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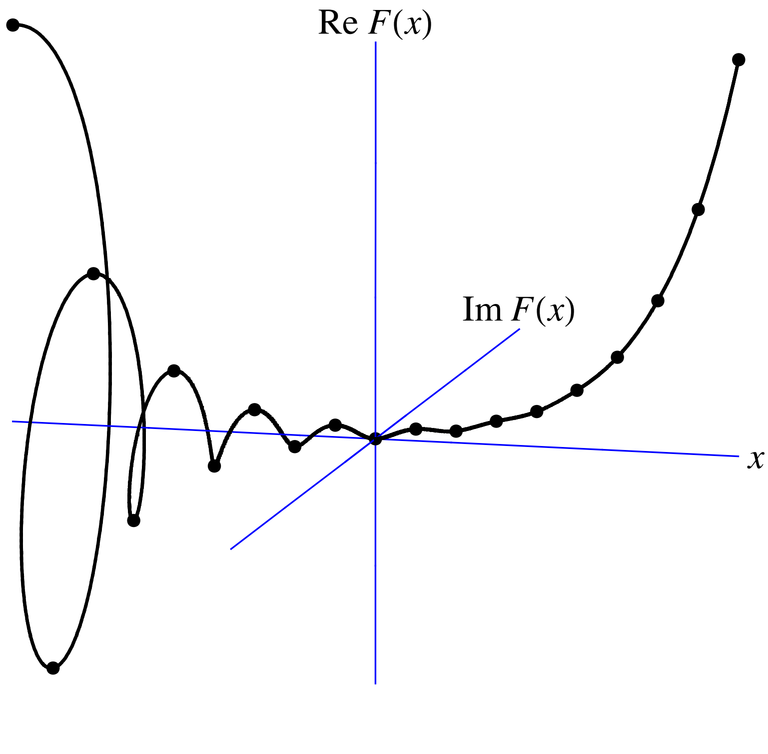


Complex Fibonacci

Martin Hansen

One of the highlights of an undergraduate mathematician's first year at university must surely be the revelation that many functions are amenable to being approximated by polynomials. In particular, there are the following three marvellous results, sometimes referred to as 'Taylor Polynomials' (about $x = 0$) or 'Maclaurin Series' or 'Power Series'.

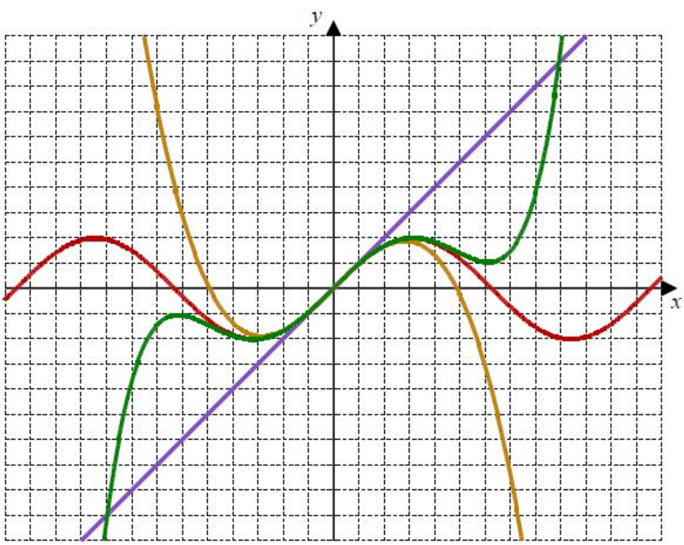
Power Series Expansions (centred on $x = 0$)

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \ldots \qquad \text{valid for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \qquad \text{valid for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \qquad \text{valid for all } x$$

Pleasingly, any one of these results can be visualised by taking successive partial sums of the series and plotting graphs. Below are shown approximations of the sine function with $y = x$, $y = x - x^3/3!$ and $y = x - x^3/3! + x^5/5!$. As the degree of the approximating polynomial is increased, it better follows the sine curve over a greater interval, centred on the origin.



