 0$ provided: $\frac{n-1}{(n-1)x^n+1} \ge \frac{x^n+1}{(1+x^2)(1+x^{2n-2})}$

The above inequality is easy, for example: $\frac{n-1}{(n-1)x^n+1} \ge \frac{1}{x^n+1} \ge \frac{x^n+1}{(1+x^2)(1+x^{2n-2})}$

Thus, for $x \ge 1$ we get $h'(x) \ge 0$ or equivalently $h(x)$ is increasing,

it follows that $h(x) \ge h(1) = 0$ (ans).

Equality occurs if $a_1 = a_2 = ... = a_n = 1$ for $n \ge 3$.

***Remarks:***

• The above problem is an open problem by Pham Kim Hung and our proof is the first presence of it.

• By considering simultaneously functions $f$, $g$, it is able to expand MV method.

**IX. INEQUALITY GENERAL INDUCTION (IGI)**

One of the mixing variable technique (($n$ – 1) equal variable), which was introduced by Pham Kim Hung in his book "Secrets in Inequalities" we should mention is the IGI. We may think about the ideas of this technique as follows. Suppose that we want to prove an inequality in n variables $a_1, a_2, ..., a_n$ with the constraint $a_1 a_2 ... a_n = 1$, where equality holds when the variables are all equal. Because of the condition of the variables, we cannot apply the induction method directly. However, if we modify the constraint to $a_1 a_2 ... a_n \ge 1$ then we may assume that $a_n = \min\{a_1, a_2, ..., a_n\}$ to obtain $a_1 a_2 ... a_{n-1} \ge 1$ and we may then use the hypothesis for these n-1 variables to reduce the problem to the case where n-1 variables are all equal. Following are some typical examples for this technique.

**Problem (Pham Kim Hung).** Given $a_1, a_2, ..., a_n > 0$ and $a_1 a_2 ... a_n = 1$.

Prove that: $$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + ... + \frac{1}{(1+a_n)^k} \ge \min\left\{1, \frac{n}{2^k}\right\}, \ \forall k > 0$$

*Solution*

• We prove a more general result:

If $k > 0$ and $a_1, a_2, ..., a_n$ are positive numbers whose product $\ge 1$ then:

$$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + ... + \frac{1}{(1+a_n)^k} \ge \min\left\{1, \frac{n}{(1+\sqrt[n]{a_1 a_2 ... a_n})^k}\right\} \quad (1)$$

• Induction on $n$.

If $n = 1$ then (1) is obviously true! Suppose (1) holds true to $n$ $(n \ge 1)$, we show that it is also true for $n+1$.

Let $a_1, a_2, ..., a_{n+1}$ be positive real numbers whose product is $s^{n+1}$ where $s \ge 1$,

we need to prove: $$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + ... + \frac{1}{(1+a_{n+1})^k} \ge \min\left\{1, \frac{n+1}{(1+s)^k}\right\} \quad (2)$$

We now fix s. First, let us simplify the problem.

• ***Clause:*** Let $k_0 = \frac{\ln(n+1)}{\ln(1+s)}$ (which means $\frac{n+1}{(1+s)^{k_0}} = 1$)

Suppose (2) holds true for $k = k_0$, then so is every $k > 0$. Moreover, we only have $k_0 \ge n$ in the case $n = k_0 = s = 1$ and then (2) is also true.

***Proof:*** The fact that (2) holds true for $k = k_0$ means that $\sum_{i=1}^{n+1} \frac{1}{(1+a)^{k_0}} \ge 1 = \frac{n+1}{(1+s)^{k_0}}$

If $k > k_0$ then: $$\frac{\sum_{i=1}^{n+1} \frac{1}{(1+a_i)^k}}{n+1} \ge \left(\frac{\sum_{i=1}^{n+1} \frac{1}{(1+a_i)^{k_0}}}{n+1}\right)^{\frac{k}{k_0}} \ge \left(\frac{\left[\frac{n+1}{(1+s)^{k_0}}\right]}{n+1}\right)^{\frac{k}{k_0}} = \frac{1}{(1+s)^k}$$

If $k < k_0$ then: $\sum_{i=1}^{n+1}\frac{1}{(1+a_i)^k} \ge \sum_{i=1}^{n+1}\frac{1}{(1+a_i)^{k_0}} \ge 1$

Moreover, if $k_0 \ge n$ then $n+1 = (1+s)^{k_0} \ge 1 + k_0 s \ge 1+n$ (since $k_0 \ge n \ge 1$ and $s \ge 1$)

Which implies $n = k_0 = s = 1$, then (2) is obviously true.

• Now we return to our problem. The preceding claim allows us to prove (2) for the only case $k = k_0$, and $k_0 < n$. From now we will consider k to be $k_0$ , which means that $k = \frac{\ln(n+1)}{\ln(1+s)}$.

Assume that $a_1 \ge a_2 \ge ... \ge a_{n+1}$. Then $a_1, a_2, ..., a_n$ are positive numbers whose product $\ge 1$, hence by induction hypothesis

$$\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + ... + \frac{1}{(1+a_n)^k} \ge \min\left\{1, \frac{n}{(1+a)^k}\right\} \text{ where } a = \sqrt[n]{a_1 a_2 ... a_n} \ge s$$

Since $a_{n+1} = \frac{s^{n+1}}{a^n}$, (2) is true if we can show that $\frac{n}{(1+a)^k} + \frac{1}{\left(1+\frac{s^{n+1}}{a^n}\right)^k} \ge \min\left\{1, \frac{n+1}{(1+s)^k}\right\} = 1$ (3)

• Consider the LHS of (3) as a function in a and we denote it by $f(a)$. In order to prove (3), we will investigate the behavior of $f$ where $a \in \mathbb{R}^+$. We have: $f'(a) = \frac{-kn}{(1+a)^{k+1}} + \frac{kn\frac{s^{n+1}}{a^{n+1}}}{\left(1+\frac{s^{n+1}}{a^n}\right)^{k+1}}$

$$f'(s) \ge 0 \Leftrightarrow g(a) = (n+1)\left[\ln(s) - \ln(a)\right] + (k+1)\left[\ln(1+a) - \ln\left(1+\frac{s^{n+1}}{a^n}\right)\right] \ge 0$$

Note that: $g'(a) = \frac{-(n-k)a^{n+1} - (n+1)a^n + (n+1)ks^{n+1}a + (kn-1)s^{n+1}}{a(1+a)(a^n + s^{n+1})}$

$$g'(a) \ge 0 \Leftrightarrow h(a) = -(n-k)a^{n+1} - (n+1)a^n + (n+1)ks^{n+1}a + (kn-1)s^{n+1} \ge 0$$

We have: $h'(a) = (n+1)\left[-(n-k)a^n - na^{n-1} + ks^{n+1}\right]$

Since $k < n$, $h'(a)$ changes its sign from the positive sign to the negative one on $\mathbb{R}^+$. Moreover $h(0) > 0$, hence $h(a)$ also changes its sign from the positive sign to the negative one on $\mathbb{R}^+$. Since $g'(a) = 0$ has the same sign with $h(a)$ , the equation $g(a) = 0$ has at most 2 roots on $\mathbb{R}^+$. It follows that $f'(a) = 0$ has at most 2 roots on $\mathbb{R}^+$. Moreover, the facts that $f'(s) = 0$ and $\lim_{a\to 0^+} f(a) = n \ge 1 = f(s) = \lim_{a\to+\infty} f(a)$ implies that the other root of $f'(a) = 0$ on $\mathbb{R}^+$ is $a_0 \in (s, +\infty)$. Therefore $f(a)$ increases on $(s, a_0)$ and decreases on the intervals $(0, s)$ and $(a_0, \infty)$.

It follows that $f(a) \ge f(s) = \lim_{x\to+\infty} f(x) = 1$, $\forall a \in \mathbb{R}^+$. The proof of the problem is completed.

**X. ENTIRELY MIXING VARIABLES – EMV**

All the MV techniques we have seen so far have one thing in common, that is if $f(a,b,c) \geq f(a',b',c')$ then the tuple $(a,b,c)$ and $(a',b',c')$ must have the same characteristic, for example sum or product, etc.

EMV technique also has unchanged quantity after "mixing variables". However, the difference with other methods is that the quantities used are the difference of variables, i.e. $a-b, b-c, c-a$".

The idea of EMV is that when we increase or decrease variables with the same value then the inequality becomes unchanged or weakened. Thus we only need to consider the case when one variable is on the boundary.

Not that if we use EMV for inequalities with $n$ variables, when force one variable to the boundary, the inequality become new inequality with n-1 variable, if this has more than 3 variables then it is still complicated. Thus, we will introduce only the application of EMV for inequalities with 3 variables

## 1. EMV with the boundary at 0:

There are many inequailities where variable has boundary at 0, especially those with three variables. In IV, we consider a technique to force the variables the the boundary at 0. Now EMV gives us another approach. We will begin with a famous inequality

**Problem 10.1 [Dao Hai Long]** Given $a,b,c \in \mathbb{R}$. Prove that

$$(a^2+b^2+c^2)\left(\frac{1}{(b-c)^2}+\frac{1}{(a-c)^2}+\frac{1}{(b-a)^2}\right) \geq \frac{9}{2}$$

*Proof*

Since $LS(a,b,c) \leq LS(|a|,|b|,|c|)$ we only need to consider the case when $a \geq b \geq 0, c \leq 0$

We have: $a^2+b^2+c^2 = \frac{(a+b+c)^2}{3} + \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3} \geq (a-b)^2+(b-c)^2+(c-a)^2$

Thus, it suffices to prove that

$$\left((a-b)^2+(b-c)^2+(c-a)^2\right)\left(\frac{1}{(b-c)^2}+\frac{1}{(a-c)^2}+\frac{1}{(b-a)^2}\right) \geq \frac{27}{2} \quad (1)$$

Let LHS (1) = $f(a, b, c)$. Notice that the above inequality contains only characteristic quantities of EMV, therefore $f(a,b,c) = f(a-x, b-x, c-x), \forall x \in \mathbb{R}$. We now assume a variable is 0, let assume $b$, i.e. $x = b$ the tuple $(a-b, 0, c-b)$ satisfies $a-b \geq 0, c-b \leq 0$.

Now, we need to prove: $\left((a-c)^2+c^2+a^2\right)\left(\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{(c-a)^2}\right)\geq\frac{27}{2}$ (2)

We have: $c^2+a^2\geq\frac{(a-c)^2}{2},\frac{1}{c^2}+\frac{1}{a^2}\geq\frac{8}{(c-a)^2}$.

Thus: LHS (2) $\geq\left(1+\frac{1}{2}\right)(a-c)^2\,\frac{(1+8)}{(a-c)^2}=\frac{27}{2}$. The problem is then solved.

Equality occurs if $a=t,b=0,\ c=-t$ for $t\in\mathbb{R}^*$ or its permutations.

***Remark:*** EMV technique is used uniquely in this problem, in which the expression is unchanged. Let come back to Schur inequality again.

**Problem 10.2 (Schur inequality)** Given $a,\ b,\ c\geq 0$. Prove that:

$$a^3+b^3+c^3+3abc\geq a^2(b+c)+b^2(c+a)+c^2(a+b)\quad(1)$$

***Proof***

$(1)\Leftrightarrow\ f(a,b,c)=(b+c-a)(b-c)^2+(a+c-b)(a-c)^2+(a+b-c)(b-a)^2\geq 0$ (2)

We will prove $f(a,b,c)\geq f(a-x,b-x,c-x),\ \forall x\in[0,\min\{a,b,c\}]$.

This is obviously true because $b+c-a\geq b+c-a-x=b-x+c-x-a+x$ and similar results.

By EMV, we only need to consider the case when one variable is 0, which is true.

***Remark:*** The solution of above problem use EMV under SOS form:

$$S_a(a,b,c)(b-c)^2+S_b(a,b,c)(c-a)^2+S_c(a,b,c)(a-b)^2\geq 0$$

This is efficient technique because it eliminates quadratic terms $(a-b)^2,(b-c)^2,(c-a)^2$.

Thus we only need to prove $S_g(a,b,c)\geq S_g(a-x,b-x,c-x),g\in\{a,b,c\}$

However, it is not always that simple:

**Problem 10.3.** Given $a,\ b,\ c\geq 0$. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{9}{a+b+c}\geq 4\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)$ (1)

***Proof***

The inequality (1) $\Leftrightarrow\ (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)+3$

$$\Leftrightarrow\sum\left(\frac{1}{bc}-\frac{2}{(a+b)(a+c)}\right)(b-c)^2\geq 0\Leftrightarrow\sum a(b+c)(a^2+ab+ac-bc)(b-c)^2\geq 0$$

Now, you can see that the technique used in previous problem is not applicable in this problem. Thus we need to find more suitable transformation. Note that with EMV technique, the more $a-b,b-c,c-a$ occur, the more efficient the method is.

