### 8.1 The Ratio Theorem

A situation that repeatedly occurs in vectors problems is that of a line *AB* being divided in a given ratio *m* : *u*. The Ratio Theorem generalises the resulting algebraic manipulations and gives a formula that can be used to skip through this recurring situation at a brisk pace.

The theorem makes use of the simple yet clever idea that for any line *AB* there must be an **O**ther point, *O*, "somewhere". This other point is often the origin, but it does not have to be and, indeed, part of the skill of using the ratio theorem quickly is to pick a good "other point". The other point is often on an associated diagram of which the line *AB* is a part but, again, it does not have to be.

The Ratio Theorem
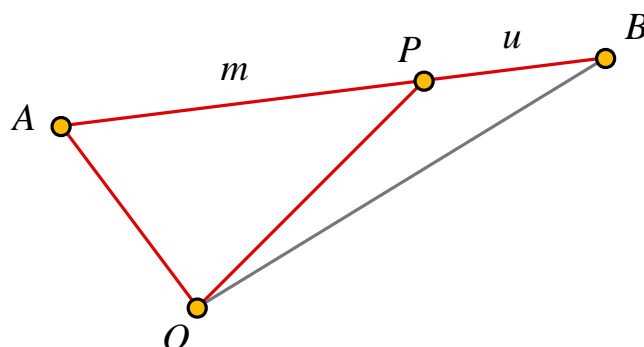
If the point *P* divides the line segment *AB* in the ratio *m* : *u* then,

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{u + m}\, \overrightarrow{AB}$$

Proof

$$\overrightarrow{AP} \ : \ \overrightarrow{PB}$$

$$m \ : \ u$$

$$u\,\overrightarrow{AP} = m\,\overrightarrow{PB}$$

$$u\left(\overrightarrow{AO} + \overrightarrow{OP}\right) = m\left(\overrightarrow{PO} + \overrightarrow{OB}\right)$$

$$u\,\overrightarrow{AO} + u\,\overrightarrow{OP} = m\,\overrightarrow{PO} + m\,\overrightarrow{OB}$$

$$u\,\overrightarrow{OP} - m\,\overrightarrow{PO} = -u\,\overrightarrow{AO} + m\,\overrightarrow{OB}$$

$$u\,\overrightarrow{OP} + m\,\overrightarrow{OP} = u\,\overrightarrow{OA} + m\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$$

$$(u + m)\,\overrightarrow{OP} = (u + m)\,\overrightarrow{OA} + m\,\overrightarrow{AB}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{m}{u + m}\,\overrightarrow{AB} \qquad \square$$

8.2 Example

A line segment *AB* has endpoints *A*(1, 4) and *B*(11, 19)

A point *P* on the line segment *AB* is such that *AP* : *PB* = 3 : 2.

Find the coordinates of *P*

Teaching Video : http://www.NumberWonder.co.uk/v9009/8.mp4

After watching the video
write out your solution.

☞

[3 marks]

To help remember the ratio theorem notice that *mu* is the noise made by a cat

and the fraction in the theorem $\dfrac{m}{u\ +\ m}$ is a "sort of" spelling of the word *mum*

8.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
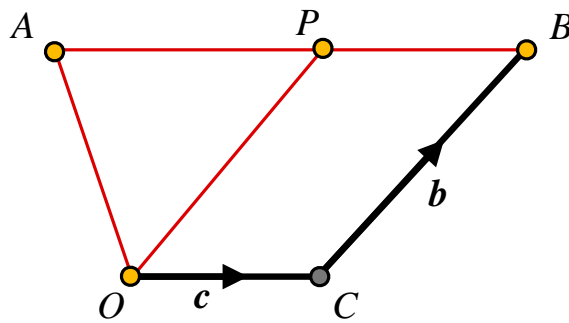*Make the method used clear.*
Marks available : 40

Question 1

A line segment *AB* has endpoints *A*(7, 3) and *B*(42, 59)
A point *P* on the line segment *AB* is such that *AP* : *PB* = 4 : 3
Find the coordinates of *P*

[**3 marks**]

Question 2

OABC is a trapezium with *AB* parallel to *OC* and *AB* = 5 *OC*.
P divides *AB* such that *AP* : *PB* = 3 : 2

$$\overrightarrow{OC} = c \quad \text{and} \quad \overrightarrow{CB} = b$$

(**i**)   Find $\overrightarrow{OA}$ in terms of *c* and *b*

[**1 mark**]

(**ii**)   By using the Ratio Theorem, find $\overrightarrow{OP}$ in terms of *c* and *b*

[**3 marks**]

