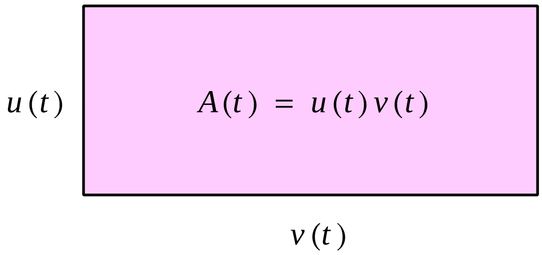
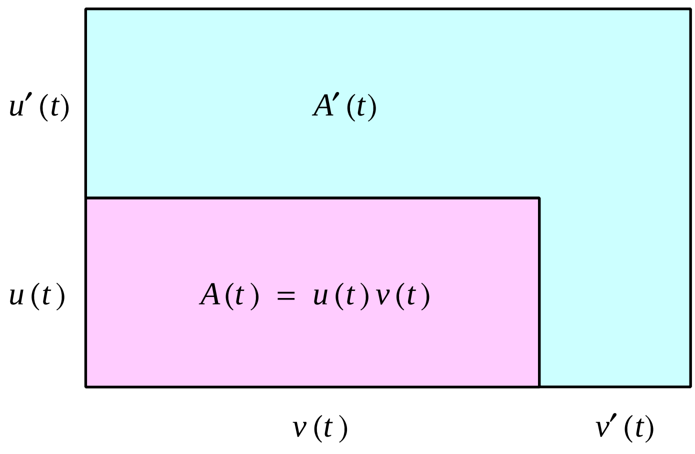
4.1 An Intuitive Proof for The Product Rule

At this stage of the A-Level course, the mathematics that is needed to rigorously prove The Product Rule has yet to be developed. However, why The Product Rule is of the form it is can be intuitively visualised.

The starting point is to imagine that a rectangle at some time, t, has a height given by a function $u(t)$ and a width given by a function $v(t)$.
The area of the rectangle, $A(t)$, is then given by $A(t) = u(t)\, v(t)$.



Some time later (for example 1 second), the size of the rectangle will have changed. Derivatives can be thought of as "rates of change". Let the rate of change in the height be given by $u'(t)$ and the rate of change in the width be $v'(t)$.

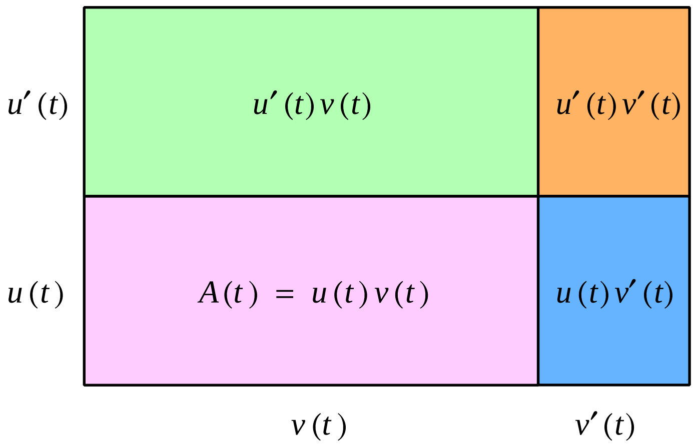


The key question is what is the rate of change of the area of the rectangle, $A'(t)$?