A key fact about power series is that any two (with the same centre) can be added, multiplied or divided in the same way as polynomials. This suggests that the exponential series can be manipulated in the following adventurous manner when the index is a complex number, $z = a + ib$, $a, b \in \mathbb{R}$, $i^2 = -1$:

$$\begin{aligned}
e^z &= e^{a+ib} \\
&= e^a e^{ib} \\
&= e^a \left(1 + ib + \frac{(ib)^2}{2!} + \frac{(ib)^3}{3!} + \frac{(ib)^4}{4!} + \frac{(ib)^5}{5!} + \frac{(ib)^6}{6!} + \dots\right) \\
&= e^a \left(1 + ib - \frac{b^2}{2!} - \frac{ib^3}{3!} + \frac{b^4}{4!} + \frac{ib^5}{5!} - \frac{b^6}{6!} + \dots\right) \\
&= e^a \left(\left(1 - \frac{b^2}{2!} + \frac{b^4}{4!} - \frac{b^6}{6!} + \dots\right) + i\left(b - \frac{b^3}{3!} + \frac{b^5}{5!} - \dots\right)\right) \\
&= e^a(\cos b + i \sin b).
\end{aligned}$$

The real exponential function is thus extended into the world of complex numbers via the beautiful result known as Euler's Relation.

Euler's Relation

$$e^{ib} = \cos b + i \sin b.$$

When $b = \pi$ Euler's Relation yields $e^{\pi i} = -1$, usually written as

$$e^{\pi i} + 1 = 0$$

and often referred to as the most beautiful equation in all of mathematics. Of course, when $b = 0$ the complex exponential function is identical to the real exponential function. In this case, a real function has been extended into the complex realm by a simple multiplication of $\cos b + i \sin b$. It is natural to wonder if there are other functions extendable in this sort of way.

A drawback for beginners in trying to appreciate what has been achieved with the extension of the exponential from the real to the complex is that the complex exponential function is tricky to visualise. Its domain is two dimensional as is its codomain. Visualising four dimensions simultaneously does not come naturally to the average person. In this article I wanted to look at a similar extension to a well-known function but one which yields results more easily visualised.

The function I have in mind is that associated with the Fibonacci numbers. These are usually defined by means of the simple recursive formula,

$$F_{n+2} \;=\; F_{n+1} + F_n, \qquad n \in \mathbb{Z}, \quad n \geq 0,$$

along with two initial terms $F_0 = 0$ and $F_1 = 1$. It gives rise to the 'world famous' sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots.$$

This has so many astounding mathematical properties that a scholarly journal, *The Fibonacci Quarterly*, is devoted to ongoing research of this and related sequences such as those of Lucas, Jacobsthal, and Pell.

The domain can be extended readily enough to include the negative integers. The resulting 'extended leftward' sequence is

$$\ldots, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

Part of the fascination of the Fibonacci sequence stems from the fact that it has a closed form formula for term n that, although we are working with the integers, contains fractions and square roots 'all over the place'. Yet it will yield an integer output no matter what integer input is assigned to n.

The Binet Formula for the Fibonacci Sequence (Version 1)

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right), \qquad n \in \mathbb{Z}$$

Having extended the domain of the Fibonacci sequence to include the negative integers, the Binet formula provides an opportunity to go further and extend it to the reals. Of course, $1 - \sqrt{5}$ is negative, and so for some values of n its powers will be complex numbers. When I reached for my calculator (a Casio Classwiz fx-991EX in complex number mode) it gave, for $n = 0.5$,

$$\begin{aligned} F_{0.5} \;&=\; \frac{1}{\sqrt{5}}\left(\sqrt{\frac{1+\sqrt{5}}{2}} - \sqrt{\frac{1-\sqrt{5}}{2}}\right) \\ &=\; 0.569 - 0.352\, i. \end{aligned}$$

However, it could not cope with the same calculation presented in the form

$$F_{0.5} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{0.5} - \left(\frac{1-\sqrt{5}}{2}\right)^{0.5}\right).$$

In retrospect, this was a blessing; rather than reaching for more powerful software a deeper understanding was called for.

The key idea is to find a way of separating the real part of the calculation from the imaginary part and an ingenious way to do this is to be found in a paper from 1968 in *The Fibonacci Quarterly* by Alan Scott, [1].