Thus we continue the transformation as following $\sum a(b+c)(a^2+ab+ac-bc)(b-c)^2 \geq 0$

$$\Leftrightarrow 2\sum a^3(b+c)(b-c)^2 - \sum a(b+c)(a-b)(a-c)(b-c)^2 \geq 0$$

$$\Leftrightarrow 2\sum a^3(b+c)(b-c)^2 + (b-c)^2(c-a)^2(a-b)^2 \geq 0$$

This is obviously true. Equality occurs $\Leftrightarrow$ $a=b=c$ or $a=b, c=0$ and its permutations.

***Comments:*** Transformation to expressions of $(b-c)^2(c-a)^2(a-b)^2$ is nice and efficient. You can see the tightness of that expression in estimation that closes to 0. It is much clearer when we come back to Iran inequality

**Problem 10.4.** Given $a,b,c \geq 0$. Prove that $(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geq \frac{9}{4}$ (1)

***(Iran TST 1996)***

This problem is the most well studied and there are many solutions have been developed for it. It is "surprised" to know that this problem can also be solved by EMV.

First of all, we carry out the following transformation

$$\sum \frac{bc+ca+ab}{(b+c)^2} - \frac{9}{4} = \sum \frac{4b(a-b)+4c(a-c)+(b-c)^2}{4(b+c)^2}$$

$$= \sum (b-c)\left(\frac{c}{(c+a)^2} - \frac{b}{(b+a)^2}\right) + \frac{1}{4}\sum \frac{(b-c)^2}{(b+c)^2}$$

$$= \sum \left(\frac{bc-a^2}{(a+b)^2(a+c)^2} + \frac{1}{4(b+c)^2}\right)(b-c)^2$$

$$= \sum \frac{(a-b)(a-c)(a^2+bc+3ab+3ac)}{4(b+c)^2(c+a)^2(a+b)^2}(b-c)^2 + \sum \frac{bc(b-c)^2}{(c+a)^2(a+b)^2}$$

$$= \sum \frac{bc(b-c)^2}{(c+a)^2(a+b)^2} - \frac{(b-c)^2(c-a)^2(a-b)^2}{4(b+c)^2(c+a)^2(a+b)^2}$$

Thus (1) $\Leftrightarrow$ $\sum \frac{bc(b-c)^2}{(c+a)^2(a+b)^2} \geq \frac{(b-c)^2(c-a)^2(a-b)^2}{4(b+c)^2(c+a)^2(a+b)^2}$

$$\Leftrightarrow 4\sum bc(b^2-c^2)^2 \geq (b-c)^2(c-a)^2(a-b)^2$$

Therefore, we only need to consider the case when one variable is 0, which is already proved in previous section. Moreover, we can tighten the inequality as following:

**Problem 10.5 [Le Trung Kien]** Given $a,b,c\ge 0$. Prove that

$$(ab+bc+ca)\left(\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\right)\ge\frac{9}{4}+\frac{15(b-c)^2(c-a)^2(a-b)^2}{4(b+c)^2(c+a)^2(a+b)^2}$$

Again, we only need to prove when $c=0$:

$$ab\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{(a+b)^2}\right)\ge\frac{9}{4}+\frac{15(a-b)^2}{4(a+b)^2}\Leftrightarrow (a-b)^4\ge 0$$

Equality occurs if and only if $a=b=c$ or $a=b,c=0$ or its permutations.

***Notes:*** The above presentation is to illustrate the idea of EMV. You can also see that the inequality is directly equivalent to

$$\left[ab(a+b)^2-4(a-c)^2(b-c)^2\right](a-b)^2+ca\left(c^2-a^2\right)^2+bc\left(b^2-c^2\right)^2\ge 0$$

This is true because $a\ge a-c\ge 0, b\ge b-c\ge 0$ (assumed $a\ge b\ge c$)

The efficiency of EMV method has been shown in above examples – symmetric inequalities with 3 variables. However, permutation inequalities with 3 variables are the one that EMV mostly applied to:

**Problem 10.6.** Given $a,b,c\ge 0$ such that $a+b+c=3$. Prove that $a^2b+b^2c+c^2a+abc\le 4$

***Proof***

First of all, in order to use EMV, we need to homogenousize the inequality:

Given $a,b,c\ge 0$. Prove that $a^2b+b^2c+c^2a+abc\le\frac{4}{27}(a+b+c)^3$

$\Leftrightarrow 8\sum a^3-6abc-3\sum bc(b+c)+27\sum bc(c-b)\ge 0$

$\Leftrightarrow f(a,b,c)=\sum(a+4b+4c)(b-c)^2+27(a-b)(b-c)(c-a)\ge 0$

Thus: $f(a,b,c)\ge f(a-x,b-x,c-x),\ \forall x\in\left[0,\min\{a,b,c\}\right]$

So we only need to prove the inequality when $c=0$: $a^2b\le\frac{4}{27}(a+b)^3$

Using AM – GM, we have: $a^2b=4\cdot\frac{a}{2}\cdot\frac{a}{2}\cdot b\le 4\left(\frac{\frac{a}{2}+\frac{a}{2}+b}{3}\right)^3=\frac{4}{27}(a+b)^3$

Therefore the problem is solved.

Equality occurs $\Leftrightarrow$ $a=b=c=1$ or $a=2,b=1,c=0$ or its cyclic permutations

***Note:*** While we use "mixing variables", the expression $(a-b)(b-c)(c-a)$ appears to be noticeable. Its role is to eliminate the permutative property of the rest of the inequality, which make transforming to SOS form easier

**Problem 10.7.** Given distinct numbers $a,b,c \ge 0$. Prove that:

$$\left(\frac{a-b}{b-c}\right)^2+\left(\frac{b-c}{c-a}\right)^2+\left(\frac{c-a}{a-b}\right)^2>\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} \quad (1)$$

***Proof***

Assume that $c=\min\{a,b,c\}$ and let $f(a,b,c)=LHS(1)-RHS(1)$. We have

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}=\frac{(c-a)(c-b)}{(b+c)(a+b)}+\frac{(a-b)^2}{(c+a)(c+b)}$$

Therefore we have: $f(a,b,c)\ge f(a-x,b-x,c-x), \forall x\in[0,c]$

We now only need to prove the inequality when $c=0$:

$$\left(\frac{a-b}{b}\right)^2+\left(\frac{b}{a}\right)^2+\left(\frac{a}{a-b}\right)^2\ge\frac{a+b}{b}+\frac{b}{a}+\frac{a}{a+b}$$

Standardize $b=1$, the inequality becomes $(a-1)^2+\left(\frac{1}{a}\right)^2+\left(\frac{a}{a-1}\right)^2>a+1+\frac{1}{a}+\frac{a}{a+1}$

This simple inequality is left for readers to solve.

**Problem 10.8 [Le Trung Kien]** Find all constants $k$ such that

$$k\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\le\frac{a^2+b^2+c^2}{ab+ac+bc}+\frac{3k-2}{2} \quad , \quad \forall a,b,c\ge 0$$

***Proof***

The inequality $\Leftrightarrow k\dfrac{(a-b)(a-c)(b-c)}{(a+b)(a+c)(b+c)}\le\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{ab+ac+bc}$

$$\Leftrightarrow k\frac{(a-b)(a-c)(b-c)}{(a-b)^2+(b-c)^2+(a-c)^2}\le\frac{(a+b)(a+c)(b+c)}{ab+ac+bc} \quad (1)$$

Let $f(a,b,c)=RHS(1)-LHS(1)$. We need to find $k$ such that $f(a,b,c)\ge 0 \forall a,b,c\ge 0$

We will prove $f(a,b,c)\ge f(a-x,b-x,c-x) \forall x\in\left[0,\min\{a,b,c\}\right]$ (2)

Indeed $f(a,b,c)\ge f(a-x,b-x,c-x)$

$$\Leftrightarrow \frac{(a+b)(a+c)(b+c)}{ab+bc+ac}\ge\frac{(a+b-2x)(a+c-2x)(b+c-2x)}{(a-x)(b-x)+(b-x)(c-x)+(c-x)(a-x)}$$

$$\Leftrightarrow a+b+c-\frac{abc}{ab+ac+bc}\ge a+b+c-3x-\frac{(a-x)(b-x)(c-x)}{(a-x)(b-x)+(b-x)(c-x)+(c-x)(a-x)}$$

$$\Leftrightarrow 3x+\frac{1}{\left(\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}\right)^2}\ge\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \quad (3)$$

Consider the function $g(x)=LHS(3)$, then $g(0)=RHS(3)$. We have

$$g'(x)=3-\frac{\frac{1}{(a-x)^2}+\frac{1}{(b-x)^2}+\frac{1}{(c-x)^2}}{\left(\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}\right)^2}>0 \Rightarrow g(x)\ge g(0)\text{. Thus (2) is true.}$$

From (2) we have: $f(a,b,c) \ge 0, \forall a,b,c \ge 0 \Leftrightarrow f(a,b,0) \ge 0, \forall a,b \ge 0$

Consider: $f(a,b,0) \ge 0 \Leftrightarrow \dfrac{kab(a-b)}{(a-b)^2+a^2+b^2} \le a+b$ (4)

We have (4) is true $\forall a,b \ge 0 \Leftrightarrow |k| \le \min h(x)$ where $h(x) = \dfrac{2(x^3+1)}{x(x-1)}, x \ge 1$

It is easy to see that $\min h(x) = 2\sqrt{9+6\sqrt{3}}$ $(x \ge 1)$ when $x = \dfrac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$

Thus (1) is true $\forall a,b,c \ge 0 \Leftrightarrow |k| \le 2\sqrt{9+6\sqrt{3}}$

Equality occurs $\Leftrightarrow a=b=c$

Furthermore if $k = \pm 2\sqrt{9+6\sqrt{3}}$ then the equality also occurs when $(a,b,c)\left[(c,b,a)\right]$ is a cyclic permutation of $\left(\dfrac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}t, t, 0\right)$ , $t \ge 0$

***Remark:*** When we need to find optimal constants and when equality occurs correspondingly, EMV is the perfect choice. All other methods either reach dead-end or are too complicated. In the next section, we will see an interesting aspect of EMV.

## 2. EMV for inequalities with triangles.

We begin with the following problem:

**Problem 10.9.** Let $a,b,c$ be the length of edges in a triangle (possible to degrade)

Find the maximum value of $f(a,b,c) = \dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b}$

***Solution***

The function $f(a,b,c)$ is associated with Nesbit inequality, and we know that

$\min f(a,b,c) = \dfrac{3}{2}$, when $a=b=c$.

However we need to find the maximal value and this problem is solved as following:

Assume $a = \max\{a,b,c\}$. After simple transformation:

$$f(a,b,c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \sum \frac{(b-c)^2}{2(a+b)(a+c)} + \frac{3}{2}$$

So $f(a,b,c) \le f(a-x, b-x, c-x)$, $\forall x \in [0,k]$. Where $k$ is a suitable number which will be found later. So, we only need to prove the inequality where one variable is on the boundary. However, is this problem's boundary similar to previois problems? Let us investigate on this issue

Let try with $x = k = \min\{a, b, c\}$ i.e. one variable is 0, for example $c = 0$.

So $f(a,b,0) = \frac{a}{b} + \frac{b}{a}$. We cannot say anything about the maximal value in this case.

Note that $c = 0$ implies $a = b$ because $0 = c \geq |a - b|$, so $f(a,b,c) = 2$.

This weakens $f(a, b, c)$ significantly, which make it impossible to find max $f(a, b, 0)$.

Therefore, we need to find better boundary. Note that, if $a = \max\{a,b,c\}$ then the sufficient and necessary condition for $a$, $b$, $c$ to be length of edges in a triangles (possible to degrade) is $a \leq b + c$. Thus $b + c$ is the boundary of $a$. We now begin to force $a$ to $b + c$.

