Lesson 7

A-Level Pure Mathematics, Year 2

Functions II

7.1 Inverse Functions

Inverse functions have been met before at GCSE level (See Functions I).
Here is a question by way of recalling some of that previous knowledge;

(i) Write down the function described by the following diagram,

$$x \to \boxed{\times 5} \to \boxed{+\ 3} \to \boxed{\div\ 4} \to f(x)$$

[2 marks]

(ii) Write down the inverse function $f^{-1}(x)$

[2 marks]

7.2 A More Advanced Example

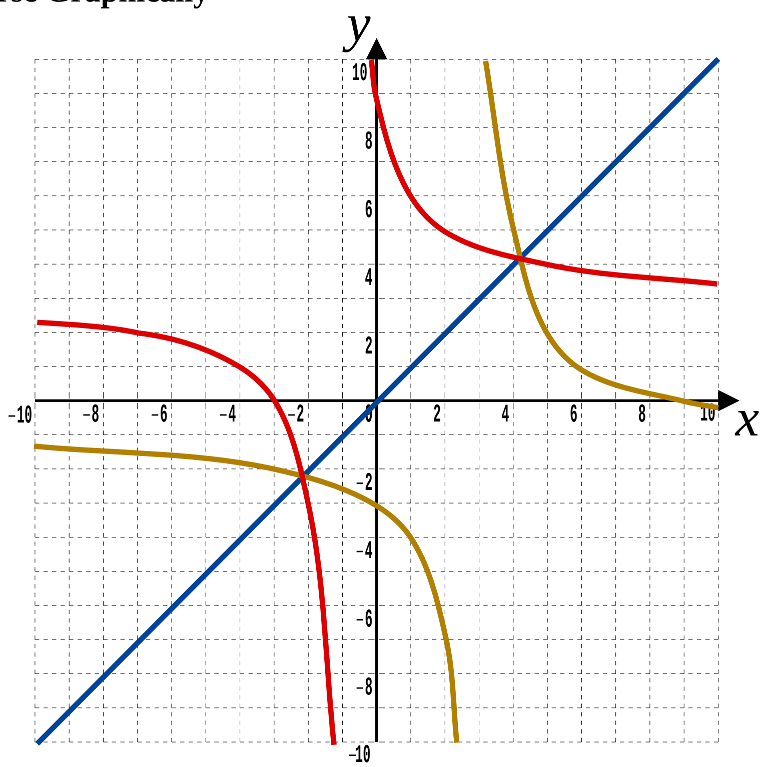
For the one-to-one function,

$$f(x) = \frac{6}{x+1} + 3$$

find $f^{-1}(x)$

[3 marks]

7.3 Inverse Graphically



Red: $f(x)$ Gold: $f^{-1}(x)$

(a) For the function, $f(x) = \dfrac{6}{x+1} + 3$

(i) Write down the domain:

[1 mark]

(ii) Write down the range:

[1 mark]

(b) For the inverse function,

$$f^{-1}(x) = \frac{6}{x-3} - 1$$

(i) Write down the domain:

[1 mark]

(ii) Write down the range:

[1 mark]

General Observations

- Geometrically, the inverse of a function is a reflection in the line $y = x$
- The domain of the function became the range of the inverse
- The range of the function became the domain of the inverse

Graphing the Function

The function, $f(x) = \dfrac{6}{x+1} + 3$ is better thought of as, $(y-3) = \dfrac{6}{(x+1)}$,

because the relationship to the graph of inverse proportion, $y = \dfrac{1}{x}$, is then obvious.