To understand Scott's result we first need to look more carefully at $e^{i\pi} = -1$. Values of the function $f(b) = e^{ib}$ are best understood by visualising them as being on the unit circle in the complex plane. The variable b then has the interpretation of being the angle (in radians) of (anticlockwise) rotation, where the positive real axis corresponds to an angle of 0. The diagram on page 4 shows a few points plotted for b between $-\pi$ and π radians. The crucial point being made by this diagram is that the result can be equally well written as $e^{-i\pi} = -1$. In fact, given any point on the unit circle, a rotation of $2\pi k$ (about the centre) for any integer k gives the same point.

In general, we have that $e^{i(2k+1)\pi} = -1$, $k \in \mathbb{Z}$.

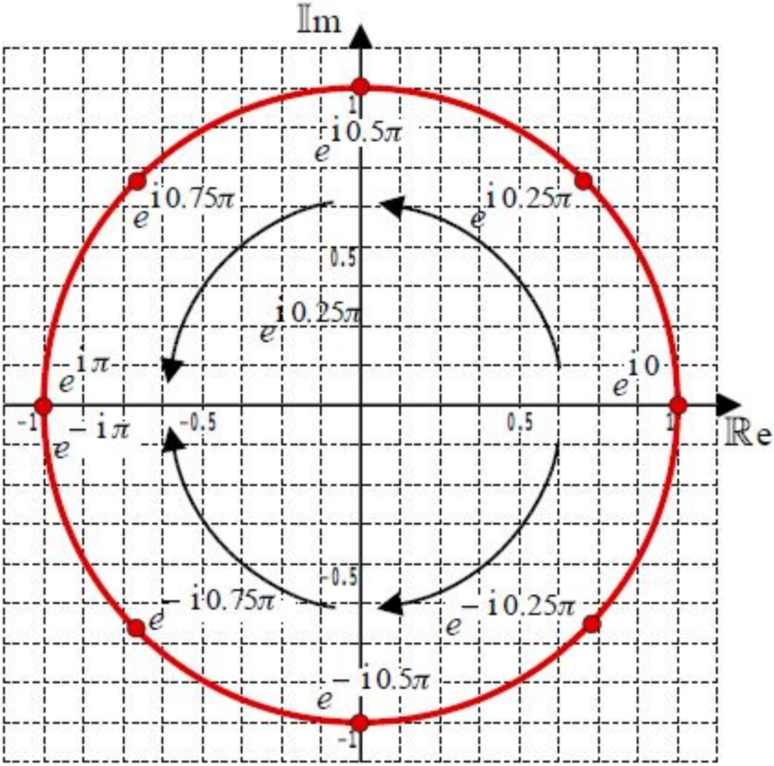
The digression over, we can pick up the main thread of the article. Observe that

$$\left(\frac{1-\sqrt{5}}{2}\right)^{n} = \left(\frac{2}{1-\sqrt{5}}\right)^{-n} = (-1)^{-n}\left(\frac{1+\sqrt{5}}{2}\right)^{-n}.$$

From this observation we obtain

The Binet Formula for the Fibonacci Sequence (Version 2)

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - (-1)^{-n}\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$



Now, we could recall here that $e^{i\pi} = -1$ (so beautiful!) but, to match the result my calculator produced earlier, let's use $e^{-i\pi} = -1$ instead:

$$(-1)^{-n} \;=\; \left(e^{-\pi}\right)^{-n} \;=\; e^{i\pi n} \;=\; \cos(\pi n) + i\sin(\pi n).$$

Thus is obtained a third version of the Binet formula for the Fibonacci numbers.

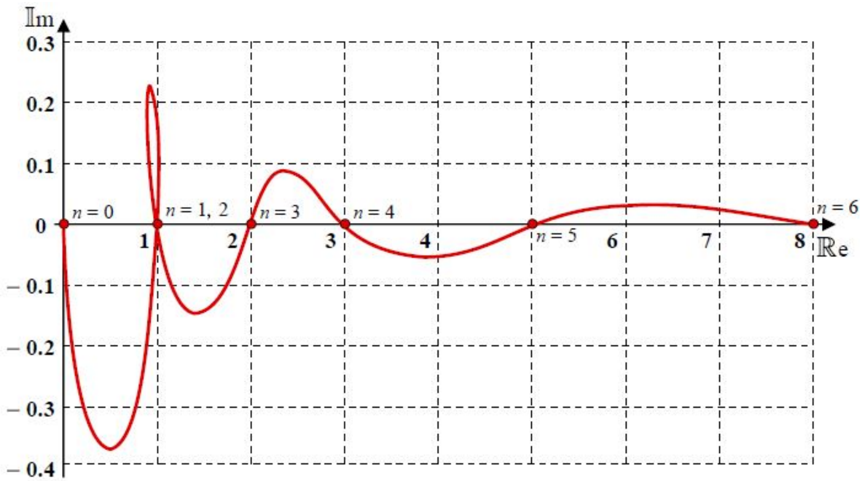
The Binet Formula for the Fibonacci Sequence (Version 3)