Question 3

Diagram NOT accurately drawn
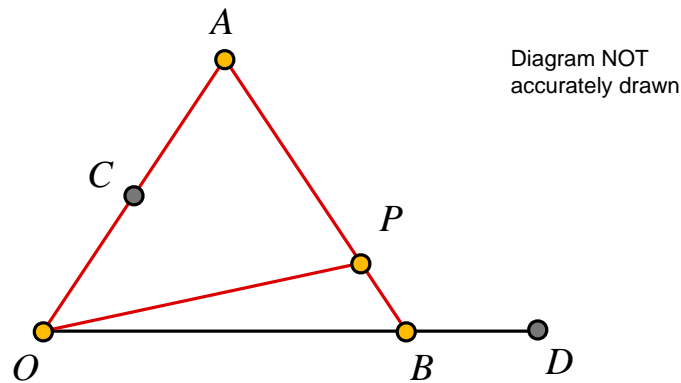
OAB is a triangle with $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$

C is the midpoint of OA and P is the point on AB such that $AP : PB = 3 : 1$

D is the point such that $\overrightarrow{OB} = 2\,\overrightarrow{BD}$

(i)　　Use the Ratio Theorem to find $\overrightarrow{OP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[2 marks]

(ii)　　Use $\overrightarrow{CP} = \overrightarrow{CO} + \overrightarrow{OP}$ to find $\overrightarrow{CP}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(iii)　　Use $\overrightarrow{PD} = \overrightarrow{PO} + \overrightarrow{OD}$ to find $\overrightarrow{PD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$

[1 mark]

(iv)　　Prove that the points C, P and D lie on the same straight line

[1 mark]

(v)　　Determine the ratio $CP : PD$

[1 mark]

Question 4

Diagram NOT accurately drawn
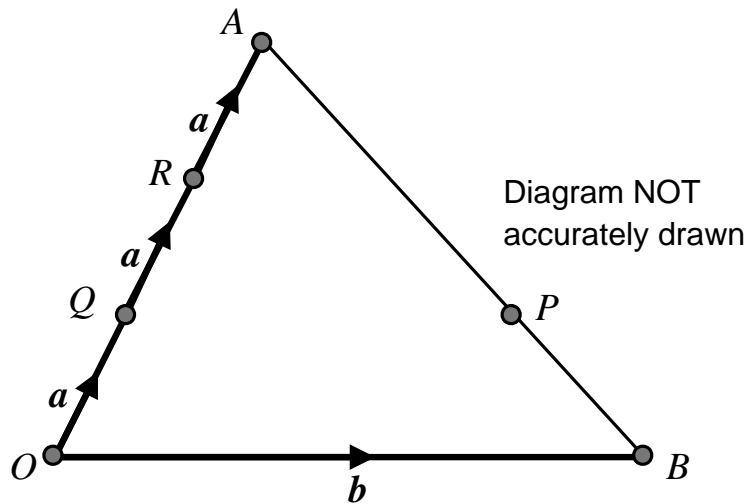
OAB is a triangle in which $\overrightarrow{OA} = 3a$ and $\overrightarrow{OB} = b$

Q is the point on OA such that $OA = 3\,OQ$

P is the point on AB such that $AB = 3\,PB$

(**i**)   Show how to use the Ratio Theorem to express $\overrightarrow{OP}$ in terms of a and b

[3 marks]

(**ii**)   Show that $\overrightarrow{QP} = k\,\overrightarrow{OB}$ where k is an integer
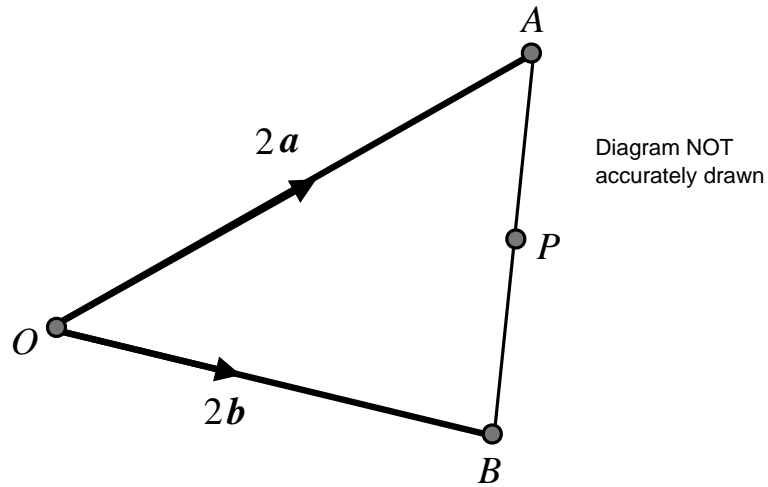
[2 marks]

(**iii**)   State two things that your answer to part (ii) tells you about the relationship between the line segments QP and OB.