We have $a - x = b - x + c - x \Leftrightarrow x = b + c - a \geq 0$

Let $x = k = b + c - a$, that is $a = b + c$. Therefore:

$$f(b+c,b,c) = 1 + \frac{b}{2c+b} + \frac{c}{2b+c} = 2 - \frac{3bc}{(2b+c)(2c+b)} \leq 2$$

Equality occurs if and only if $a = b$, $c = 0$. So max $f(a, b, c) = 2$

**Problem 10.10.** Given $a,b,c \in [2,3]$. Prove that:

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 6\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \quad (1)$$

***Proof***

To use EMV, we first need to change the given conditions to suitable. Note that for all $x, y, z \in [2,3]$ then $x + y \geq z, 3x \geq 2y$. We will prove the inequality with the condition that $a,b,c$ are length of edges of a triangle such that $3\min\{a,b,c\} \geq 2\max\{a,b,c\}$. We have,

$$(1) \Leftrightarrow \sum\left(\frac{1}{bc} - \frac{3}{(a+b)(a+c)}\right)(b-c)^2 \geq 0 \Leftrightarrow \sum a(b+c)\left(a^2 + ab + ac - 2bc\right)(b-c)^2 \geq 0$$

$$\Leftrightarrow \sum a^2(3a-b-c)(b+c)(b-c)^2 - 2\sum a(b+c)(a-b)(a-c)(b-c)^2 \geq 0$$

$$\Leftrightarrow \sum a^2(3a-b-c)(b+c)(b-c)^2 + 2(b-c)^2(c-a)^2(a-b)^2 \geq 0 \quad (2)$$

Let $f(a,b,c) = LHS(2)$ and assume $a = \max\{a,b,c\}$.

We will prove: $f(a,b,c) \geq f(a-x, b-x, c-x), \ \forall x \in [0, b+c-a]$.

This is obvious because $3a - b - c \geq 3(a-x) - (b-x) - (c-x), 3a - b - c \geq 0$ and similar inequalities.

So we only consider the case when $a = b + c$:

$$2(b+c)\left(\frac{1}{b+c} + \frac{1}{b} + \frac{1}{c}\right) \geq 6\left(1 + \frac{b}{2c+b} + \frac{c}{2b+c}\right) \Leftrightarrow (b-c)^4 \geq 0$$

The inequality is proved. Equality occurs if and only if $a = b = c$.

**Problem 10.11.** Let $a$, $b$, $c$ be length of edges of a triangle. Prove that

$$4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 3\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)+3 \quad (1)$$

***Proof***

$$(1) \Leftrightarrow 7\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\frac{a}{c}-\frac{c}{b}-\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-6\right)\geq 0$$

$$\Leftrightarrow 7(a-b)(b-c)(c-a)+a(b-c)^2+b(c-a)^2+c(a-b)^2\geq 0$$

So we only need to prove when $c=a+b$, that is:

$$7ab(b-a)+(a+b)(a-b)^2+a^3+b^3\geq 0 \Leftrightarrow 2a^3+2b^3-7ab^2+7a^2b\geq 0$$

$\Leftrightarrow 2b\left(b-\frac{7}{4}a\right)^2+\frac{7}{8}a^2b+2a^3\geq 0$. The inequality is thus solved.

Equality occurs if and only if $a=b=c$

***Remark:*** Using EMV we can solved the generalized problem:

Let $a$, $b$, $c$ be length of edges of a triangle. Find the best constant k such that:

$$(k+1)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq k\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3$$

**Problem 10.12 [Le Trung Kien]**

Let $a$, $b$, $c$ be length of sides of a triangle (possible to degrade).

Prove that: $9\leq \frac{a+5b}{b+c}+\frac{b+5c}{c+a}+\frac{c+5a}{a+b}\leq 9,5 \quad (1)$

***Proof***

$$\frac{a+5b}{b+c}+\frac{b+5c}{c+a}+\frac{c+5a}{a+b}\geq 9 \Leftrightarrow \sum (b+c)(b-c)^2\geq 10(a-b)(b-c)(c-a)$$

It is easy to see that we only need to consider the case $a=b+c$, that is:

$$(b+c)(b-c)^2+(2b+c)c^2+(b+2c)b^2\geq 10bc(c-b)$$

$$\Leftrightarrow 2b^3+11b^2c-9bc^2+2c^3\geq 0 \Leftrightarrow 2c\left(c-\frac{9}{4}\right)^2+\frac{7}{8}b^2c+2b^3\geq 0$$

Equality occurs when $a=b=c$.

On the other hand $\frac{a+5b}{b+c}+\frac{b+5c}{c+a}+\frac{c+5a}{a+b}=\sum\frac{(b-c)^2}{2(a+b)(a+c)}+\frac{5(a-b)(b-c)(c-a)}{2(a+b)(b+c)(c+a)}=f(a,b,c)$

Now $f(a,b,c)\leq f(c,b,a)$ if $a\leq b\leq c$. So we can assume that $a\geq b\geq c$.

It is then obvious that $f(a,b,c) \le f(a-x,b-x,c-x),\ \forall x \in [0, b+c-a]$.

So again, we only need to consider the case when $a = b+c$, that is:

$$\frac{6b+c}{b+c} + \frac{b+5c}{2c+b} + \frac{6c+5b}{2b+c} \le 9{,}5 \Leftrightarrow bc(b+11c) \ge 0$$

The problem is then solved. Equality occurs $\Leftrightarrow$ $a = b, c = 0$ and its permutation.

*Note:* It is interesting to see that 2-sided inequality can be solved completely using EMV

**Problem 10.13. (Le Trung Kien)** Let *a*, *b*, *c* be length of edges of a triangle

Prove that: $2\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \ge \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} + a + b + c \quad (1)$

***Proof***

$$(1) \Leftrightarrow 3\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} - \frac{a^2}{c} - \frac{b^2}{a} - \frac{c^2}{b}\right) + \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b} - 2(a+b+c) \ge 0$$

$$\Leftrightarrow \frac{3(a-b)(b-c)(c-a)(a+b+c)}{abc} + \frac{(b+c)(b-c)^2}{bc} + \frac{(a+c)(a-c)^2}{ac} + \frac{(b+a)(b-a)^2}{ba} \ge 0$$

$$\Leftrightarrow 3(a-b)(b-c)(c-a) + \frac{a(b+c)(b-c)^2 + b(a+c)(a-c)^2 + c(b+a)(b-a)^2}{a+b+c} \ge 0 \qquad (2)$$

Assume $c = \max\{a,b,c\}$. We have: $\frac{a(b+c)}{a+b+c} = \frac{1}{\frac{1}{a} + \frac{1}{b+c}} \ge \frac{1}{\frac{1}{a-x} + \frac{1}{b+c-2x}}$, $\forall x \in [0, a+b-c]$

and similar inequalities, so we only need to prove (2) when $c = a+b$, that is:

$$3ab(b-a) + \frac{a^3(2b+a) + b^3(2a+b) + (a+b)^2(a-b)^2}{2(a+b)} \ge 0$$

$$\Leftrightarrow a^4 - 2a^3b - a^2b^2 + 4ab^3 + b^4 \ge 0 \Leftrightarrow (a^2 - ab - b^2)^2 + 2ab^3 \ge 0$$

The problem is thus solved. Equality occurs $\Leftrightarrow$ $a = b = c$

***Remark:*** Using EMV we can solve the generalized problem:

Let *a*, *b*, *c* be length of edges of a triangle. Find the best constant k such that:

$$(k+1)\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) \ge k\left(\frac{a^2}{c} + \frac{b^2}{a} + \frac{c^2}{b}\right) + a + b + c$$

**Problem 10.14 [Le Trung Kien]**

Let *a*, *b*, *c* be length of edges of a triangle (possible to degrade).

Prove that: $\frac{(a+b+c)^2}{ab+bc+ca} + 20 \le 16\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right) \quad (1)$

***Proof***

We can see that equality in (1) does not occur when $a = b = c$. Our transformation has no effect at all, therefore we need to come up with another way

$$(1) \Leftrightarrow \frac{a^2+b^2+c^2-2(ab+bc+ca)}{ab+bc+ca} \le \frac{8(a-b)(b-c)(a-c)}{(a+b)(a+c)(b+c)}$$

As $a,b,c$ are length of sides of a triangle so $a^2+b^2+c^2-2ab-2bc-2ca \le 0$. Thus

If $a \ge b \ge c$ then the inequality is true.

If $c \ge b \ge a$, then the inequality is equivalent to

$$\frac{(a+b)(a+c)(b+c)\left(2ab+2bc+2ca-a^2-b^2-c^2\right)}{ab+ac+bc} \ge 8(a-b)(b-c)(c-a)$$

Consider $0 \le x \le a+b-c$ and $f(x)=(a+b-2x)(b+c-2x)(a+c-2x)+$

$$+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\left(3x-a-b-c+\frac{1}{\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}}\right)$$

So $f(0)=$ LHS (1). We also have:

$$f'(x)=-24x^2+16(a+b+c)x-2\left(a^2+b^2+c^2+3ab+3ac+3bc\right)$$

$$+\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}\left[3-\frac{\frac{1}{(a-x)^2}+\frac{1}{(b-x)^2}+\frac{1}{(c-x)^2}}{\left(\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}\right)^2}\right]$$

$$\le -24x^2+16(a+b+c)x-2\left(a^2+b^2+c^2+3ab+3ac+3bc\right)+\frac{4}{3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$\le -24(a+b-c)^2+16(a+b+c)(a+b-c)$$
$$-2\left(a^2+b^2+c^2+3ab+3ac+3bc\right)+\frac{4}{3}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$=\frac{2}{3}\left[-59c^2+59c(a+b)-11\left(a^2+b^2\right)-37ab\right]=\frac{2}{3}\left[-59(c-b)(c-a)-11(a-b)^2\right] \le 0$$

Function $f(x)$ is decreasing for $0 \le x \le a+b-c$, therefore $f(0) \ge f(a+b-c)$ so we only need to consider (1) when $c=a+b$, that is:

$$\frac{(a+b)(2a+b)(2b+a)}{a^2+b^2+3ab}4ab \ge 8ab(b-a) \Leftrightarrow 4a^3+3a^2b+11ab^2 \ge 0$$

The problem is thus solved. Equality occurs $\Leftrightarrow$ $a=b, c=0$ or its permutations.

***Note:*** The power of EMV is fully illustrated in above problem. Note that the evaluation $f(a,b,c) \ge f(a-x,b-x,c-x)$ in previous problems are quite loose, while in this problem the tightness is shown via the proof of $f'(x) \le 0$.

**XI. SOME SPECIAL MIXING VARIABLES TECHNIQUES.**

**One of the most attractive features of MV method is the diversity of techniques. This can be illustrated via a number of problems that can be solved using special MV techniques:**

**Problem 11.1. [Phan Thanh Nam]** Given $x,y,z\in[-1,1]$ and $x+y+z=0$.

Prove that: $\sqrt{1+x+y^2}+\sqrt{1+y+z^2}+\sqrt{1+z+x^2}\ge 3$ (1)

***Proof***

***Lemma:*** If $ab\ge 0;a,b,a+b\ge -1$ then $\sqrt{1+a}+\sqrt{1+b}\ge 1+\sqrt{1+a+b}$

***Proof:*** Indeed, square both sides of the inequality we have

$$2+a+b+2\sqrt{(1+a)(1+b)}\ge 2+a+b+2\sqrt{1+a+b}\Leftrightarrow(1+a)(1+b)\ge 1+a+b\Leftrightarrow ab\ge 0$$

***Application:*** Among there numbers $x+y^2,y+z^2,z+x^2$ there must be at least two numbers with the same sign. WLOG, assume $(x+y^2)(y+z^2)\ge 0$. Using the lemma we have:

$$\text{LHS (1)}\ \ge 1+\sqrt{1+x+y^2+y+z^2}+\sqrt{1+z+x^2}=1+\sqrt{\left(\sqrt{1-z+z^2}\right)^2+y^2}+\sqrt{\left(\sqrt{1+z}\right)^2+x^2}$$

$$\ge 1+\sqrt{\left(\sqrt{1-z+z^2}+\sqrt{1+z}\right)^2+(x+y)^2}=1+\sqrt{\left(\sqrt{1-z+z^2}+\sqrt{1+z}\right)^2+z^2}$$

Finally, it suffices to prove that $\left(\sqrt{1-z+z^2}+\sqrt{1+z}\right)^2+z^2\ge 4$

$\Leftrightarrow 2z^2+2\sqrt{1-z^3}\ge 2\Leftrightarrow 1-z^2\ge 1-2z^2+z^4\Leftrightarrow z^2(z^2-1)\le 0$ which is true because $-1\le z\le 1$,

The problem is then solved. Equality occurs $\Leftrightarrow$ $x=y=z=0$.

