3.1 Differentiation from First Principles

In Lessons 1 and 2 good use was made of this rule:

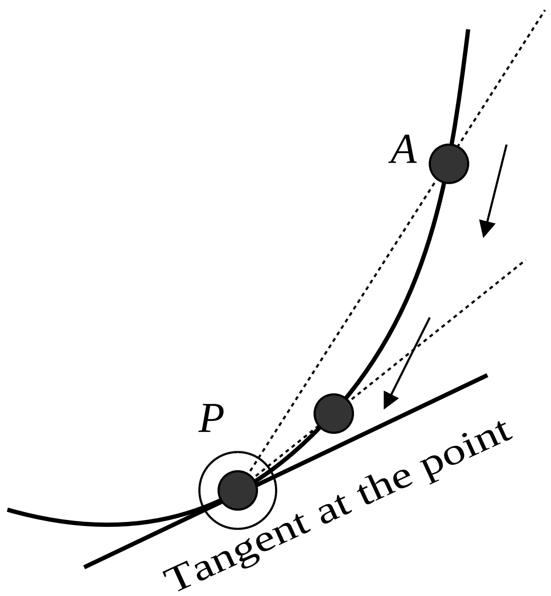
If $y = x^n$ then $\dfrac{dy}{dx} = n\,x^{n-1}$ for any constant n

---

It is time to gain some understanding of where this rule comes from and why it works. A key idea stems from our work with the tangent to a curve.

Definition: The Gradient of a Curve
The gradient of a curve at any given point on that curve is the same as the gradient of the tangent to the curve at that point

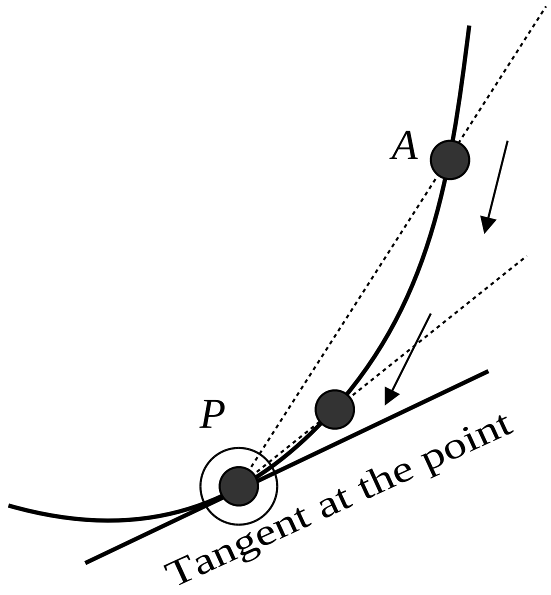
---



Suppose the gradient at the point P is sought.
Another point on the curve a short step away, A, is selected and the line between P and A is considered. The line between P and A is a poor approximation to what is wanted, the tangent at P.
However, as A slides along the curve towards P the approximating line becomes a better and better approximation of the desired line, the tangent at P.

Translating this argument into mathematics results in a 'first principles' mathematical definition of what a derivative is.

So, here is the same argument written as mathematics.



Let the curve be of the function, $f(x)$

Let the x coordinate of P also be x.

The point P then has coordinates $(x,\ f(x))$

Let the x coordinate at A be $x + h$

The point A then has coordinates $(x + h,\ f(x + h))$

The gradient of the line between P and A will be

$$\begin{aligned} m &= \frac{\Delta y}{\Delta x} \\ &= \frac{f(x+h) - f(x)}{x + h - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

To slide the point A towards P, h has to become smaller and smaller;

Definition of The Gradient Function (The Derivative)

$$f'(x) = \underset{h \to 0}{limit} \frac{f(x+h) - f(x)}{h}$$

Using this definition to find a derivative is “differentiating from first principles”.

3.2 Example

The point *P* with coordinates $(3, \ 36)$ lies on the curve with equation $y = 4x^2$
By differentiating from first principles, determine the gradient at *P*.

[5 marks]

3.3 Exercise

Marks Available : 40

Question 1

(i) Try to write down from memory, the first principles definition of a curve's gradient equation.

[2 marks]

(ii) The point *P* with coordinates $(5,\ 25)$ lies on the curve $y = x^2$
By differentiating from first principles, determine the gradient at *P*.

[4 marks]

Question 2

The point *P* with coordinates $(4,\ 48)$ lie on the curve $y = 3x^2$
By differentiating from first principles, determine the gradient at *P*.

[5 marks]

Question 3

The point P with coordinates $(2,\ 7)$ lie on the curve $y = x^2 + 3$
By differentiating from first principles, determine the gradient at P.

[6 marks]

Question 4

The point P with coordinates $(1,\ 5)$ lie on the curve $y = x^3 + 4x$
By differentiating from first principles, determine the gradient at P.

[8 marks]

Question 5

(i) By using the binomial theorem, or otherwise, expand $(3 + h)^4$

[2 marks]

(ii) The point P with coordinates $(3,\ 81)$ lie on the curve $y = x^4$
By differentiating from first principles, determine the gradient at P.

[5 marks]

Question 6

Consider the function

$$f(x) = \frac{1}{x}$$

The point P with coordinates (5, 0.2) lies on the curve of $f(x)$

By differentiating from first principles, determine the gradient at P.

[8 marks]