Lesson 2

A-Level Pure Mathematics : Year 2
Trigonometric Identities

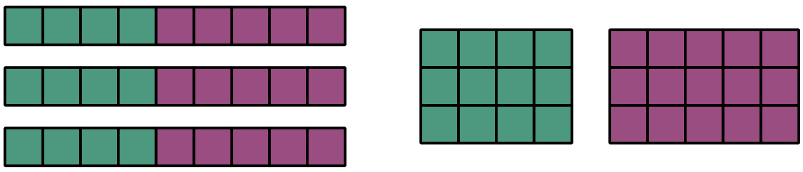
2.1 Law Breaker

Many brackets in mathematics can be expanded by using the Distributive Law. It states that multiplying a number by a group of numbers added together is the same as doing each multiplication separately.

The Distributive Law

$$a \times (b + c) = a \times b + a \times c$$

For example,



$$3 \times (4 + 5) \quad = \quad 3 \times 4 \quad + \quad 3 \times 5$$

To show that the distributive law does not apply to trigonometric functions, let $A = 30°$ and $B = 60°$ such that $A + B = 90°$ in which case,

$$\sin (A + B) \neq \sin A + \sin B$$

because

$$\sin 30 = \frac{1}{2}, \ \sin 60° = \frac{\sqrt{3}}{2}, \ \sin 90° = 1 \text{ and } \frac{1}{2} + \frac{\sqrt{3}}{2} \neq 1$$

The correct way that the brackets do expand is as follows,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

A visual proof of this is given over the page.

This formula for expanding the brackets of $\sin (A + B)$ is one of six formulae that are collectively known as The Addition Formulae,

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The Addition Formulae

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

2.2 Visual Proof

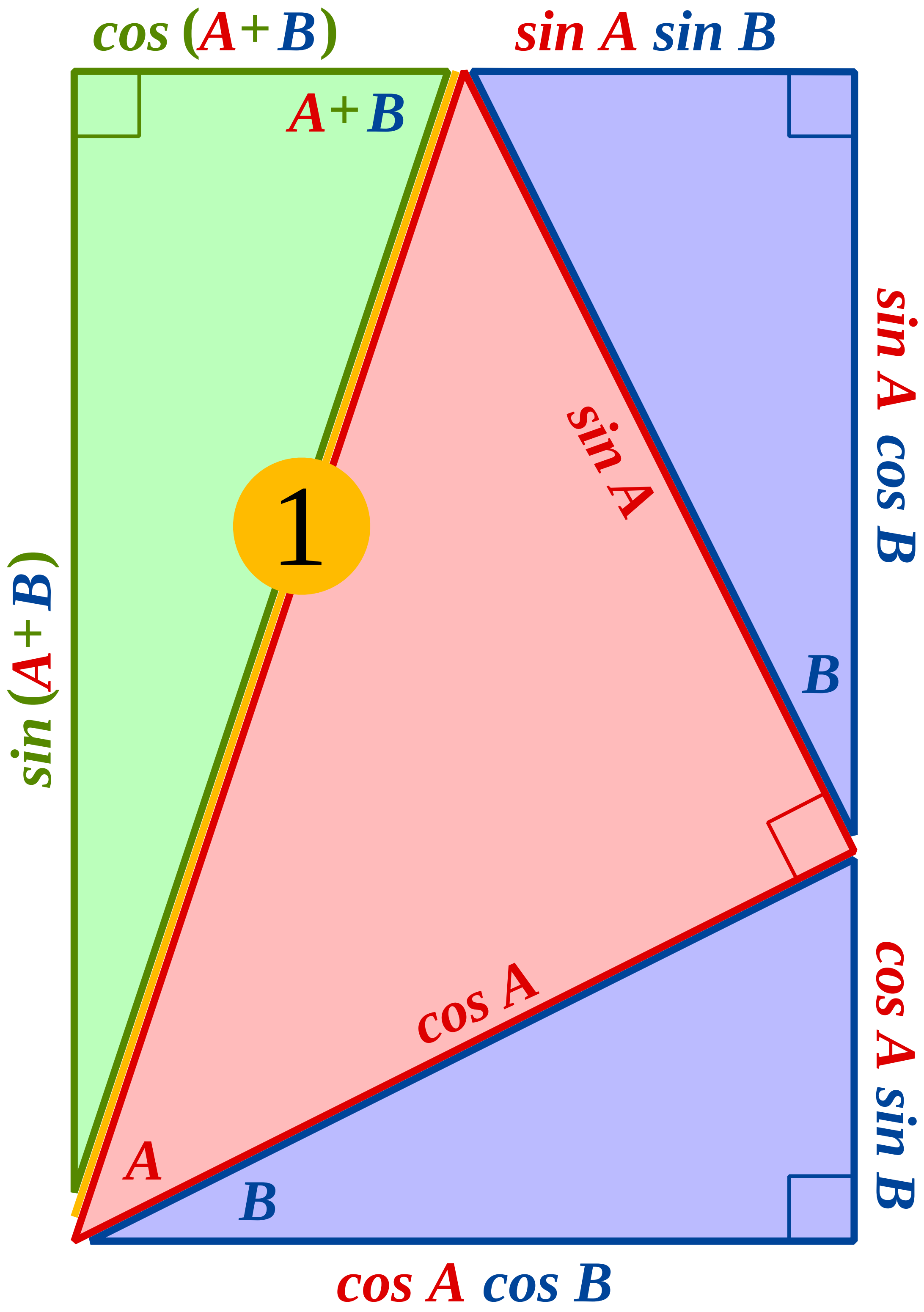
The diagram below is a “proof without words” that,