**Problem 11.2. [Pham Kim Hung]** Given $a,b,c,d\ge 0;a+b+c+d=3$. Find the maximum value of $ab(a+2b+3c)+bc(b+2c+3d)+cd(c+2d+3a)+da(d+2a+3b)$

***Solution***

It is easy to see that $f(a,b,c,d)\le f(a,b,c+d,0)$ $(d=\min\{a,b,c,d\})$

The problem becomes: Given $a,b,c,d\ge 0;a+b+c+d=3$. Find max:

$$f(a,b,c)=ab(a+2b+3c)+bc(b+2c)=(2a+c)b^2+(a+2c)(a+c)b$$

We have $f(a,b,c)-f(a+c,b,0)=bc(a+c-b)$; $f(a,b,c)-f(0,b,a+c)=ab(b-a-c)$

Therefore $f(a,b,c)\le\max\{f(a+c,b,0),f(0,b,a+c)\}$

In addition: $f(a+c,b,0)=2(a+c)b^2+(a+c)^2b=2(3-b)b^2+(3-b)^2b=g(b)$

$$f(0,b,a+c)=(a+c)b^2+2(a+c)^2b=(a+c)(3-a-c)^2+2(a+c)^2(3-a-c)=g(a+c)$$

where $g(x)=9x-x^3=-(x-\sqrt{3})^2(x+2\sqrt{3})+6\sqrt{3}\le 6\sqrt{3},\ x\ge 0$.

Therefore: $f(a,b,c) \le \max\{f(a+c,b,0), f(0,b,a+c)\} = 6\sqrt{3}$

Equality occurs $\Leftrightarrow$ $a = 3-\sqrt{3}, b = \sqrt{3}, c = d = 0$ or its permutations

So $\max f(a,b,c) = 6\sqrt{3}$

**Problem 10.3.** Given $x, y, z \in \left[\frac{1}{2}, 2\right]$. Prove that $8\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) \ge 5\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)+9$

***Proof***

Let $f(x,y,z) = 8\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) \ge 5\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)-9$

Assume that $x = \max\{x,y,z\}$. We have:

$$f(x,y,z) - f(x,\sqrt{xz},z) = 8\left(\frac{x}{y}+\frac{y}{z}-2\sqrt{\frac{x}{z}}\right) - 5\left(\frac{y}{x}+\frac{z}{y}-2\sqrt{\frac{z}{x}}\right) = \frac{\left(y-\sqrt{xz}\right)^2(8x-5z)}{xyz} \ge 0$$

$\Rightarrow f(x,y,z) \ge f(x,\sqrt{xz},z)$. Now let $t = \sqrt{\frac{x}{z}}, 1 \le t \le 2$ we have:

$$f(x,\sqrt{xz},z) = 8\left(2\sqrt{\frac{x}{z}}+\frac{z}{x}-3\right) - 5\left(2\sqrt{\frac{z}{x}}+\frac{x}{z}-3\right) = 8\left(2t+\frac{1}{t^2}-3\right) - 5\left(\frac{2}{t}+t^2-3\right)$$

$$= \frac{8}{t^2}(t-1)^2(2t+1) - \frac{5}{t}(t-1)^2(t+2) = \frac{(t-1)^2(8+6t-5t^2)}{t^2} = \frac{(t-1)^2(4+5t)(2-t)}{t^2} \ge 0$$

The problem is thus solved.

Equality occurs $\Leftrightarrow$ $x = y = z$ or $(x,y,z) = \left(2,1,\frac{1}{2}\right); \left(1,\frac{1}{2},2\right); \left(\frac{1}{2},2,1\right)$

## XII. GENERAL MIXING VARIABLES THEOREM

In this section, we will introduce GMV theorem (General Mixing Variables) for n variables. This theorem almost contains all possibilities of "mixing variables"

1. We begin with some definition in $\mathbb{R}^n$.

**Definition 1:**

• The space $\mathbb{R}^n$ is the set of $x = (x_1, x_2, ... x_n)$ where $x_i \in \mathbb{R}, \forall i$.

• A sequence $\{x_m = (x_{1,m}, x_{2,m}, ..., x_{n,m})\}$ in $\mathbb{R}^n$ is defined to converge to $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$ if each sequence $x_{i,m}$ converges to $z_i$ when $m \to \infty, \forall i = 1, 2, ..., n$.

• Let $D \subset \mathbb{R}^n$. A function $f : D \to \mathbb{R}$ is continuous on D if: for any sequence $\{x_m\} \subset D$ and $z \in D$ such that $\{x_m\}$ converges to $z$, then we have: $f(x_m)$ converges to $f(z)$.

**Definition 2**: Let $D \subset \mathbb{R}^n$. We say:

• D is closed if for any sequence $\{x_m\} \subset D$ and $z \in \mathbb{R}^n$ such that $\{x_m\}$ converges to $z$, then we have $z \in D$.

• D is bounded if there exists $M$ such that: $\forall x = (x_1, x_2, ... x_n) \in D$, then $|x_i| \le M, \forall i = 1, 2, ..., n$.

For example, a finite set is closed and bounded

The following theorem is fundamental to our results (the proof will be given at the end of this section)

**Weierstrass theorem:** Let $D$ be a closed and bounded set in $\mathbb{R}^n$, and $f: D \to \mathbb{R}$ is continuous function. Then $f$ has global minima in $D$, i.e. there exists $x_0 \in D$ such that: $f(x_0) \le f(x), \forall x \in \mathbb{R}$. We also have similar result for global maxima.

• ***Comment:*** This theorem is a generalization of a familiar result: "Let [*a, b*] be a closed interval in R and $f:[a,b] \to \mathbb{R}$ is continuous, then $f$ has global minima in [*a, b*]". Therefore, Weierstrass theorem is intuitively understandable for us.

**2. GMV Theorem:** We assume that:

• D is a subset of $\mathbb{R}^n$, and $\Lambda$ is a closed subset of *D*.

• $f: D \to \mathbb{R}$ is an arbitrary continuous function such that $f$ has a minima in $\Lambda$.

• $T_1,...,T_k : D \to D$ are transformations such that $T_1(x) = ... = T_k(x) = x, \forall x \in \Lambda$.

We will give some criteria such that the minima of $f$ on $\Lambda$ is also the global minima of $f$ on *D*.

**Theorem GMV1:** If

• $f(x) \ge \min\limits_{j=\overline{1,k}} \left\{ f(T_j(x)) \right\}, \forall x \in D \setminus \Lambda$

• $\forall j = \overline{1,k}, \forall x \in D$ we have $\lim\limits_{m\to\infty} T_j^m(x) \in \Lambda$, where $T_j^0(x) = x, T_j^m(x) = T_j\left(T_j^{m-1}(x)\right)$

then $f(x) \ge \min\limits_{y\in\Lambda} \left\{ f(y) \right\}, \forall x \in D$.

Moreover, equality does not occur on $D \setminus \Lambda$ if $f(x) > \min\limits_{j=\overline{1,k}} f(T_j(x)), \forall x \in D \setminus \Lambda$.

**Theorem GMV2:** If

• D is closed and bounded in $\mathbb{R}^n$.

• $f(x) > \min\limits_{j=\overline{1,k}} f(T_j(x)), \forall x \in D \setminus \Lambda$.

then $f(x) \ge \min\limits_{y\in\Lambda} \left\{ f(y) \right\}, \forall x \in D$, moreover, equality does not occur on $D \setminus \Lambda$.

**Theorem GMV3:** If

• D is closed and bounded in $\mathbb{R}^n$.

• $f(x) \ge \min\limits_{j=\overline{1,k}} f(T_j(x)), \forall x \in D \setminus \Lambda$

• There exists a continuous function $h_j$: $D \to \mathbb{R}$ such that: $h_j(x) > h_j\left(T_j(x)\right)$, $\forall \mathrm{x} \in \mathrm{D} \setminus \Lambda$.

then $f(x) \ge \min\limits_{y\in\Lambda} \left\{ f(y) \right\}, \forall x \in D$.

***Proof***

**GMV1:** Let's consider arbitrary $x \in D$. We have: $f(x) \geq \min_{j=\overline{1,k}} f(T_j(x))$. By induction we have $f(x) \geq \min_{j=\overline{1,k}} f(T_j^m(x)), \forall x \in D$. Since $f$ is continuous and $\lim_{m\to\infty} T_m(x) \in \Lambda$:

$$f(x) \geq \lim_{m\to\infty}\left(\min_{j=\overline{1,k}} f(T_j^m(x))\right) = \min_{j=\overline{1,k}} f\left(\lim_{m\to\infty} T_j^m(x)\right) \geq \min_{y\in\Lambda} f(y)$$

Furthermore, if $f(x) > \min_{j=\overline{1,k}} f(T_j(x))$ then we have

$f(T_j(x)) \geq \min_{y\in\Lambda} f(y)$ implies $f(x) > \min_{y\in\Lambda} f(y)$.

**GMV2:** Using Weierstrass Theorem, there exists $x_0 \in D$ such that $f(x_0) \leq f(x)$, $\forall x \in D$. If $x_0 \notin \Lambda$ then $f(x_0) > \min_{j=\overline{1,k}} f(T_j(x_0))$, contradiction. Therefore $x_0 \in \Lambda$.

**GMV3:** Consider $y_0 \in \Lambda$ such that $f(y_0) = \operatorname{Min}_{y\in\Lambda}\{f(y)\}$. We assume that there exists $z \in D$ such that $f(z) < f(y_0)$. Since $h_j(x)$ is lower-bounded on D, by adding necessary constant, we may assume that $h_j(x) \geq 0, \forall x \in D, \forall j = 1,...,k$.

Select ε > 0 small enough we will have $f(z) + \varepsilon\sum_{j=1}^{k} h_j(z) < f(y_0)$.

Let $g(x) = \min_{j=\overline{1,k}}\{f(x) + \varepsilon h_j(x)\}, \forall x \in D$ then $g: D \to \mathbb{R}$ is continuous and

$g(x) > \min_{j=\overline{1,k}} g(T_j(x)), \forall x \in D\setminus\Lambda$ and $g(z) < f(y_0) \leq \operatorname{Min}_{y\in\Lambda}\{g(y)\}$.

This is contradiction with Theorem 2.

• ***Comment:*** GMV theorems look simple however it has a wide range of applications. For each transformation, we have a corresponding "mixing variables" technique. For instance, we have following collaries

***Consequence 1:*** (Pham Kim Hung, SMV-Strongly Mixing Variables) Let:

• $D \subset \mathbb{R}^n$ is a closed and bounded set and $s_0 = (s, s, ..., s) \in D$.

• Transformation $T$ is defined as: For each tuple $(a_1, a_2, ...a_n) \in D$, we select the largest and smallest and then replace them by their average number. We assume that $T$: $D \to D$.

• $f: D \to \mathbb{R}$ be a symmetric and continuous function such that: $f(a) \geq f(T(a)), \forall a \in D$.

Then $f(a) \geq f(s_0), \forall a \in D$

***Proof:*** Let $h(a_1, a_2, ..., a_n) = \sum_{i=1}^{n} a_i^2$ , we have $h: D \to R$ is continuous and $h(a) > h(T(a)), \forall a \in D\setminus\{s_0\}$. Then the theorem is proved using GMV3.

• ***Remark:***

We can also use GMV1, we only need to check for all $a \in D$ then $\lim_{m\to\infty} T^m(a) = s_0$.

This is intuitively clear, so we leave the proof as a small exercise for readers. In fact, we can overlook the assumption that D is closed and bounded (i.e. *Consequence 1* is true for all subset D of $\mathbb{R}^n$ such that $T(D) \subset D$).

***Consequence 2:*** (Dinh Ngoc An, UMV – Undefined Mixing Variables) Given:

• $D \subset \left\{ x = (x_1, x_2, ... x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = \text{const} \right\}$. Let $\Lambda$ is a set of elements in $D$ such that $t$ coordinates are 0 and the rest are equal $(t \geq 0)$.

• $f : D \to \mathbb{R}$ is continuous and symmetric such that:

$$f(a_1, a_2, ..., a_n) \geq \min \left\{ f\left( \frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, ..., a_n \right), f(0, a_1 + a_2, a_3, ..., a_n) \right\}$$

then $f(x) \geq \min_{y \in \Lambda} \{ f(y) \}, \forall x \in D$

***Proof:*** Consider the transformations $T_1, T_2 : D \to D$ as following:

$T_1(a) = T_2(a) = a, \forall a \in \Lambda$ and for each $a = (a_1, a_2, ... a_n) \in D \setminus \Lambda$, select two indices $i \neq j$ such that $a_i = \min\{a_t > 0, t = 1, ..., n\}$ and $a_j = \max\{a_1, a_2, ... a_n\}$, corresponding to $T_1$ we replace $a_i, a_j$ by its average, corresponding to $T_2$ we replace $a_i, a_j$ by $(0, a_i + a_j)$. Then $f(a) \geq \min\{f(T_1(a)), f(T_2(a))\}, \forall a \in D$.