Focussing on the two asymptotes, one along the x-axis, the other along the y, it's then deduced that replacing x with $(x+1)$ will translate the y-axis asymptote to $x = -1$ and that replacing the y with $(y-3)$ will translate the x-axis asymptote to $y = 3$

Keeping in mind the fact that, for example,

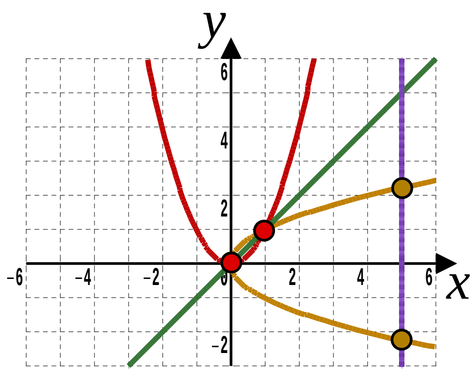
$y = \dfrac{1}{x}$, $\quad y = \dfrac{6}{x}$, $\quad y = \dfrac{12}{x}$, and indeed $y = \dfrac{k}{x}$ where k is a constant,

all have asymptotes along the x and y axis explains why the 6 has no effect on where the asymptotes of the transformed graph lie.

7.4 Existence of an Inverse

To have an inverse, a function must be one-to-one

The graph below shows (in red) the many-to-one function $f(x) = x^2$, $x \in \mathbb{R}$ and also (in gold) its reflection in the line $y = x$. However the reflection is not a function as a vertical line can be drawn that cuts it more than once.



If the inverse of a many-to-one function is sought, it must be broken up into pieces each of which is one-to-one and then, separately, the inverse of each piece found.

So, for example, to find the inverse of the many-to-one function $f(x) = x^2$ requires breaking it into two separate pieces,

$$g(x) = x^2, \qquad x \geqslant 0$$

$$h(x) = x^2 \qquad x < 0$$

and separately finding the inverse of each piece;

$$g^{-1}(x) = \sqrt{x} \qquad x \geqslant 0$$

$$h^{-1}(x) = -\sqrt{x} \quad x \geqslant 0$$

7.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available: 50

Question 1
Consider the following diagram,

$$x \rightarrow \boxed{+\ 2} \rightarrow \boxed{\div\ 3} \rightarrow \boxed{\times\ 8} \rightarrow f(x)$$

(a) Write down the function described by the diagram

[1 mark]

(b) Find the inverse function in two different ways,

(i) By looking at the diagram and working through it 'backwards'
(As in section 7.1)

[2 marks]

(ii) By algebraic manipulation.
(As in section 7.2)

[2 marks]

Question 2
Using a method of your choice, find the inverse of the function

$$g(x) = 8x + 3$$

[2 marks]

Question 3
Using a method of your choice, find the inverse of the function

$$h(x) = \frac{x}{5} - 4$$

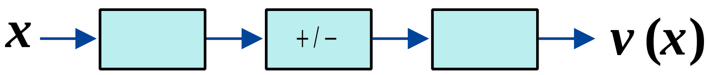
[2 marks]

Question 4

The function $v(x)$ is described by,

$$v(x) = 11 - 6x$$

A flow diagram for the function, shown below, includes the "change sign" instruction, which reverses the sign of whatever is fed into it.



(a) Complete the flow diagram by filling in the two empty boxes with appropriate instructions.

[2 marks]

(b) Find the inverse function in two different ways,

(i) By looking at the diagram and working through it 'backwards' (As in section 7.1)

[2 marks]

(ii) By algebraic manipulation (As in section 7.2)

[2 marks]

Question 5

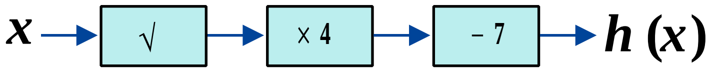
Using a method of your choice, find the inverse of the function

$$u(x) = \frac{3x}{5} + 4$$

[3 marks]

Question 6

Consider the following diagram,



(a) **(i)** Write down the function described by the diagram

[1 mark]

(ii) What is the domain of the function ?

