Lesson 3

A-Level Pure Mathematics, Year 2

Functions II

3.1 Transformation Of Graphs (Part 1)

All mathematicians have a knowledge of the shape of many basic graphs. Attached to each of these graphs is the algebraic description of that graph.

It's useful to be able to start with one of these basic graphs, transform it, and then write down the algebra that describes the new graph. Alternatively, given a new piece of algebra, spot if it's the transformation of a basic graph.

The transformations to be considered include

- ◇ Translations
- ◇ Reflections
- ◇ Stretches parallel to the *x*-axis
- ◇ Stretches parallel to the *y*-axis

Note : If stretches parallel to the *x*-axis and *y*-axis of equal scale factor are applied to a graph then the result is an enlargement.

3.2 An Example Utilising Existing Knowledge

A circle, centre (0, 0) and radius *r* has equation;

$$x^2 + y^2 = r^2$$

A circle, centre (4, 7) and radius *r* has equation;

$$(x - 4)^2 + (y - 7)^2 = r^2$$

The general principal is this;

◇ Replacing all occurrences of x with $(x - a)$ in ***any*** equation translates the graph of the equation $\begin{pmatrix} a \\ 0 \end{pmatrix}$

◇ Replacing all occurrences of y with $(y - b)$ in ***any*** equation translates the graph of the equation $\begin{pmatrix} 0 \\ b \end{pmatrix}$

For the example of the circle with centre (a, b) and radius r, the equation is thus;

$$(x - a)^2 + (y - b)^2 = r^2$$

which was used in the Year 1 part of the A-level course but without viewing it from this broader perspective.

Note : A circle is not a function because a vertical line can be found that cuts its graph in more than one point but these results apply ***all*** equations.

3.3 An Inverse Proportion Example

Consider this inverse proportion function;

$$f(x) = \frac{12}{x} \qquad x \in \mathbb{R},\ x \neq 0, \qquad f(x) \in \mathbb{R},\ f(x) \neq 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y - 3 = \frac{12}{x + 5}$$

[2 marks]

(c) Consider the function given by;

$$g(x) = \frac{12}{x + 5} + 3$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

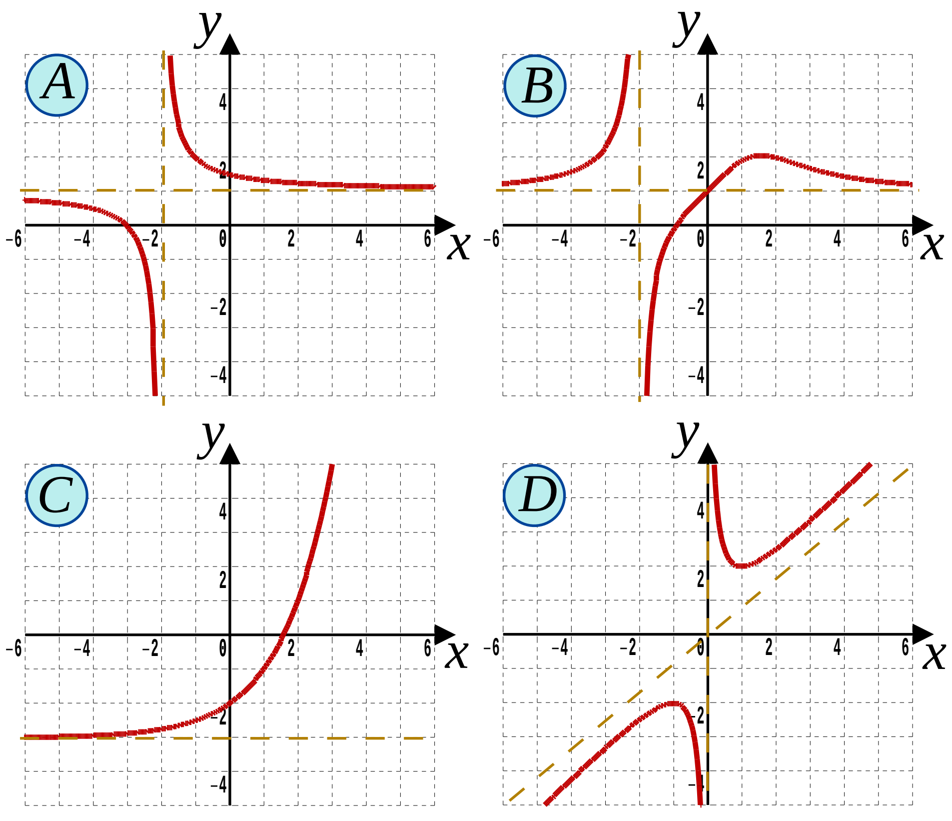
[1 mark]

3.4 What is an Asymptote ?

In the inverse proportion example the vertical and horizontal asymptotes were particularly useful when it came to translating the graph.
A top tip when a graph with asymptotes has to be translated is to use the asymptotes.
However, what exactly is an asymptote ?