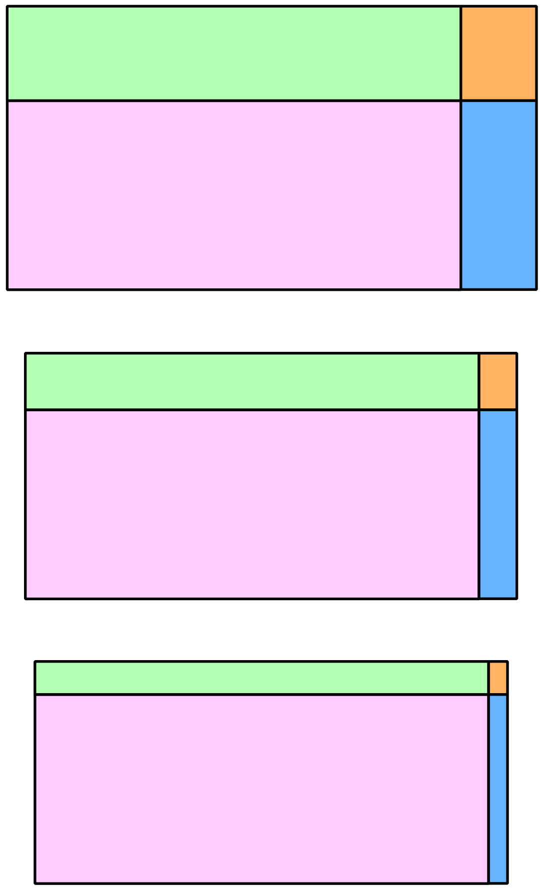
In the diagram above, this is the region shaded blue. It can be divided up into three rectangles one with area $u(t)\, v'(t)$, one with area $u'(t)\, v'(t)$ and one with area $u'(t)\, v(t)$.

In other words,

$$A'(t) = u(t)\, v'(t) + u'(t)\, v'(t) + u'(t)\, v(t)$$



The final step is to let the time interval tend towards zero; to get the "instantaneous rate of change".



The interesting observation is that the contribution of the orange rectangle, representing $u'(t)\ v'(t)$ is becoming increasingly insignificant in comparison to the contributions from the blue rectangle, representing $u(t)\ v'(t)$, and the green rectangle, representing $u'(t)\ v(t)$.

Thus $\quad A'(t) = u(t)\ v'(t) + u'(t)\ v'(t) + u'(t)\ v(t)$

becomes $\quad A'(t) = u(t)\ v'(t) + u'(t)\ v(t)$

which is The Product Rule $\square$

4.2 Exercise

Marks Available : 40

Question 1

Given that, $f(x) = \left(2x^4 - 3x + 6\right)^5$ use The Chain Rule to show $f'(1) = 5^6$

[3 marks]

Question 2

Given that, $g(x) = \sqrt{5x^2 - 4}$ use The Chain Rule to show $g'(2) = 2.5$

[3 marks]