Let $h_1(a_1, a_2, ... a_n) = \sum_{i=1}^{n} a_i^2$ and $h_2(a_1, a_2, ..., a_n) = -\max\{a_1, a_2, ..., a_n\}$ then $h_1, h_2 : D \to R$ are continuous and $h_j(a) > h_j(T_j(a)), \forall a \in D \setminus \{s_0\}, \forall j = 1,2$. The problem is then solved using GMV3.

• ***Remark:*** We can also check if for all $a \in D$ then $\lim_{m\to\infty} T_1^m(a) = (s, s, ..., s)$ and

$\lim_{m\to\infty} T_2^m(a) = (0, ..., 0, r, ..., r)$ and apply GMV1.

## 3. Following problems will illustrate the power of GMV.

**Problem 12.1. (Cauchy inequality)** Given $n$ non-negative numbers $x_1, x_2, ... x_n$.

Prove that: $x_1 + x_2 + ... + x_n \geq n \cdot \sqrt[n]{x_1 x_2 ... x_n}$

***Proof***

By normalizing, we assume that $x_1 x_2 ... x_n = 1$ thus we need to prove $x_1 + x_2 + ... + x_n \geq n$.

Obviously we only need to consider $x_i \leq n, \forall i$.

Let $D = \{ x = (x_1, x_2, ... x_n) \mid x_i \in [0, n],\ x_1 x_2 ... x_n = 1 \}$, D is closed and bounded

Let $\Lambda = \{ x_0 = (1, 1, 1, ..., 1) \}$. Consider a continuous function $f : D \to \mathbb{R}$ such that:

For each $x = (x_1, x_2, ... x_n) \in D$ then $f(x) = x_1 + x_2 + ... + x_n$.

Let $T$: $D\backslash\Lambda \to D$ be: For each $x=\left(x_1,x_2,...x_n\right)\in D\setminus\Lambda$ then there exists $x_i \neq x_j$ and we $T(x)$ is the tuple $x$ after replacing $x_i$ and $x_j$ by their geometric mean.

Then $f(x)-f(T(x))=\left(\sqrt{x_i}-\sqrt{x_j}\right)^2>0$. Using GMV2 we have $f(x)\geq f\left(x_0\right), \forall x\in D$,

Equality occurs if $x=x_0$.

**Problem 12.2. (Dinh Ngoc An)** Given $k > 0$, $x_1,x_2,...x_n\geq 0$ and $x_1+x_2,...+x_n=n$. Prove that:

$$(x_2x_3...x_n)^k+(x_1x_3...x_n)^k+...+(x_1x_2...x_{n-1})^k\leq \max\left\{n,\left(\frac{n}{n-1}\right)^k\right\}$$

***Proof***

Define $f(x_1,x_2,...,x_n)=(x_2x_3...x_n)^k+(x_1x_3...x_n)^k+...+(x_1x_2...x_{n-1})^k$.

WLOG, we may suppose that $x_1\leq x_2$. Then $x_1=s-t, x_2=s+t$ where $t\in[0,s]$.

Consider $f(x_1,x_2,...,x_n)=g(t)=A\left[(s+t)^k+(s-t)^k\right]+B(s^2-t^2)^k$

where $A=(x_3...x_n)^k$, $B=(x_4x_5...x_n)^k+(x_3x_5...x_n)^k+...+(x_3x_4...x_{n-1})^k$.

We have: $g'(t)=Ak\left[(s+t)^{k-1}-(s-t)^{k-1}\right]-2Bkt(s^2-t^2)^{k-1}\geq 0$

$$\Leftrightarrow h(t)=(s-t)^{1-k}-(s+t)^{1-k}-\frac{2Bt}{A}\geq 0$$

$$h'(t)=(k-1)\left[(s-t)^{-k}+(s+t)^{-k}\right]-\frac{2B}{A},\ \ h''(t)=k(k-1)\left[(s-t)^{-k-1}-(s+t)^{-k-1}\right]>0$$

Hence $h(t)$ is convex, moreover $0=h(0), \lim\limits_{t\to s^-}h(t)=+\infty$. Therefore, either $h(t)$ is always positive (0,s), or $h(t)$ changes its sign from the negative sign to the positive one on (0,s).

But g’$(t)$ has the same sign with $h(t)$, *so* $g(t)\leq \max\{g(0),g(s)\}$. It follows that:

$$f(x_1,x_2,...,x_n)\leq \max\left\{f\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2},x_3,...,x_n\right),f(0,x_1+x_2,x_3,...,x_n)\right\}$$

Applying the UMV theorem, we have the proof.

• ***Remark:*** We observe that if there is an equality dependency among n variables, we can fix ($n$ – 2) variables and let other two vary. Therefore, consequence 3 allow us see how $n$ variables vary by looking at how two variables vary. If there is more equality dependency among variables, the number of varied variables can be more to preserve the dependency. We often see only inequality with one equality dependency. We will introduce two problems where there is more than one such dependency.

**Problem 12.3. (Phan Thanh Viet)**

Let $n \ge 3$, consider the set D consisting of all tuples of $n$ positive numbers satisfying $a_1 + a_2 + ... + a_n = A$, $a_1^2 + a_2^2 + ... + a_n^2 = B$, where A, B are given constants such that $nB > A^2$

Prove that:

**a)** D has a unique element $(b_1, b_2, ..., b_n)$ satisfying $b_2 = b_3 = ... = b_n$.

**b)** The expression $f(a_1, a_2, ..., a_n) = a_1 a_2 ... a_n$ attains its maximum value in D if $(a_1, a_2, ..., a_n)$ is a permutation of $(b_1, b_2, ..., b_n)$.

***Proof***

**a)** Assume that $a = (a_1, a_2, ..., a_n)$ satisfying $a_2 = a_3 = ... = a_n$. To have $a \in D$ we need $a_1, a_2 > 0$, $a_1 + (n-1)a_2 = A$, $a_1^2 + (n-1)a_2^2 = B$. It follows that $a_2$ is a positive root of the equation $\left(A - (n-1)a_2\right)^2 = B - (n-1)a_2^2 \Leftrightarrow n(n-1)a_2^2 - 2A(n-1)a_2 - B = 0$

Since this equation has a unique positive root, we are done.

**b)**

• ***Step 1:*** We first consider the case $n = 3$.

The equality $a_1^3 + a_2^3 + a_3^3 - 3a_1 a_2 a_3 = \frac{1}{2}\left(a_1 + a_2 + a_3\right)\left[3(a_1^2 + a_2^2 + a_3^2) - (a_1 + a_2 + a_3)^2\right]$ implies that $f(a_1, a_2, a_3) = 3a_1 a_2 a_3 + \frac{1}{2}A(3B - A)$, hence $f(a_1, a_2, a_3)$ attains its maximum value on D if two of the three variables $a_i$ are equal (readers can prove this by investigating a function in one variable – see the article on ABC method of Nguyen Anh Cuong).

• ***Step 2:*** We now consider the general case. Let $\Lambda$ be the set of all tuples in D with ($n$ – 1) equal components. Consider the transformation $T : D \setminus \Lambda \to D$ as follows. Assume that $a = (a_1, a_2, ..., a_n) \in D \setminus \Lambda$, we may always choose 3 numbers, say $(a_1, a_2, a_3)$, such that they are pairwise distinct. Then, there exists a unique triple of positive numbers $(e_1, e_2, e_3)$ such that $e_1 + e_2 + e_3 = a_1 + a_2 + a_3$, $e_1^2 + e_2^2 + e_3^2 = a_1^2 + a_2^2 + a_3^2$. The transformation $T$ replaces $(a_1, a_2, a_3)$ by $(e_1, e_2, e_3)$. Obviously, $T(a) \in D$, so by the result in the case n=3 we have $f(T(a)) > f(a)$. It follows that $f(a)$ attains its maximum value if $a \in \Lambda$.

That is exactly what we want to prove.

**Problem 12.4. (Phan Thanh Nam)**

Let $n \geq 3$, consider the set D consisting of all tuples of n positive real numbers whose sum is $A$ and whose product is $B$, where $A$, $B$ are given constants such that $A^n > n^n B$. Prove that

**a)** D has precisely two elements $(b_1, b_2, ..., b_n)$, $(c_1, c_2, ..., c_n)$ satisfying

$$b_1 > b_2 = b_3 = ... = b_n \text{ and } c_1 < c_2 = c_3 = ... = c_n.$$

**b)** The expression $f(a_1, a_2, ..., a_n) = a_1^2 + a_2^2 + ... + a_n^2$ attains its maximum value (or minimum) in D if $(a_1, a_2, ..., a_n)$ is a permutation of $(b_1, b_2, ..., b_n)$ (or $(c_1, c_2, ..., c_n)$, respectively).

***Proof***

**a)** Assume that $a = (a_1, a_2, ..., a_n)$ satisfying $a_2 = a_3 = ... = a_n$. To have $a \in D$ we need $a_1, a_2 > 0$, $a_1 + (n-1)a_2 = A$, $a_1 a_2^{n-1} = B$. It follows that $a_2$ is a positive root of the equation $A - (n-1)a_2 = \frac{B}{a_2^{n-1}} \Leftrightarrow (n-1)a_2^n - Aa_2^{n-1} + B = 0$. Consider $d(x) = (n-1)x^n - Ax^{n-1} + B$

$d'(x) = (n-1)x^{n-2}(nx - A) \Rightarrow d(x)$ decreases on $\left(0; \frac{A}{n}\right)$ and increases on $\left(\frac{A}{n}; \infty\right)$.

Moreover, $d(0) = B > 0 > -\left(\frac{A}{n}\right)^n + B = d\left(\frac{A}{n}\right)$, hence the polynomial $d(x)$ has precisely two positive roots, one is in $\left(0; \frac{A}{n}\right)$ while the other is in $\left(\frac{A}{n}; \infty\right)$. We then have the proof.

**b)**

• ***Step 1:*** We first consider the case $n = 3$. We show that for all $a = (a_1, a_2, a_3) \in D$, we will have $a_i \in [c_1, b_1]$ where $i = 1, 2, 3$. It suffices to consider $i = 1$.

We have : $\frac{4B}{a_1} = 4a_2 a_3 \leq (a_2 + a_3)^2 = (A - a_1)^2 \Leftrightarrow a_1(A - a_1)^2 \geq 4B$

The polynomial $g(x) = x(A - x)^2 - 4B$ has 3 roots, where the two smaller roots are $b_1$ and $c_1$

Indeed, since $b_1 + 2b_2 = A, b_1 b_2^2 = B \Rightarrow (A - b_1)^2 = 4b_2^2 = \frac{4B}{b_1}$, $b_1$ is a root of g($x$) and similarly so is $c_1$, moreover by Viete's theorem the sum of three roots is $2A > 2b_1 + c_1$ (otherwise $2A \leq 2b_1 + c_1 \Rightarrow c_1 \geq 2A - 2b_1 = 4b_2 \Rightarrow c_2 = \frac{b_1 b_2^2}{c_1 c_2} \leq \frac{b_1 b_2^2}{c_1^2} \leq \frac{b_1}{16}$, contradicts $3c_2 \geq c_1 + 2c_2 = A > b_1$) so the other root is greater than $b_1$.

Hence $g(a_1) \geq 0 \Leftrightarrow a_1 \in [c_1, b_1]$. In particular, $b_2, c_2 \in [c_1, b_1] \Rightarrow c_1 < b_2 < c_2 < b_1$.

We now have $a_1^2+a_2^2+a_3^2=a_1^2+(a_2+a_3)^2-2a_2a_3=a_1^2+(A-a_1)^2-\frac{2B}{a_1}$.

Consider $h(x)=x^2+(A-x)^2-\frac{2B}{x}$ where $x\in[c_1,b_1]$. We have: $h'(x)=\frac{2(2x^2-Ax+B)}{x^2}$

As in the remark, we see that the equation $h'(x)=0$ has exactly 2 positive roots, those are $b_2$ and $c_2$, moreover $h'(x)$ is negative on $(b_2,c_2)$ and positive on $(c_1,b_2)\cup(c_2,b_1)$

On the other hand, if $a_1\in\{b_1,b_2\}$ then $(a_1,a_2,a_3)$ is a permutation of $(b_1,b_2,b_3)$, hence $h(b_1)=f(b_1,b_2,b_3)=h(b_2)$, similarly $h(c_1)=f(c_1,c_2,c_3)=h(c_2)$. It then follows that $f(c_1,c_2,c_3)\le f(a_1,a_2,a_3)\le f(b_1,b_2,b_3)$. Moreover, equality occurs if $(a_1,a_2,a_3)$ is a permutation of $(b_1,b_2,b_3)$ or $(c_1,c_2,c_3)$ respectively (q.e.d).

