Lesson 8

A-Level Pure Mathematics, Year 2

Functions II

8.1 The Möbius Function

A Möbius function is of the form,

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R},\ ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c and d are constants.

Finding the inverse of this one-to-one function requires a method for dealing with the fact that the the x appears in two different places, in both numerator and denominator.

Example

(i) Find the inverse of,

$$h(x) = \frac{5x + 2}{3x + 4} \qquad x \in \mathbb{R}, \quad x \neq -\frac{4}{3}$$

[4 marks]

(ii) State the domain of the inverse function.

[1 mark]

8.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available: 54

Question 1

(i) Find the inverse of, $m(x) = \dfrac{7x + 1}{2x + 3} \qquad x \in \mathbb{R}, \quad x \neq -\dfrac{3}{2}$

[4 marks]

(ii) State the domain of the inverse function

[1 mark]

Question 2

Prove that the inverse of

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R},\ ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c and d are constants, is

$$f^{-1}(x) = \frac{-dx + b}{cx - a} \qquad x \in \mathbb{R}, \quad x \neq \frac{a}{c}$$

[6 marks]

Question 3

(i) Find the inverse of, $s(x) = \dfrac{2x - 1}{x - 2} \qquad x \neq 2$

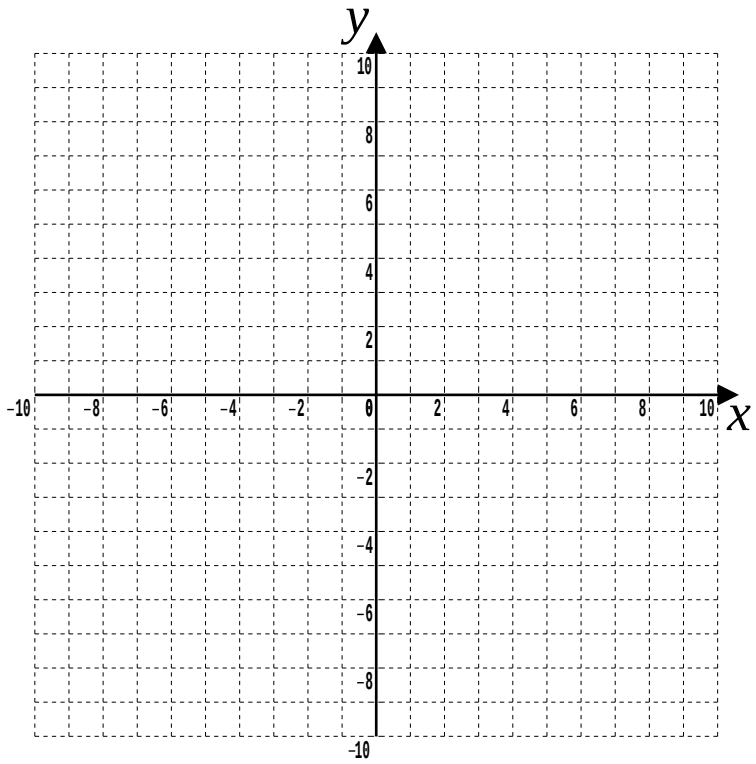
by using the result proven in question 2

[2 marks]

(ii) This is an example of a "self inverse" function.
Explain what this means.

[1 mark]

(iii) Plot an accurate graph of $s(x)$ and also the line $y = x$
Mark on the horizontal and the vertical asymptotes of $s(x)$



[3 marks]

(iv) What property do the graphs of all self-inverse functions have ?

[2 marks]

Question 4

(i) Write down the equations of the two asymptotes of the function,

$$f(x) = \frac{2}{x} \qquad x \in \mathbb{R}, \quad x \neq 0$$

[2 marks]

(ii) With your part (i) answer in mind, write down the equations of the two asymptotes of the graph of the function,

$$h(x) = \frac{2}{x + 1} + 3$$

[2 marks]

(iii) What number must be excluded from the domain of $h(x)$?

[1 mark]

(iv) Sketch the graph of $h(x)$
(No need for an accurate graph, just the essential shape and the asymptotes)

[3 marks]

(v) Show that $h(x)$ is a Möbius transformation by writing it in the form,

$$f(x) = \frac{ax + b}{cx + d} \qquad x \in \mathbb{R},\ ad - bc \neq 0, \quad x \neq -\frac{d}{c}$$

where a, b, c, and d are constants the values of which you have determined.

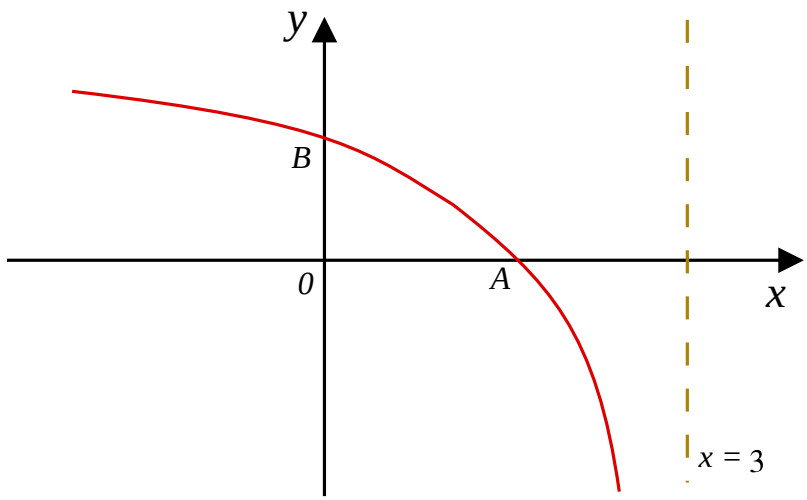
[3 marks]

(vi) Find $h^{-1}(x)$, stating its domain.

[4 marks]

Question 5

A-Level Examination Question from November 2018, Paper C34, Q10 (Edexcel)



The sketch is of the graph with equation $y = g(x)$, where,

$$g(x) = \frac{3x - 4}{x - 3}, \quad x \in \mathbb{R}, \ x < 3$$

The graph cuts the x-axis at the point A and the y-axis at the point B, as shown.

(a) State the range of g

[1 mark]

(b) State the coordinates of,

(i) point A

[1 mark]

(ii) point B

[1 mark]

(c) Find $gg(x)$ in its simplest form.

[3 marks]

(d) Sketch the graph with equation $y = |g(x)|$

On your sketch, show the coordinates of each point at which the graph meets or cuts the axes and state the equation of each asymptote.

[3 marks]

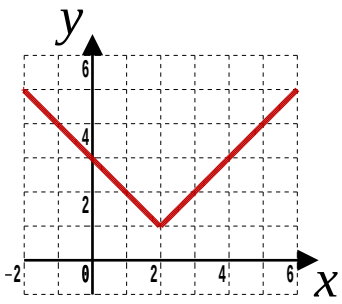
(e) Find the exact solution of the equation $|g(x)| = 8$

[3 marks]

Question 6

$$f(x) = |x - 2| + 1,\ x \in \mathbb{R}$$

The function $f(x)$ has a "critical value" when $|x - 2| = 0$ in the sense that there will be a "drastic change" in the smoothness of the graph at $x = 2$



This function can also be written in hybrid form as, $f(x) = \begin{cases} x - 1 & x \geqslant 2 \\ -x + 3 & x < 2 \end{cases}$

Consider now the following function,

$$g(x) = |x + 1| + |2x + 1| - |x - 2|$$

Determine the critical values of $g(x)$, sketch $g(x)$ and express $g(x)$ in hybrid form.

[8 marks]