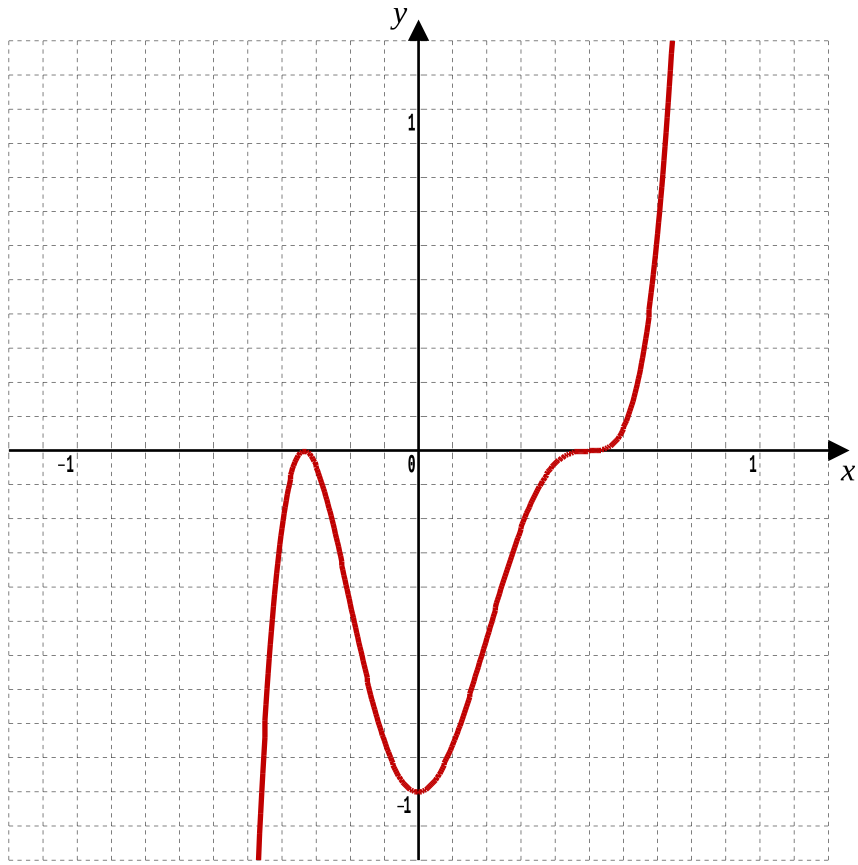
Question 3

A curve C has equation $y = (5 - 2x)^3$

Find the tangent to the curve at the point P with x-coordinate 2

[4 marks]

Question 4

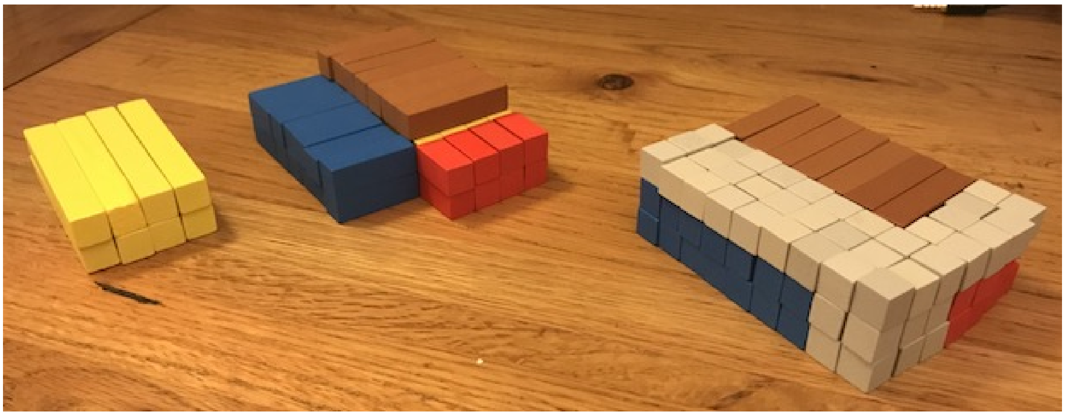


Use The Product Rule to find the stationary points of the curve with equation,

$$y = (2x - 1)^3 (3x + 1)^2$$

[6 marks]

Question 5



The yellow cuboid to the left of the photograph measures $7 \times 2 \times 4$. It's extended to measure $(7 + 2) \times (2 + 1) \times (4 + 3)$ as shown to the right of the photograph.

(i) Calculate the increase in the volume of the cuboid.

[2 marks]

(ii) Calculate the percentage of the extra volume that is given by the red, brown and blue parts.

[4 marks]

(iii) If the original $7 \times 2 \times 4$ cuboid had been extended to measure $(7 + 0.2) \times (2 + 0.1) \times (4 + 0.3)$ what percentage of the extra volume would be given by the new red, brown and blue parts.

[5 marks]

(iv) Explain the significance of the answers to parts (ii) and (iii) in relation to the circumstances in which the calculation of the extra volume can be approximated by just the red, brown and blue cuboids, without a significant loss of answer accuracy.

[2 marks]

(v) A function $V(x)$ is the product of three other functions,

$$V(x) = u(x)\ v(x)\ w(x)$$

Keeping in mind the earlier parts of this question, and the proof of The Product Rule at the start of this lesson, make an inspired guess as to what the formula for the derivative of $V(x)$ is likely to be.

[2 marks]

Question 6

Show that if $f(x) = x^2\sqrt{3x - 1}$ then $f'(x) = \dfrac{x(15x - 4)}{2\sqrt{3x - 1}}$

[5 marks]

Question 7

The curve C has equation $y = \dfrac{1}{(1 - 2x)^2}$ where $x \neq \dfrac{1}{2}$

The point A on C has x-coordinate $\dfrac{1}{4}$

Find an equation of the tangent to C at A in the form $y = mx + c$

[4 marks]