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

and

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

It's a great diagram to pin up on a wall and think about.



2.3 Proving Your Identity

A trigonometric identity is a relationship between trigonometric functions that is essentially true for all values of angle. There may be a necessary technical restriction, on the domain to avoid, for example, a division by zero.

All six of the Addition Formulae are trigonometric identities and so too are,

$$cos^2\,\theta + sin^2\,\theta = 1, \qquad \theta \in \mathbb{R}$$

and

$$tan\,\theta = \frac{sin\,\theta}{cos\,\theta}, \qquad \theta \in \mathbb{R},\ \ \theta \neq 90°\,(1 + 2n) \text{ for } n \in \mathbb{Z}$$

Faced with an unfamiliar trigonometric identity, the immediate task is often to prove that it is true.

2.4 Example

Prove that $cos\,(\,90° - \theta\,) = sin\,\theta$

Teaching Video : http://www.NumberWonder.co.uk/v9040/2.mp4

Watch the Teaching Video which shows the correct technical manner in which to write out the proof.

☞

[3 marks]

2.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 38

Question 1
Prove the following identity.
Be sure to start "LHS = " and conclude with "= RHS" along with the special symbol used to indicate the conclusion of the proof.

$$sin\ (90° + \theta) = cos\ \theta$$

[3 marks]

Question 2
Prove the following identity.
Be sure to start "LHS = " and conclude with "= RHS" along with the special symbol used to indicate the conclusion of the proof.

$$sin\ (180° - \theta) = sin\ \theta$$

[3 marks]

Question 3
Prove that

$$sin(A + B) + sin(A - B) = 2\, sin\, A\, cos\, B$$

[3 marks]

Question 4
Prove this identity;

$$cos(A - B) - cos(A + B) = 2\, sin\, A\, sin\, B$$

[3 marks]

Question 5

Prove,

$$\frac{sin\,(A + B)}{cos\,A\,cos\,B} = tan\,A + tan\,B, \qquad A,\ B \neq 90°\,(1 + 2n),\ n \in \mathbb{Z}$$

[3 marks]

Question 6

By letting $B = A$ in the addition formula $sin\ (A + B) = sin\,A\ cos\,B + cos\,A\,sin\,B$ prove that;

$$sin\,2A = 2\,sin\,A\,cos\,A$$

[3 marks]

Question 7

Using a similar technique to that of question 6, prove that;

$$\cos 2A = \cos^2 A - \sin^2 A$$

[3 marks]

Question 8

Prove that $\quad \tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A} \qquad A \neq 45°(1 + 2n), \ n \in \mathbb{Z}$

[3 marks]

Question 9
The three formulae, proved in questions 6, 7 and 8 are called “The Double Angle Formulae”. They are important and useful, and need to be memorised for recall at any moment.

Write out the three double angle formulae.
You may claim a bonus mark if you can do it from memory !

[3 marks]
[1 Bonus]

Question 10
Explain carefully, giving an example of each, the difference between a trigonometric identity and a trigonometric equation.

[4 marks]

Question 11

Solve the following trigonometric equation for x between 0° and 360°

$$2\,sin\,x \;=\; cos\;(\;x\;+\;60°\;)$$

[6 marks]

The Visual Proof of *sin*(*A*+*B*) seems to have it's origins in a diagram from *Inspired By Maths*