• ***Step 2:*** We now consider the general case. We prove for the minimum case, the maximum one is similar. Let Λ be the set of all permutations of $(c_1,c_2,...,c_n)$, that is the set of all tuples in D with ($n$ – 1) equal components and greater than the remaining one. Consider the transformation $T:D\setminus\Lambda\to D$ as follows.

Assume that $a=(a_1,a_2,...,a_n)\in D\setminus\Lambda$, we may always choose 3 numbers , say $(a_1,a_2,a_3)$, such that the case that two numbers are equal and greater than the remaining one doesn't hold. Then there exists a unique triple $(e_1,e_2,e_3)$ such that $e_1<e_2=e_3$,

$e_1+e_2+e_3=a_1+a_2+a_3$, $e_1e_2e_3=a_1a_2a_3$. The transformation $T$ replaces $(a_1,a_2,a_3)$ by $(e_1,e_2,e_3)$. Obviously $T(a)\in D$. So by the result in the case $n$ = 3, we have $f(T(a))>f(a)$

It follows that $f(a)$ attains its maximum value if $a\in\Lambda$.

That is exactly what we want to prove.

• To end this section, we give a proof for Weierstrass Theorem.

**Definition 3:** Given a sequence $\{a_m\}_{m=1}^{\infty}$ (in $\mathbb{R}$ or $\mathbb{R}^n$). A sequence $\{a_{m_k}\}_{k=1}^{\infty}$ is called a sub-sequence of $\{a_m\}_{m=1}^{\infty}$ if $\{m_k\}_{k=1}^{\infty}$ is a strictly increasing sequence.

*For example:* $\{a_{2m}\}_{m=1}^{\infty}$ is a subsequence of $\{a_m\}_{m=1}^{\infty}$.

***Lemma 1:*** (Weierstrass) Any bounded sequence $\{a_m\}$ in $\mathbb{R}$ has a convergent subsequence

***Proof:*** We know that "if a sequence is monotonous and bounded then it converges", thus we only need to show there exist a monotonous subsequence.

Consider $T = \{ m \in \mathbb{Z}^+ \mid \exists m' > m : a_{m'} \ge a_m \}$

If $T$ is finite then $\{a_m\}$ is decreasing from some index. If $T$ is infinite then we can extract an increasing subsequence. In both cases, we can show there exists a monotonous subsequence.

***Lemma 2:*** (Weierstrass) Any bounded sequence $\{a_m\}$ in $\mathbb{R}^n$ has a convergent subsequence

***Proof:*** Let $\{a_m = (x_{1,m}, ..., x_{n,m})\}$ be a bounded sequence in $\mathbb{R}^n$. Since $\{x_{1,m}\}$ is bounded in $\mathbb{R}$, there exists a subsequence $\{x_{1,m_{k_1}}\}$ that converges. The sequence $\{x_{2,m_{k_1}}\}$ is also bounded in $\mathbb{R}$ so there exists a subsequence $\{x_{2,m_{k_2}}\}$ that converges. By selecting "subsequence of subsequence" continuously, we finally have a subsequence $\{a_{m_k} = (x_{1,m_k}, ..., x_{n,m_k})\}$ such that $\forall i = \overline{1,n}$, $\{x_{i,m_k}\}$ converges in $\mathbb{R}$. Therefore $\{a_{m_k}\}$ converges in $\mathbb{R}^n$.

***Lemma 3:*** (Completeness of $\mathbb{R}$) Let A be a bounded set in $\mathbb{R}$. There exists $M \in \mathbb{R}$ such that: $M \le A$ (i.e. $M \le a, \forall a \in A$) and a sequence $\{a_k\}$ in A converges to $M$. We will denote this by $M = \inf A$.

***Proof:*** We will prove that $\forall \varepsilon > 0, \exists a \in A,\ a - \varepsilon \le A$. Assume this is not correct. Consider arbitrary $x_1 \in A$, by induction, we can build a sequence $\{x_m\}$ in A such that $x_{m+1} \le x_m - \varepsilon, \forall m \in \mathbb{Z}^+$

So $x_m \le x_1 - (m-1)\varepsilon, \forall m \in \mathbb{Z}^+$ this is contradictory to the fact that A is lower-bounded.

Thus, $\forall m \in \mathbb{Z}^+$, there exists $a_m \in A$ such that $a_m - \frac{1}{m} \le A$. Since $\{a_m\}$ is bounded, there exists a subsequence $\{a_{m_k}\}$ that converges to $M$ in $\mathbb{R}$. What is left to prove is that $M \le A$. Indeed, consider arbitrary $a \in A$ we have $a_{m_k} - \frac{1}{m_k} \le a, \forall k \in \mathbb{Z}^+$, when $k \to \infty$ then $M \le a$.

***Proof of Weierstrass Theorem:*** Let $A = f(D)$. We will prove that A has a minimum value. We will show that $A$ has following property: if the sequence $\{a_m\}$ is contained in $A$ and $a_m \to \alpha$ then $\alpha \in A$. Indeed, by definition we have $x_m \in D$ such that $f(x_m) = a_m \to \alpha$. Since $\{x_m\}$ is bounded (contained in D), there exists a subsequence $\{x_{m_k}\}$ that converges to $c$ in $\mathbb{R}^n$. Since D is closed, we have $c \in D$. Since $f(x_m) \to \alpha$ we have $f(x_{m_k}) \to \alpha$. On the other hand, $\{x_{m_k}\} \to c$ and $f$ is continuous we have $f(x_{m_k}) \to f(c)$. This limit is unique, therefore $f(c) = \alpha$.

Now, $A$ is lower-bounded (since it will lead to contradiction if $\alpha = -\infty$). Therefore, there exists $M = \inf A$. By the definition of *inf* and the property of $A$ we have shown, we have $M \in A$. So $A$ has a minimum value which is $M$. The theorem is proved.

### XIII. REVIEW

Dear reader, it is necessary to review our journey so far. As mentioned in VI, MV method is known early when considering convex function together with a lot of nice results. Jensen inequality can be seen as a criteria for mixing variables to centre globally. For details, see Inequalities by *Hardy – Polya – Littewood*.

Our purpose is not just to describe a wide variety of "mixing variables" techniques but also to let the readers grasp the idea of the method. Similar to many other methods, sometimes we might not solve the problem directly by MV method; we need to transform it to a suitable form. For example

**Problem 13.1. (VMEO I – Phan Thanh Nam)**

Find all constant *k* such that there exists $c_k > 0$ satisfies:

$$(1+x^2)(1+y^2)(1+z^2) \geq c_k\,(x+y+z)^k \ , \forall x,\ y,\ z > 0$$

For each *k*, find the best constant $c_k$.

Using limit, it is easy to see that $k \in [0,\ 2]$, however it much more complicated to find best. You might try and see that it is difficult to use MV method here with variables *x, y, z*. However by transforming $x = \operatorname{tg}\alpha, y = \operatorname{tg}\beta, z = \operatorname{tg}\gamma$ where $\alpha, \beta, \gamma \in \left(0; \frac{\pi}{2}\right)$ then it is surprised that we can "force" the inequality to one variable by replacing $\alpha, \beta, \gamma$ by their average. The rest is just to investigate the behavior of an one-variable function or simply use generalized AM – GM inequality.

MV techniques we have introduced hopefully give the overview about MV method. This method is widely applicable, from 3-variable, 4-variable to n-variable inequalities. Generally speaking, inequalities with 3 or 4 variables are still the first customers of MV method. For many general problems with n variables, "mixing variables" is really difficult. Theorems such as SMV, UMV could not solve those problems. Sometimes, MV techniques could not solve even symmetric and homogeneous inequalities with 3 variables:

**Problem 13.2.** Given *a, b, c* ≥ 0. Prove that

$$\left(a^3+b^3+c^3-6abc\right)^2+\left[(a+b+c)^3-36abc\right]^2 \geq 0$$

Equality occurs when (*a, b, c*) = (*t,* 2*t,* 3*t*), *t* ≥ 0 (and its permutations).

Then all MV techniques we have covered could not solve this problem.

Another example is the famous inequality Vasile.

**Problem 13.3.** Given real numbers $a,b,c$. Prove that: $\left(a^2+b^2+c^2\right)^2 \geq 3\left(a^3b+bc^3+c^3a\right)$

Beside the case $a=b=c$ , equality also occurs when $a=\sin^2\frac{4\pi}{7}, b=\sin^2\frac{2\pi}{7}, c=\sin^2\frac{\pi}{7}$

From this query, it is nature to raise a question that: "Is there any MV technique for those problems?" This topic will be discussed with readers in near future.

With the possibility that optima can be at the centre or on the boundary, it is difficult to reduce the number of variables to 1. Thus we hope that there is a global "mixing variables" method, similar to Jensen inequality. With this goal, we presented two beautiful SMV (strong mixing variables) and UMV (undefined mixing variables). These two theorems can be thought to be "twin": SMV is "specialized" in inequalities whose optima are at the centre, UMV allows us to combine both cases – optima are either at the centre or on the boundary. For simplicity, these two theorems are only considered for symmetric functions.

We realize that it is not necessary to separate two cases, in fact we can merge the strengths of these theorems (as shown in consequence 3 / VIII). However, GMV theorem is not just simply the generalization of SMV and UMV; it opens a new horizon with numberless ways of "mixing variables". What is required here is just: if a tuple $x=\left(x_1,x_2,...x_n\right)$ is not in Λ, it can be replaced by another tuple ($T(x)$). In SMV ("classical"), the event of forcing two variables - the biggest and the smallest to be the same, we can have the feeling that n variables will reach the average value; in this case it is not the case anymore. However, we are still able to achieve the result without extra conditions.

## *Proposed Problems*

**Problem 1.** (Mathlinks) Given *a, b, c* > 0 and *abc* = 1. Prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{6}{a+b+c}\geq 5$

**Problem 2.** (MOSP 2001) Let *a, b, c* be positive numbers such that *abc* = 1.

Prove that $(a+b)(b+c)(c+a)\geq 4(a+b+c-1)$

**Problem 3.** Given $a,b,c\geq 0$. Prove that:

$$2\left(a^3+b^3+c^3\right)+(ab+bc+ca)+2abc+1\geq 2\left(a^2+b^2+c^2\right)+bc(b+c)+ac(a+c)+ab(a+b)$$

**Problem 4.** Given *x, y,* z ≥ 0 satisfy $x^2+y^2+z^2=3$. Prove that $7(xy+yz+zx)\leq 12+9xyz$

**Problem 5.** Given *a, b, c* > 0 and *abc* = 1. Prove that $a^2+b^2+c^2+6\geq\frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

**Problem 6.(Pham Kim Hung)**

Prove that $\frac{1}{\sqrt{4a^2+bc}}+\frac{1}{\sqrt{4b^2+ca}}+\frac{1}{\sqrt{4c^2+ab}}\ge\frac{4}{a+b+c}$ , $\forall\ a,\ b,\ c\ge 0$

**Problem 7. (Murray Klamkin)** Given $a,\ b,\ c\ge 0$ such that $a+b+c=2$. Prove that

$$\left(a^2+ab+b^2\right)\left(b^2+bc+c^2\right)\left(c^2+ca+a^2\right)\le 3$$

**Problem 8. (China 2005)** Let $a,\ b,\ c>0$ and $ab+bc+ca=\frac{1}{3}$. Prove that

$$\frac{1}{a^2-bc+1}+\frac{1}{b^2-ca+1}+\frac{1}{c^2-ab+1}\le 3$$

**Problem 9.** Let $a,\ b,\ c\in\ [p,\ q]$ where $0<p\le q$. Find the maximum value of

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

**Problem 10 (A generalized of RMO 2000)** Let $a,\ b,\ c\ge 0$ and $a+b+c=3$.

Find the lowest constant $k>0$ such that the following inequality holds:

$$a^k+b^k+c^k\ge ab+bc+ca$$

**Problem 11. (Mathlinks)** Let $a,\ b,\ c\ge 0$ and $ab+bc+ca=1$. Prove that:

$$\frac{1+a^2b^2}{(a+b)^2}+\frac{1+b^2c^2}{(b+c)^2}+\frac{1+c^2a^2}{(c+a)^2}\ge\frac{5}{2}$$

**Problem 12.(Vasile Cirtoaje)** Prove that for $a,\ b,\ c>0$: $\sqrt{\frac{a}{8b+c}}+\sqrt{\frac{b}{8c+a}}+\sqrt{\frac{c}{8a+b}}\ge 1$