Definition : Asymptote
An asymptote of a curve is a straight broken line such that the distance between the curve and the line approaches zero as one, or both, of the *x* or *y* coordinates tends to plus infinity or minus infinity.

3.4.1 Examples



$A \,:\, y - 1 = \dfrac{1}{x+2}$: Vertical asymptote $x = -2$, Horizontal asymptote $y = 1$

$B \,:\, y - 1 = \dfrac{8x}{x^3+8}$: Vertical asymptote $x = -2$, Horizontal asymptote $y = 1$

$C \,:\, y + 3 = 2^x$: No vertical asymptote , Horizontal asymptote $y = -3$

$D \,:\, y = x + \dfrac{1}{x}$: Vertical asymptote $x = 0$, Oblique asymptote $y = x$

3.5 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 50

Question 1

Consider the parabola;

$$f(x) = x^2 \qquad x \in \mathbb{R}$$

(a) Sketch the graph of $y = f(x)$

[1 mark]

(b) Sketch the graph of the equation,

$$y + 7 = (x + 4)^2$$

[2 marks]

(c) Consider now the function given by;

$$g(x) = (x + 4)^2 - 7$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 2

Consider the positive square root curve;

$$f(x) = \sqrt{x} \qquad x \in \mathbb{R},\ x \geqslant 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y + 1 = \sqrt{x + 2}$$

[2 marks]

(c) Consider now the function given by;

$$g(x) = \sqrt{x + 2} - 1$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 3

Consider the trigonometric curve;

$$f(x) = \sin x \qquad x \in \mathbb{R}, \qquad -90° \leqslant x \leqslant 360°$$

(i) Sketch the graph of $f(x)$ over the specified domain.

[1 mark]

(ii) Sketch the graph of the function,

$$g(x) = \sin(x + 90°) \qquad x \in \mathbb{R}, \qquad -90° \leqslant x \leqslant 360°$$

[2 marks]

(iii) A interesting relationship between the sine and cosine functions should be suggested by your part (b) sketch. State what this might be.

[1 mark]

Question 4

Sketch the graph of $y = |2x + 3|$ clearly indicating all axis contact points.

[3 marks]

Question 5

Consider “The Loch Ness Monster Function” given by;

$$f\,(x) \;=\; 1 \;+\; \sqrt{cos\, x}$$

Clearly, care is needed over which values of x are in the domain of this function, as attempting to take negative square roots is to be avoided.

(i) Sketch the cosine function over an interval that includes $0° \leqslant x \leqslant 720°$.
Along the x-axis mark: 0°, 90°, 180°, 270°, 360°, 450°, 540°, 630°, 720°.

[2 marks]

(ii) Using your part (a) sketch, and working over the interval $0° \leqslant x \leqslant 720°$, state two intervals for which the cosine function is negative.

[2 marks]

(iii) With parts (a) and (b) in mind, sketch the function $f\,(x)$.
Along the x-axis mark: 0°, 90°, 180°, 270°, 360°, 450°, 540°, 630°, 720°.

[2 marks]

(iv) Why is this function known as “The Lock Ness Monster Function” ?

[1 mark]

Question 6
Solve the equation

$$|2x - 4| = x$$

[3 marks]

Question 7
Sketch the function

$$f(x) = |x^3 - x|$$

[3 marks]

Question 8
Solve the equation

$$|3x + 2| = 2 - x$$

[3 marks]

Question 9

Consider the function

$$f(x) = -\sqrt{x} \qquad x \in \mathbb{R},\ x > 0$$

(a) Sketch the graph of $y = f(x)$

[2 marks]

(b) Sketch the graph of the equation,

$$y - 4 = -\sqrt{x + 7}$$

[3 marks]

(c) Consider now the function given by;

$$g(x) = 4 - \sqrt{x + 7}$$

(i) State the domain of $g(x)$

[1 mark]

(ii) State the range of $g(x)$

[1 mark]

Question 10

A-Level Examination Question from June 2019, Paper 1, Q5 (Edexcel)

$$f(x) = 2x^2 + 4x + 9, \qquad x \in \mathbb{R}$$

(a) Write $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are integers.

[3 marks]

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

[3 marks]

(c) **(i)** Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where,

$$g(x) = 2(x - 2)^2 + 4x - 3, \qquad x \in \mathbb{R}$$

[2 marks]

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \qquad x \in \mathbb{R}$$

[2 marks]

Question 11

$$f(x) = (x - 3)^2 + 7, \qquad x \in \mathbb{R}$$

(i) Sketch $f(x)$ paying particular attention to the vertex of the parabola.

[2 marks]

(ii) State the range of $f(x)$

[1 mark]

(iii) Solve

$$f(x) = 2x$$

[3 marks]

(iv) With the assistance of your part (c) answer, describe the relationship between $f(x)$ and the line $y = 2x$

[3 marks]