**Chapter 7**

**GCSE** and **A-Level Pure Mathematics**
**Vectors I**

## 7.1 Change of Basis

When a point is specified, such as $P(3, 4)$, it can be thought of as a vector description of $P$'s location from the origin. Such vectors that are tied to a location, the origin in this case, are called position vectors, rather than free vectors.

$$\boldsymbol{p} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{or} \quad \boldsymbol{p} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

The description of $\boldsymbol{p}$ in the style $\boldsymbol{p} = 3\boldsymbol{i} + 4\boldsymbol{j}$ emphasises that the vector $\boldsymbol{p}$ is expressed in terms of two other vectors; the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in the $x$ and $y$-axis directions respectively. The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are said to form a basis for the Cartesian coordinate system. Any other location on the $XY$ plane can be specified using some combination of $\boldsymbol{i}$ and $\boldsymbol{j}$.
Other vectors can be used as a basis for a different coordinate system.

## 7.2 Example

A coordinate system has basis vectors $\boldsymbol{A} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

**( i )** Write $\boldsymbol{s} = 6\boldsymbol{A} + 4\boldsymbol{B}$ in the form $\boldsymbol{s} = k\begin{pmatrix} 20 \\ 21 \end{pmatrix}$ for some constant $k$.

**[ 1 mark ]**

**( ii )** Write $\boldsymbol{t} = 9\boldsymbol{A} + 6\boldsymbol{B}$ in the form $\boldsymbol{t} = K\begin{pmatrix} 20 \\ 21 \end{pmatrix}$ for some constant $K$.

**[ 1 mark ]**

**( iii )** What do your answers to part (i) and (ii) show ?

**[ 1 mark ]**

**( iv )** Show that $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel for all other vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.
(In other words, $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel in any linear coordinate system)

**[ 2 marks ]**

Teaching Video : http://www.NumberWonder.co.uk/v9009/7.mp4

<= Watch the video, complete the above example.

**7.3 Exercise**

*Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.*

Marks available : 40

**Question 1**

Two vectors, $\boldsymbol{E}$ and $\boldsymbol{Z}$, are $\boldsymbol{E} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\boldsymbol{Z} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

**( i )** Write $\boldsymbol{a} = 9\,\boldsymbol{E} + 3\,\boldsymbol{Z}$ in the form $\boldsymbol{a} = k \begin{pmatrix} 9 \\ 19 \end{pmatrix}$ for some constant $k$.

**[ 1 mark ]**

**( ii )** Write $\boldsymbol{b} = 15\,\boldsymbol{E} + 5\,\boldsymbol{Z}$ in the form $\boldsymbol{b} = K \begin{pmatrix} 9 \\ 19 \end{pmatrix}$ for some constant $K$.

**[ 1 mark ]**

**( iii )** What do your answers to part (i) and (ii) show ?

**[ 1 mark ]**

**( iv )** Show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel for all other vectors $\boldsymbol{E}$ and $\boldsymbol{Z}$.

**[ 2 marks ]**

**Question 2**

Two vectors, $\boldsymbol{E}$ and $\boldsymbol{T}$, are given by $\boldsymbol{E} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\boldsymbol{T} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

**( i )** Write $\boldsymbol{p} = 8\,\boldsymbol{E} - 20\,\boldsymbol{T}$ in the form $\boldsymbol{p} = k \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ for some constant $k$.

**[ 1 mark ]**

**( ii )** Write $\boldsymbol{h} = 6\,\boldsymbol{E} - 15\,\boldsymbol{T}$ in the form $\boldsymbol{h} = K \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ for some constant $K$.

**[ 1 mark ]**

**( iii )** Show that $\boldsymbol{p}$ and $\boldsymbol{h}$ are parallel for all vectors $\boldsymbol{E}$ and $\boldsymbol{T}$.

**[ 2 marks ]**

**Question 3**

Two vectors, $\boldsymbol{X}$ and $\boldsymbol{Y}$ are parallel if one can be written as a multiple of the other.
In each question decide if the two vectors given are parallel or not.
For those that are parallel, write in the form $\boldsymbol{X} = k\,\boldsymbol{Y}$ for some constant $k$.

**( i )** $\boldsymbol{X} = \begin{pmatrix} -15 \\ 9 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

**[ 1 mark ]**

**( ii )** $\boldsymbol{X} = 4\boldsymbol{i} - 2\boldsymbol{j}$ $\boldsymbol{Y} = 4\boldsymbol{i} + 2\boldsymbol{j}$

**[ 1 mark ]**

**( iii )** $\boldsymbol{X} = \begin{pmatrix} 8 \\ -4 \\ 6 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 12 \\ -6 \\ 9 \end{pmatrix}$