**Problem 13. (Phan Thanh Nam)** Let $x,\ y,\ z\ge 0$ such that $a+b+c=1$. Prove that

$$\sqrt{x+y^2}+\sqrt{y+z^2}+\sqrt{z+x^2}\ge 2$$

**Problem 14.** Given $a,b,c\ge 0$ . Prove that: $\frac{ab+4bc+ca}{b^2+c^2}+\frac{ac+4ab+bc}{b^2+a^2}+\frac{ab+4ac+bc}{a^2+c^2}\ge 4$

**Problem 15. (Vasile Cirtoaje)** Let $x,\ y,\ z\ge 0$ such that $a^2+b^2+c^2=1$. Prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}\le\frac{9}{2}$$

**Problem 16. [Le Trung Kien]** Let $a$, $b$, $c$ be length of sides of a triangle.

Find the best constant $k$ such that: $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+k\frac{a^2+b^2+c^2}{ab+bc+ca}\ge\frac{a}{c}+\frac{b}{a}+\frac{c}{b}+k$

**Problem 17. (Phan Thanh Viet)** Let $x,\ y,\ z\in\ [-1,\ 1]$ and $x+y+z=0$. Prove that

$$\sqrt{1+x+\frac{7}{9}y^2}+\sqrt{1+y+\frac{7}{9}z^2}+\sqrt{1+z+\frac{7}{9}x^2}\ge 3$$

**Problem 18 [Le Trung Kien]**

Given $a,b,c \geq 0$. Prove that: $\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}+\frac{3(a^2b+b^2c+c^2a)}{ab^2+bc^2+ca^2} \geq 4$

**Problem 19.** Given $a,b,c \geq 0$. Prove that: $\sqrt{1+\frac{48a}{b+c}}+\sqrt{1+\frac{48b}{a+c}}+\sqrt{1+\frac{48c}{b+a}} \geq 15$

**Problem 20. (Pham Kim Hung)** Given $a, b, c \geq 0$ and $a + b + c = 3$. Prove that:

$$a^4+b^4+c^4-3abc \geq 6\sqrt{2}(a-b)(b-c)(c-a)$$

**Problem 21.** Given $a, b, c \geq 0$, $a + b + c = 2$. Find the maximum value of

$$(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2)$$

**Problem 22. (Le Trung Kien)**

Given $a,b,c \geq 0$. Determine the best constant for the following inequality

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+k\frac{a^2+b^2+c^2}{(a+b+c)^2} \geq \frac{3}{2}+\frac{k}{3}$$

**Problem 23.** Let $a,b,c \geq 0$. Determine the best constant for the following inequality

$$\frac{a^3+b^3+c^3}{a^2b+b^2c+c^2a}+\frac{kabc}{ab^2+bc^2+ca^2} \geq 1+k; \qquad k=\frac{3}{\sqrt[3]{4}}-1$$

**Problem 24.** Given $a, b, c \geq 0$ such that $ab+bc+ca+abc=4$.

Prove that: $a+b+c \geq ab+bc+ca$

**Problem 25.** Given $a, b, c \geq 0$ such that $\min\{ab,bc,ca\} \geq \frac{1}{4}$. Prove that:

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2} \geq \frac{3}{\left(1+\sqrt[3]{abc}\right)^2}$$

**Problem 26.** Given $a, b, c \geq 0$ such that $ab+bc+ca+abc=4$.

Find the best constant $k$ such that: $a^2+b^2+c^2+3k \geq (k+1)(ab+bc+ca)$

**Problem 27. (Le Trung Kien)** Given $a,b,c \geq 0$. Prove that

$$a^2+b^2+c^2+\sqrt{2}abc+\left(\sqrt{2}+1\right)^2 \geq (2+\sqrt{2})(ab+bc+ca)$$

**Problem 28.** Given $a,b,c \geq 0$ such that $abc=1$ Prove that

$$a^2+b^2+c^2+9(ab+bc+ca) \geq 10(a+b+c)$$

**Problem 29. (Phan Thanh Viet)** Prove in any triangle that

$$m_a+m_b+m_c \leq \sqrt{3p^2+\frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]}$$

**Problem 30.(Jackgarfulkel)** Let ABC be an acute triangle. Prove that:

**a)** $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\ge\frac{4}{3}\left(1+\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

**b)** $\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\ge\frac{4}{\sqrt{3}}\left(1+\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)$

**Problem 31. (Jackgarfulkel)** Let ABC be a triangle. Prove that:

$$\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}\ge\frac{2}{\sqrt{3}}(\sin A+\sin B+\sin C)$$

**Problem 32. (Phan Thanh Nam)** Let ABC be a non-obtuse triangle.

**a)** Prove that: $\frac{\sin B\sin C}{\sin A}+\frac{\sin C\sin A}{\sin B}+\frac{\sin A\sin B}{\sin C}\ge\frac{5}{4}\sqrt{2+\sin^2 A+\sin^2 B+\sin^2 C}$

**b)** Prove or disprove the following inequality:

$$\frac{\sin B\sin C}{\sin A}+\frac{\sin C\sin A}{\sin B}+\frac{\sin A\sin B}{\sin C}\ge\frac{5}{2}+4\left(3\sqrt{3}-5\right)\cos A\cos B\cos C$$

**Problem 33. [Pham Kim Hung]** Given $a,b,c,d$ and $a+b+c+d=2$. Prove that:

$$a^4+b^4+c^4+d^4-4abcd\ge(a-b)(c-d)$$

**Problem 34.** Let *a, b, c, d* be real numbers such that $a^2+b^2+c^2+d^2=1$.

Prove that: $\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-cd}+\frac{1}{1-da}+\frac{1}{1-db}+\frac{1}{1-ca}\le 8$

**Problem 35.(Phan Thanh Nam)**

Let *x, y, z, t* be real numbers such that $\max\{xy, yz, zt, tx\}\le 1$. Prove that:

$$\sqrt{1-xy+y^2}+\sqrt{1-yz+z^2}+\sqrt{1-zt+t^2}+\sqrt{1-tx+x^2}\ge\sqrt{16+(x-y+z-t)^2}$$

**Problem 36. (Phan Thanh Nam)**

Let $x,y,z,t\in[-1;1]$ satisfying $x+y+z+t=0$. Prove that:

$$\sqrt{1+x+y^2}+\sqrt{1+y+z^2}+\sqrt{1+z+t^2}+\sqrt{1+t+x^2}\ge 4$$

**Problem 37.** Given $a,b,c,d\ge 0$ and $a^2+b^2+c^2+d^2=1$.Prove that:

$$(1-a)(1-b)(1-c)(1-d)\ge abcd$$

**Problem 38.** Given $a,b,c,d\ge 0$. Prove that:

$$4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)\ge(a+b+c+d)^3$$

**Problem 39.** Given $a_1,a_2,...,a_n\ge 0$ such that $a_1a_2...a_n=1$. Prove that:

$$(n-1)\left(x_1^2+x_2^2+...+x_n^2\right)+n(n+3)\ge(2n+2)\left(x_1+x_2+...+x_n\right)$$

**Problem 40.**Find the best constant $k_m$ such that the following inequality holds

$$\left(1+mx_1\right)\left(1+mx_2\right)...\left(1+mx_n\right)\le(m+1)^n+k_m\left(x_1x_2...x_n-1\right)$$

for every $x_1,x_2,...x_n$ such that $x_1+x_2+...+x_n=n$ and *m* is an arbitrary constant.

**Problem 41. (Phan Thanh Viet)**

Let $a_1, a_2, ...a_n, s, k > 0$ such that: $a_1 a_2 ... a_n = s^n$ and $n-1 = \frac{n}{(1+s)^k}$.

Consider the inequality: $\frac{1}{(1+a_1)^k} + \frac{1}{(1+a_2)^k} + ... + \frac{1}{(1+a_n)^k} \le n-1$

***a***) Prove that the above inequality doesn't hold true in general.

***b***) Prove that the above inequality is true if $k = 1$.

***c***) Find all values of $k$ (depending on $n$) such that the inequality holds true.

**Problem 43.** Given $a_1, a_2, ..., a_n \ge 0$ such that $a_1 + a_2 + ... + a_n = n$. Prove that

$$n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \ge 4(n-1)(a_1{}^2 + a_2{}^2 + ... + a_n{}^2) + n(n-2)^2$$

**Problem 43.** Given $a_1, a_2, ..., a_n \ge 0$ such that $a_1 + a_2 + ... + a_n = 1$. Prove that:

$$\sqrt{\frac{1-a_1}{1+a_1}} + \sqrt{\frac{1-a_2}{1+a_2}} + ... + \sqrt{\frac{1-a_2}{1+a_2}} \le n - 2 + \frac{2}{\sqrt{3}}$$

**Problem 44. (Phan Thanh Viet)**

Consider the set D consisting of $n$ positive real numbers $a_1, a_2, ..., a_n$ such that $a_1 + a_2 + ... + a_n = a_1^2 + a_2^2 + ... + a_n^2$. Find the maximum value of $f(a_1, a_2, ..., a_n) = a_1 a_2 ... a_n$.

**Problem 45. (Phan Thanh Viet)**

Consider the set D consisting of $n$ positive real numbers $a_1, a_2, ..., a_n$ such that

$a_1 + a_2 + ... + a_n = A$, $a_1^2 + a_2^2 + ... + a_n^2 = B$, where A, B > 0 are given and $A^2 < nB$.

**a)** Find the smallest value of $\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_3}$.

**b)** Find the smallest value of $\frac{1}{a_1^2} + \frac{1}{a_2^2} + ... + \frac{1}{a_n^2}$.

**Problem 46. (Tran Ho Thanh Phu)**

Consider the set D consisting of $n$ positive real numbers $a_1, a_2, ..., a_n$ such that

$$a_1 + a_2 + ... + a_n = A,\ a_1^2 + a_2^2 + ... + a_n^2 = B,\ a_1^3 + a_2^3 + ... + a_n^3 = C,$$

where A, B are given positive constants such that the set D is nonempty.

Find the greatest value of the expression $a_1 a_2 ... a_n$

**Problem 47 (Phan Thanh Viet)**

Consider the set D consisting of $n$ positive real numbers $a_1, a_2, ..., a_n$ such that

$\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_n}{a_1} = n+1$. Find the maximum and the minimum values of $\frac{(a_1 + a_2 + ... + a_n)^2}{a_1^2 + a_2^2 + ... + a_n^2}$

# §24.1 NESBIT – SHAPIRO INEQUALITY

## I. BRIEF HISTORY ABOUT NESBIT – SHAPIRO INEQUALITY

***In 1905*** an English mathematician named Nesbit introduced the inequality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}\quad\forall\ a,b,c>0$$

In this paper fifteen different ways to prove the above inequality will be given to enrich readers' thinking of inequality. ***Nesbit Inequality*** is not difficult; but it is not easy to extend the inequality for $n$ positive real numbers.

***In 1954,*** the mathematician Shapiro proposed the general form of the inequality:

• ***Problem****:* Given $a_1,a_2,...a_n>0$. Determine the inequality is true or false:

$$f_n(a_1,a_2,...,a_n)=\frac{a_1}{a_2+a_3}+\frac{a_2}{a_3+a_4}+...+\frac{a_{n-1}}{a_n+a_1}+\frac{a_n}{a_1+a_2}\geq\frac{n}{2}\quad(1)$$

This problem is so famous that in the year ***1990***, the mathematicians must say again about its proof history at Oberwolfach mathematical seminar about inequality. This is not really a surprise because the Shapiro inequality is not valid for all integer n. During 53 years a lot of mathematicians tried to find the solution for proving the inequality (1) in specific and general cases but it was not until ***1958*** that a pupil of Nesbit named Moocden found the solution for $n=4,5,6$ stopping a long time of failure.

***In 1963,*** Diananda proved that:

***If (1) is wrong for some odd $n=n_0$ then it is also wrong for all $n\geq n_0$; and if (1) is correct for some even $n=n_0$ then it is also correct for all $n\leq n_0$.***

He either gave a counter example to claim that (1) is wrong for $n=27$, means (1) is also wrong for all odd $n\geq 27$. After that Djokovic proved (1) is valid for $n=8$ and Bajsanski proved that (1) is valid for $n=7$.

***In 1968,*** Nowosad proved that (1) is valid for $n=10$ so (1) is also valid for $n=9$ according to Diananda theorem.

***In 1971,*** Kristiansen proved that (1) is valid for $n=12$; (1) is also valid $n=11$ Daykin and Malcoln proved that (1) is wrong for $n=25$.

