Lesson 7

A-Level Pure Mathematics : Year 2
Differentiation III

7.1 The Logarithm Function

The inverse of the exponential function is the natural logarithm function.
It has a surprising derivative.

The Derivative of $y = \ln x$

$$\text{If } y = \ln x \text{ then } \frac{dy}{dx} = \frac{1}{x}$$

Proof :

$$y = \ln x$$

exponentiate both sides

$$e^y = e^{\ln x}$$

$$e^y = x$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dx}{dy} = x$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \square$$

Notes on the Proof

The proof relied upon knowing that the exponential function is its own derivative.
Proving that from first principles is beyond A-Level as it needs the result that,

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln(a)$$

"Easy" proofs that the derivative of e^x is e^x need the result that the derivative of $\ln x$ is $\frac{1}{x}$ but proving that from first principles is not any easier.

7.2 Differentiating Logarithm Functions

The Product Rule and The Quotient Rule can be applied to situations where the natural logarithm function is involved. So too can The Chain Rule, as follows,

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The Chain Rule for $y = ln(f(x))$

$$\text{If } y = ln(f(x)) \text{ then } \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

7.3 Examples

Differentiate each of the following,

(i) $y = 5\,ln\left(x^2 + e^x\right)$ (Chain Rule Example)

(ii) $y = x^3\,ln\,x^2$ (Product Rule Example)

(iii) $y = \frac{ln\,x}{2x}$ (Quotient Rule Example)

Teaching Video : http://www.NumberWonder.co.uk/v9028/7.mp4

Watch the video and then write out the solutions here

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[3, 3, 3 marks]

7.4 Exercise

Marks Available : 50

Question 1

Differentiate each of the following with respect to x;

(i) $y = 3 \ln x$

(ii) $y = 4 \ln x^3$

[2, 2 marks]

(iii) $y = \ln(5x)$

(iv) $y = 6 \ln(7x)$

[2, 2 marks]

(v) $y = \ln\left(4x^3 + 3\right)$

(vi) $y = (\ln x)^3$

[2, 2 marks]

(vii) $y = (\ln x)^{-4}$

(viii) $y = \dfrac{4}{\ln x}$

[2, 2 marks]

Question 2

Use The Product Rule to differentiate the following with respect to x,

$$y = x \ln x^7$$

[3 marks]

Question 3

Use The Quotient Rule to differentiate the following with respect to x;

$$y = \frac{x}{\ln x}$$

[3 marks]

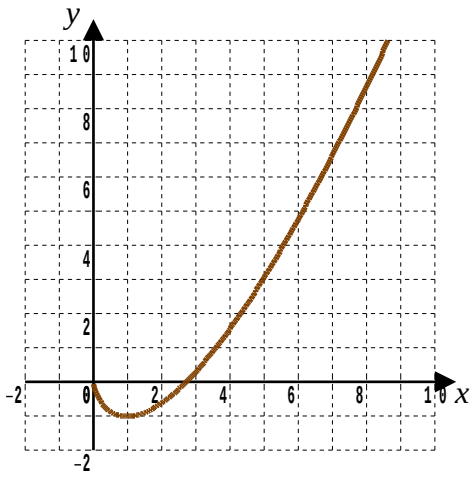
Question 4

Use The Quotient Rule to differentiate the following with respect to x,

$$y = \frac{\ln x^3}{x^3}$$

[3 marks]

Question 5



The graph is of the function

$$f(x) = x(\ln(x) - 1) \qquad x > 0$$

(i) Find the exact coordinates of the point, *A*, where the gradient is 1

[3 marks]

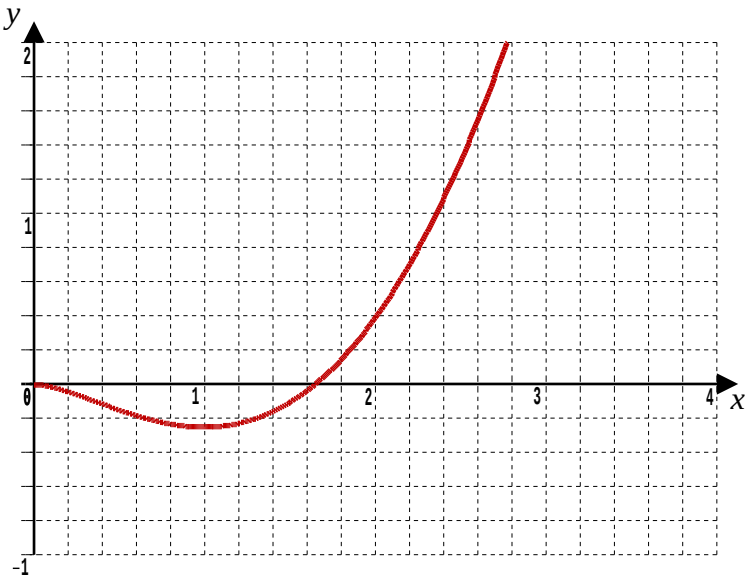
(ii) Find the exact coordinates of the turning point, *B*, and prove that it is a minimum.

[3 marks]

(iii) Find the exact coordinates of where the tangent to the curve at point *A*, meets the tangent to the curve at point *B*

[3 marks]

Question 6



The graph is of the function

$$g(x) = \frac{1}{4}x^2(2\ln(x) - 1) \qquad x > 0$$

(i) Find the exact value of $g'(e)$

[4 marks]

(ii) Find the exact coordinates of the turning point and prove that it is a minimum.

[3 marks]

Question 7

Find the exact equation of the tangent to the following curve where $x = 1$

$$y = e^x \ln x$$

[3 marks]

Question 8

A curve has equation $y = \dfrac{\ln x}{x^2}$

(i) Find an expression for $\dfrac{dy}{dx}$

[3 marks]

(ii) Find the equation of the tangent to the curve when $x = 1$

[3 marks]