$$\operatorname{Re} F_n \;=\; \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$

$$\operatorname{Im} F_n \;=\; -\frac{1}{\sqrt{5}}\left(\sin(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right)$$

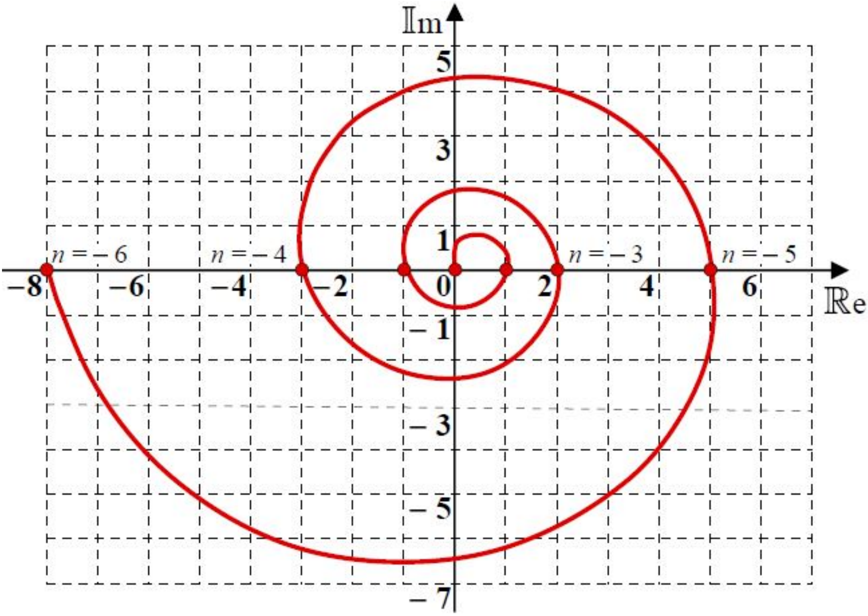
These can be interpreted as parametric equations where x is the real component and y the imaginary, and plotted as a smooth continuous curve where n is considered to be a variable over $\mathbb{R}$.

Such a plot for non-negative real n is presented below. The diagram is of the two dimensional output from the Fibonacci function with one dimensional real number domain and the ease with which it can be understood is aided considerably by knowing the integer sequence from which it was derived. Each time the curve crosses the real number axis as we move along the curve, the domain has integer value incremented by 1. So, three dimensions can be easily visualised in spite of having only the two dimensional codomain plot to study.



The loop from $n = 1$ to $n = 2$ is an attractive feature corresponding to $F_1 = F_2 = 1$ and, indeed, it can be viewed as a part of the interesting transition between the alternating sign of the integer outputs when n is a negative integer and the all positive integer outputs when n is a positive integer. The plot for the non-positive real n is given on the next page.

The earliest plots of this Fibonacci curve that I know of occurred in 1974 [2]. For any reader wishing to investigate the curve further, many clever mathematical results are to be found in a 1988 paper [3].



References (All from *FQ*, *The Fibonacci Quarterly*)

[1] A. M. Scott, Continuous Extensions of Fibonacci Identities, *FQ*, vol. 6(4), Oct 1968.

[2] F. J. Wunderlich, D. E. Shaw and M. J. Hones, Argand Diagrams of Extended Fibonacci and Lucas Numbers, *FQ*, vol. 12(3), Oct 1974.

[3] A. F. Horadam and A. G. Shannon, Fibonacci and Lucas Curves, *FQ*, vol. 26(1), Feb 1988.

Problem 312.1 – Product

Tony Forbes

Show that

$$P_N(x) \;=\; \prod_{i=1}^{N} \frac{4i+x}{4i+1}\,\frac{4i+3}{4i+2}, \qquad -1 < x < 1,$$

converges to something non-zero as $N \to \infty$ if and only if $x = 0$.