***In 1976,*** Godunova and Leni again proved that (1) is valid for $n=12$; Bushell also gave a different counter example to claim that (1) is wrong for $n=25$. Bushell and Craven either proved that (1) is valid for all odd $n\leq 23$.

***In 1979,*** Troesch and Searcy propose a proof for a very impressive result is that (1) is wrong for all even $n \geq 14$ and using computer, he pointed out that (1) is true for all even $n \leq 12$ and odd $n \leq 23$ (this is not a mathematical proof).

***In 1985,*** Troesch published his mathematical proof (1) for $n = 13$ in the Math Computing journal; and in the ***year 1989*** he continued his proof for $n = 23$. It shows that nearly 40 years after Shapiro proposed his problem, Troesch was the person who finished a long history period with the result: "***Inequality (1) was true for all even n ≤ 12 and odd n ≤ 23. And for all another n, (1) is".***

Another approach for this problem is that we will find the minimum for (1).

**II. BEST ESTIMATION FOR SHAPIRO INEQUALITY IN PRIMARY MATHEMATICS**

***In 1958*** Ranikin proved that: For $f(n) = \inf\limits_{a_i > 0} f_n(a_1, a_2, ..., a_n)$ then

$$\lambda = \lim_{n\to\infty} \frac{f(n)}{n} = \inf_{n\geq 1} \frac{f(n)}{n} < \frac{1}{2}.7.10^{-8} \text{ i.e. (1) is false for } n \text{ big enough}$$

***In 1969***, a Russian 11th grade pupil named Drinfeljd used advanced mathematical methods to introduce an impressive result as follows:

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + ... + \frac{a_{n-1}}{a_n+a_1} + \frac{a_n}{a_1+a_2} > 0.989133 \times \frac{n}{2}, \ \forall a_1, a_2, ...a_n > 0$$

Drinfeljd's method was the creation $\varphi(x)$ as the convex envelope of the functions

$$y_1 = e^{-x};\ y_2 = \frac{2}{e^x + e^{x/2}}.\ \text{Then}\ \lambda = \frac{1}{2}\varphi(0) = 0.4945668...$$

He, probaly thanks to the result, received mathematical reward Fields in ***1990***.

***In 4 - 1991,*** Kvant Magazine was successful in proving the two following results:

**1.** $\dfrac{a_1}{a_2+a_3} + \dfrac{a_2}{a_3+a_4} + ... + \dfrac{a_{n-1}}{a_n+a_1} + \dfrac{a_n}{a_1+a_2} > \left(\sqrt{2}-1\right)n = 0.41421356n, \forall a_i > 0$

**2.** $\dfrac{a_1}{a_2+a_3} + \dfrac{a_2}{a_3+a_4} + ... + \dfrac{a_{n-1}}{a_n+a_1} + \dfrac{a_n}{a_1+a_2} > \dfrac{5}{12}n = 0.41666666n, \forall a_i > 0$

Hereunder we will consider the best estimation of ***Shapiro Inequality*** in primary mathematics.

**3.** Let be given $a_1, a_2, ...a_n > 0$ for $3 \leq n \in \infty$. Prove that:

$$S = \frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_4} + ... + \frac{a_{n-1}}{a_n+a_1} + \frac{a_n}{a_1+a_2} > 0.4577996n \quad (1)$$

## *Proof*

Consider $0 < \alpha < 1$; $k\in \mathrm{N}^*$. Let $a_{n+1} = a_1; a_{n+2} = a_2$; $\beta = \dfrac{\alpha^k}{k(1-\alpha)}$. We have:

$$S = \sum_{i=1}^{n} \frac{a_i}{a_{i+1} + a_{i+2}} = \sum_{i=1}^{n} \left[ \frac{a_i + \alpha^k a_{i+1}}{a_{i+1} + a_{i+2}} + k \cdot \frac{\beta(\alpha a_{i=1} + a_{i+2})}{a_{i+1} + a_{i+2}} - k\beta \right]$$

$$= \sum_{i=1}^{n} \left[ \frac{a_i + \alpha^k a_{i+1}}{a_{i+1} + a_{i+2}} + k \cdot \frac{\beta(\alpha a_i + a_{i+1})}{a_i + a_{i+1}} \right] - k\beta n \geq \sum_{i=1}^{n} \left[ (k+1) \cdot \sqrt[k+1]{\frac{a_i + \alpha^k a_{i+1}}{a_{i+1} + a_{i+2}} \cdot \frac{\beta^k (\alpha a_i + a_{i+1})^k}{(a_i + a_{i+1})^k}} \right] - k\beta n \quad (2)$$

We will prove: $\left(a_i + \alpha^k a_{i+1}\right)\left(\alpha a_i + a_{i+1}\right)^k \geq \alpha^k \left(a_i + a_{i+1}\right)^{k+1}$ (3)

$$\Leftrightarrow \left(1 + \alpha^k t_i\right)\left(\alpha + t_i\right)^k \geq \alpha^k \left(1 + t_i\right)^{k+1} \text{ with } t_i = \frac{a_i + 1}{a_i} \geq 0$$

Consider the function $f(t) = (1 + \alpha^k t)(\alpha + t)^k - \alpha^k (1+t)^{k+1}$ with $t \geq 0$.

We have: $f^{(m)}(t) = (1 + \alpha^k t)(\alpha + t)^{k-m} k(k-1)....[k-(m-1)] +$

$$+m \cdot \alpha^k (\alpha + t)^{k-(m-1)} k(k-1)....[k-(m-2)] - \alpha^k (1+t)^{k+1-m} (k+1)k....[k-(m-2)]$$

$$\Rightarrow f^{(m)}(0) = k(k-1)....[k-(m-2)]\left[(k-m+1)\alpha^{k-m} + m\alpha^{2k-m+1} - (k+1)\alpha^k\right]$$

$$\geq k(k-1)....(k-m+2)\left[(k+1)\sqrt[k+1]{\alpha^{(k-m)(k-m+1)+m(2k-m+1)}} - (k+1)\alpha^k\right] = 0$$

Since $f(t)$ is a $k$-degree polynomial then $f^{(k)}(t) = const = f^{(k)}(0) \geq 0$

$$\Rightarrow f^{(k-1)}(t) \geq f^{(k-1)}(0) \geq 0 \Rightarrow f^{(k-2)}(t) \geq f^{(k-2)}(0) \geq 0, \ldots, f(t) \geq f(0) = 0$$

We have $f(t) \geq 0 \ \ \forall t \geq 0 \Rightarrow f(t_i) \geq 0 \ \ \forall t_i \geq 0 \Rightarrow$ (3) is true.

From (2) and (3) follow:

$$S \geq \sum_{i=1}^{n} \left[ (k+1) \cdot \sqrt[k+1]{\beta^k \alpha^k \cdot \frac{a_i + a_{i+1}}{a_{i+1} + a_{i+2}}} \right] - k\beta n = (k+1)(\alpha\beta)^{\frac{k}{k+1}} \cdot \sum_{i=1}^{n} \sqrt[k+1]{\frac{a_i + a_{i+1}}{a_{i+1} + a_{i+2}}} - k\beta n$$

$$\geq \left[(k+1)(\alpha\beta)^{\frac{k}{k+1}} - k\beta\right] n = \left[ (k+1) \frac{\alpha^k}{[k(1-\alpha)]^{\frac{k}{k+1}}} - \frac{\alpha^k}{1-\alpha} \right] n$$

Let $x = \sqrt[k+1]{\dfrac{1}{1-\alpha}} \Rightarrow \alpha = 1 - \dfrac{1}{x^{k+1}}$, then $g(x) = (k+1) \cdot \dfrac{\alpha^k}{[k(1-\alpha)]^{\frac{k}{k+1}}} - \dfrac{\alpha^k}{1-\alpha}$

$$g(x) = \left(1 - \frac{1}{x^{k+1}}\right)^k x^k \left[ \frac{k+1}{k^{\frac{k}{k+1}}} - x \right] = \frac{\left(x^{k+1} - 1\right)^k}{x^{k^2}} \left[ (k+1) k^{\frac{-k}{k+1}} - x \right]$$

$$g'(x)=\frac{k\left(x^{k+1}-1\right)^{k-1}(k+1)x^{k}.x^{k^{2}}-k^{2}x^{k^{2}-1}\left(x^{k+1}-1\right)^{k}}{x^{2k^{2}}}\left[(k+1)k^{\frac{-k}{k+1}}-x\right]-\frac{\left(x^{k+1}-1\right)^{k}}{x^{k^{2}}}$$

We have: $g'(x)=0\Leftrightarrow\frac{(k+1)\mathrm{k}\,x^{k+1}-k^{2}\left(x^{k+1}-1\right)}{x^{k^{2}+1}}\left[(k+1)k^{\frac{-k}{k+1}}-x\right]-\frac{x^{k+1}-1}{x^{k^{2}}}=0$

$$\Leftrightarrow\left[(k+1)\mathrm{k}\,x^{k+1}-k^{2}x^{k+1}+k^{2}\right]\left[(k+1)k^{\frac{-k}{k+1}}-x\right]-x\left(x^{k+1}-1\right)=0$$

$$\Leftrightarrow k\left(x^{k+1}+k\right)\left[(k+1)k^{\frac{-k}{k+1}}-x\right]-x^{k+2}+x=0\Leftrightarrow\left(x^{k+1}+k\right)\left[(k+1)k^{\frac{1}{k+1}}-kx\right]=x^{k+2}-x$$

$$\Leftrightarrow(k+1)k^{\frac{1}{k+1}}x^{k+1}+(k+1)k^{\frac{k+2}{k+1}}-kx^{k+2}-k^{2}x=x^{k+2}-x$$

$$\Leftrightarrow(k+1)x^{k+2}-(k+1)k^{\frac{1}{k+1}}x^{k+1}+\left(k^{2}-1\right)x-(k+1)k^{\frac{k+2}{k+1}}=0$$

$$\Leftrightarrow h(x)=x^{k+2}-k^{\frac{1}{k+1}}x^{k+1}+(k-1)x-k^{\frac{k+2}{k+1}}=0$$

We have: $h'(x)=(k+2)x^{k+1}-(k+1)k^{\frac{1}{k+1}}x^{k}+(k-1)$

$$\geq(k+1)\cdot\sqrt[k+1]{\left(\frac{k+2}{k}\cdot x^{k+1}\right)^{k}(k-1)}-(k+1)k^{\frac{1}{k+1}}x^{k}=(k+1)\left[\left(\frac{k+2}{k}\right)^{\frac{k}{k+1}}(k-1)^{\frac{1}{k+1}}-k^{\frac{1}{k+1}}\right]x^{k}$$

We will prove: $\left(\frac{k+2}{k}\right)^{\frac{k}{k+1}}(k-1)^{\frac{1}{k+1}}-k^{\frac{1}{k+1}}\geq0\Leftrightarrow(k+2)^{\frac{k}{k+1}}(k-1)^{\frac{1}{k+1}}\geq k$

$$\Leftrightarrow(k+2)^{k}(k-1)\geq k^{k+1}\Leftrightarrow\left(\frac{k+2}{k}\right)^{k}\geq\frac{k}{k-1}\Leftrightarrow\left(1+\frac{2}{k}\right)^{k}\geq1+\frac{1}{k-1}$$

We have: $\left(1+\frac{2}{k}\right)^{k}>1+\frac{2}{k}\cdot k=3>1+\frac{1}{k-1},\ \forall k\geq2$

$\Rightarrow h'(x)>0\Rightarrow h(x)$ is increasing. But $h(x)$ is continuous; $\lim\limits_{n\to+\infty}h(x)=+\infty$;

$h(1)=k-k^{\frac{1}{k+1}}-k^{\frac{k+2}{k+1}}<-k^{\frac{1}{k+1}}<0$ then $h(x)=0$ has unique root $x_0>1$

$\Rightarrow g'(x)=0$ has unique root $x_0>1\Rightarrow$ Variation Table $\Rightarrow$ Max $g(x)=g(x_0)$

| $x$ | $1$ | $x_0$ | $+\infty$ |
|---|---|---|---|
| $g'(x)$ | | $0$ | |
| $g(x)$ | $\nearrow$ | Max | $\searrow$ |

where $x_0$ is a root of the inequation

$$h(x)=x^{k+2}-k^{\frac{1}{k+1}}x^{k+1}+(k-1)x-k^{\frac{k+2}{k+1}}=0$$

and $g(x)=\frac{\left(x^{k+1}-1\right)^{k}}{x^{k^{2}}}\left[(k+1)k^{\frac{-k}{k+1}}-x\right]$

If $k=500$ then $x_0=1.013294063$ and $g(x_0)=0.4577996$