[2 marks]

Question 5

Specimen GCSE Examination Question for the 2018 Examinations

Diagram NOT accurately drawn

OAB is a triangle with $\overrightarrow{OA} = 2\,a$ and $\overrightarrow{OB} = 2\,b$

P is the point on AB such that $AP : PB = 5 : 3$

$\overrightarrow{OP} = k\,(\,3\,a + 5\,b\,)$ where k is a scalar quantity

Find the value of k

[3 marks]

Question 6

Diagram NOT accurately drawn
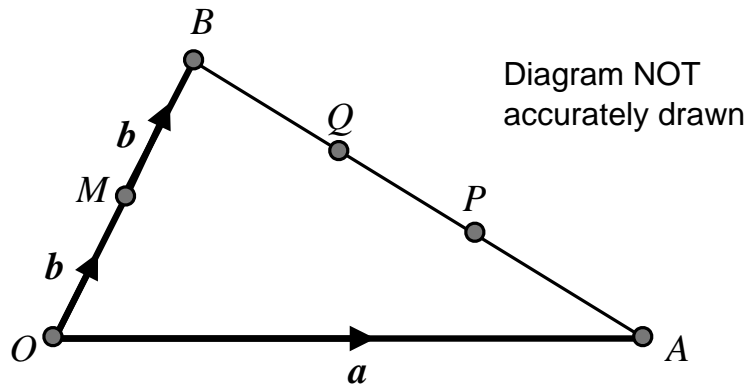
OAB is a triangle where *M* is the mid-point of *OB*
P and *Q* are points on *AB* such that *AP* = *PQ* =*QB*

$\overrightarrow{OA}$ = $\boldsymbol{a}$ and $\overrightarrow{OB}$ = 2 $\boldsymbol{b}$

(**a**)　　Find, in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, expressions for

　　　　(**i**)　　$\overrightarrow{BA}$

[1 mark]

　　　　(**ii**)　　$\overrightarrow{MQ}$

[2 marks]

　　　　(**iii**)　　$\overrightarrow{OP}$

[2 marks]

(**b**)　　What can you deduce about quadrilateral *OMQP* ?
　　　　Give a reason for your answer

[2 marks]

Question 7

Diagram NOT accurately drawn
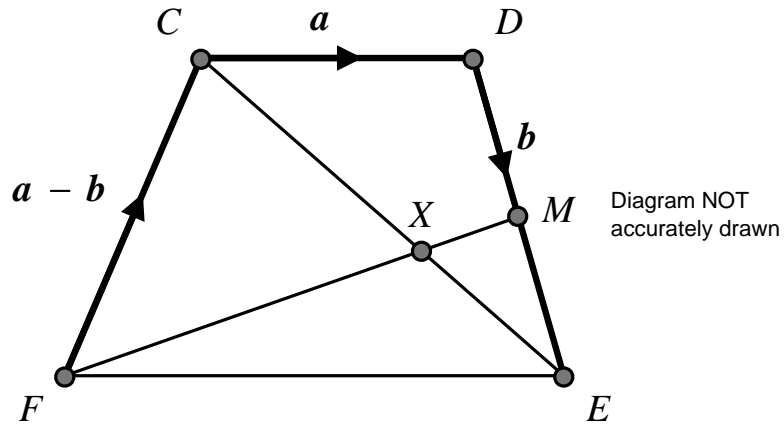
CDEF is a quadrilateral with $\overrightarrow{CD} = a$ $\overrightarrow{DE} = b$ and $\overrightarrow{FC} = a - b$

(**i**)   Express $\overrightarrow{CE}$ in terms of *a* and *b*

[**1 mark**]

(**ii**)   Prove that $\overrightarrow{FE}$ is parallel to $\overrightarrow{CD}$

[**1 mark**]

M is the midpoint of *DE*

(**iii**)   Express $\overrightarrow{FM}$ in terms of *a* and *b*

[**1 mark**]

X is the point on *FM* such that *FX* : *XM* = 4 : 1
(**iv**)   Prove that *C*, *X* and *E* lie on the same straight line

[**3 marks**]

Question 8
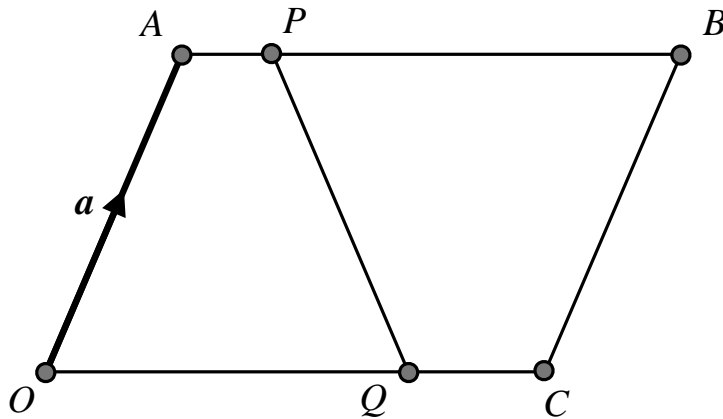
GCSE Examination Question from May 2014, Paper 3HR, Q21

Diagram NOT accurately drawn

OABC is a parallelogram with $\overrightarrow{OA} = a$ and $\overrightarrow{OC} = c$

P is the point on *AB* such that $AP : PB = 1 : 3$

Q is the point on *OC* such that $OQ : QC = 2 : 1$

Find, in terms of *a* and *c*, $\overrightarrow{PQ}$

Give your answer in its simplest form.

[4 marks]