[1 mark]

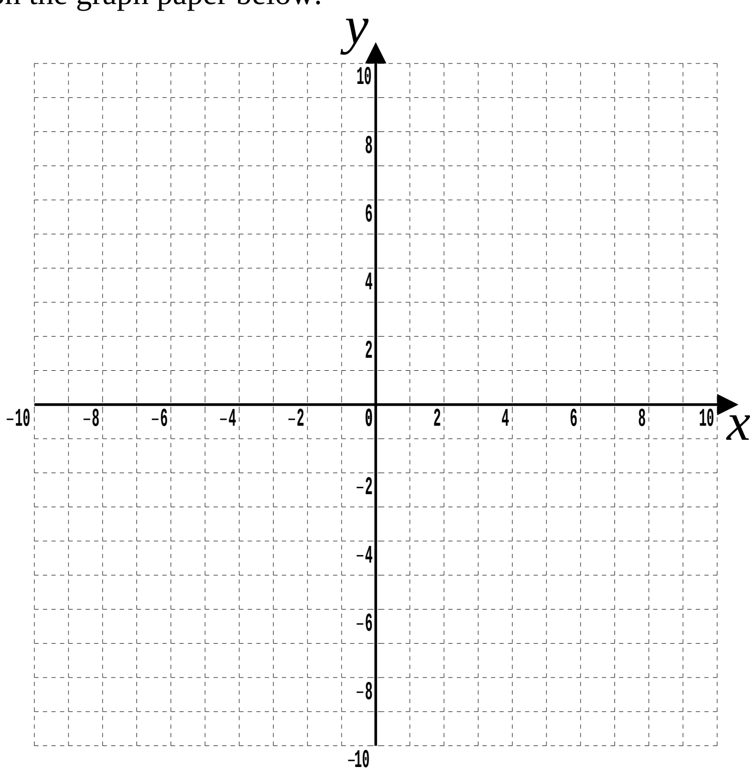
(b) Find the inverse function.

[3 marks]

(c) What is the domain of the inverse function ?

[1 mark]

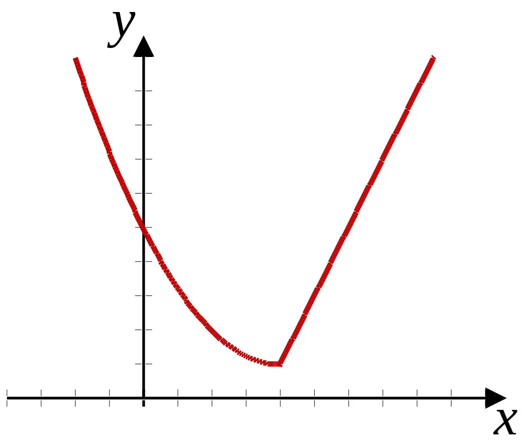
(d) Provide a fairly accurate plot of both the original function and the inverse on the graph paper below.



[4 marks]

Question 7

A-Level Examination Question from June 2019, Paper 2, Q6 (Edexcel)



The diagram shows a sketch of $y = g(x)$, where,

$$g(x) = \begin{cases} (x - 2)^2 + 1 & x \leqslant 2 \\ 4x - 7 & x > 2 \end{cases}$$

(a) Find the value of $gg(0)$

[2 marks]

(b) Find all values of x for which $g(x) > 28$

[4 marks]

The function h is defined by, $h(x) = (x - 2)^2 + 1, \quad x \leqslant 2$

(c) Explain why h has an inverse but g does not.

[1 mark]

(d) Solve the equation $h^{-1}(x) = -\dfrac{1}{2}$

[3 marks]

Question 8

A-Level Examination Question from June 2018, Paper C34, Q5 (Edexcel)

(i) The functions *f* and *g* are defined by

$$f \ : \ x \ \rightarrow \ e^{2x} \ - \ 5, \qquad x \ \in \ \mathbb{R}$$

$$g \ : \ x \ \rightarrow \ ln(\,3x \ - \ 1\,), \qquad x \ \in \ \mathbb{R},\ x \ > \ \frac{1}{3}$$

(a) Find f^{-1} and state its domain.

[3 marks]

(b) Find $fg(3)$, giving your answer in its simplest form.

[2 marks]

(ii) **(a)** Sketch the graph with equation $y \ = \ |4x \ - \ a|$ where *a* is a positive constant. State the coordinates of each point where the graph cuts or meets the coordinate axes.

[2 marks]

Given that, $|4x \ - \ a| \ = \ 9a$, where *a* is a positive constant,

(b) find the possible values of,

$$|x - 6a| + 3|x|$$

giving your answers, in terms of a, in their simplest form.

[5 marks]