**[ 1 mark ]**

**( iv )** $\boldsymbol{X} = \begin{pmatrix} 16 \\ 9 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

**[ 1 mark ]**

**( v )** $\boldsymbol{X} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$

**[ 1 mark ]**

**( vi )** $\boldsymbol{X} = 14\boldsymbol{i} - 21\boldsymbol{j}$ $\boldsymbol{Y} = 21\boldsymbol{i} - 14\boldsymbol{j}$

**[ 1 mark ]**

**( vii )** $\boldsymbol{X} = 3\boldsymbol{i} - \boldsymbol{j}$ $\boldsymbol{Y} = -6\boldsymbol{i} + 2\boldsymbol{j}$

**[ 1 mark ]**

**( viii )** $\boldsymbol{X} = 4\boldsymbol{a} - 12\boldsymbol{b}$ $\boldsymbol{Y} = 6\boldsymbol{a} - 18\boldsymbol{b}$

**[ 1 mark ]**

**( ix )** $\boldsymbol{X} = 15\boldsymbol{a}$ $\boldsymbol{Y} = 16\boldsymbol{a}$

**[ 1 mark ]**

**( x )** $\boldsymbol{X} = \boldsymbol{i} - \boldsymbol{j}$ $\boldsymbol{Y} = -\boldsymbol{i} + \boldsymbol{j}$

**[ 1 mark ]**

**( xi )** $\boldsymbol{X} = \begin{pmatrix} 1 \\ -7 \\ 3 \\ 0 \\ 11 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} -2 \\ 14 \\ 6 \\ 0 \\ -22 \end{pmatrix}$ Five dimensions is blowing my mind !

**[ 1 mark ]**

**( xii )** $\boldsymbol{X} = \frac{1}{2}a + 2b$ $\boldsymbol{Y} = \frac{1}{3}a + 3b$

**[ 1 mark ]**

**Question 4**

#VectorsFascinatingFact

Here is a fascinating fact about vectors !

The vectors $\boldsymbol{X} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} c \\ d \end{pmatrix}$

are mutually perpendicular (each at 90° to the other) if and only if $ac + bd = 0$

**( a )** For each of the following pairs of vectors state if they are mutually perpendicular or not.

**( i )** $\boldsymbol{X} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

**[ 1 mark ]**

**( ii )** $\boldsymbol{X} = \begin{pmatrix} 0 \\ 17 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

**[ 1 mark ]**

**( iii )** $\boldsymbol{X} = \begin{pmatrix} 0.5 \\ 16 \end{pmatrix}$ and $\boldsymbol{Y} = \begin{pmatrix} 8 \\ 0.25 \end{pmatrix}$

**[ 1 mark ]**

**( b )** Given that the following two vectors are mutually perpendicular.

$\boldsymbol{X} = \begin{pmatrix} -6 \\ 11 \end{pmatrix}$ $\boldsymbol{Y} = \begin{pmatrix} w \\ 9 \end{pmatrix}$

Find the value of $w$.

**[ 2 marks ]**

**Question 5**

Consider the vector $\boldsymbol{X} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

**( i )** Show that $|\boldsymbol{X}| = 5$

**[ 1 mark ]**

**( ii )** Write down a vector of magnitude 5 which is perpendicular to $\boldsymbol{X}$

**[ 2 marks ]**

**( iii )** Write down another vector of magnitude 5 which is perpendicular to $\boldsymbol{X}$

**[ 2 marks ]**

**Question 6**

Let $\boldsymbol{a} = 5\boldsymbol{p} + 4\boldsymbol{q}$ $\boldsymbol{b} = -2\boldsymbol{p} + 2\boldsymbol{q}$ and $\boldsymbol{c} = \boldsymbol{p} + 6\boldsymbol{q}$

**( i )** Find an expression for vector, $\boldsymbol{v}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$, if $\boldsymbol{v} = 5\boldsymbol{a} - \boldsymbol{b}$
Your expression should not contain any brackets.

**HINT** : $\boldsymbol{v} = 5 \quad \boldsymbol{a} \quad - \quad \boldsymbol{b}$

$\boldsymbol{v} = 5(5\boldsymbol{p} + 4\boldsymbol{q}) - (-2\boldsymbol{p} + 2\boldsymbol{q})$ Be careful with the double minus.

**[ 1 marks ]**

**( ii )** Find an expression for vector, $\boldsymbol{w}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$, if $\boldsymbol{w} = 8\boldsymbol{a} - \boldsymbol{c}$.
Your expression should not contain any brackets.

**[ 1 marks ]**

**( iii )** Prove that the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel.

**[ 2 marks ]**

**Question 7**

Let $\boldsymbol{d} = \boldsymbol{p} + 3\,\boldsymbol{q}$ $\boldsymbol{e} = \boldsymbol{p} - 3\,\boldsymbol{q}$ and $\boldsymbol{f} = 2\,\boldsymbol{p} - \boldsymbol{q}$

**( i )** Find an expression for vector, $\boldsymbol{r}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{r} = 3\,\boldsymbol{d} + 6\,\boldsymbol{e}$.
Your expression should not contain any brackets.

**[ 1 marks ]**

**( ii )** Find an expression for vector, $\boldsymbol{s}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{s} = -2\,\boldsymbol{d} + 8\,\boldsymbol{f}$.
Your expression should not contain any brackets.

**[ 1 marks ]**

**( iii )** Find an expression for vector, $\boldsymbol{t}$, in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ only, if $\boldsymbol{t} = 4\,\boldsymbol{e} - 3\,\boldsymbol{f}$.
Your expression should not contain any brackets.

**[ 1 mark ]**

**( iv )** Which of the vectors $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are parallel ?
Prove your answer.

**[ 2 marks ]**