Lesson 4

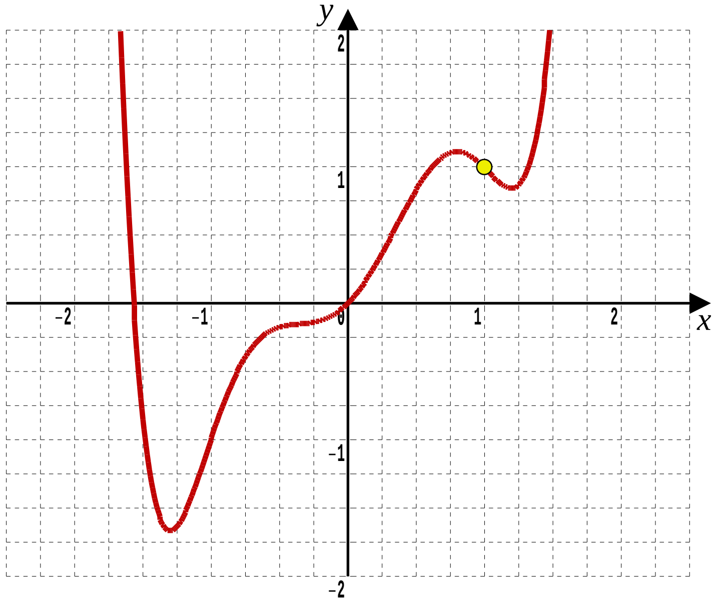
4.1 Point, Gradient, Bend

Consider the graph of the function, $f(x) = x^6 - 3x^4 + 2x^2 + x$

This is a **POINT** equation because, given a value of x, substituting that value into the function yields a **POINT** $(x, f(x))$ on the curve.

For example, if $x = 1$, then $f(1) = 1^6 - 3 \times 1^4 + 2 \times 1^2 + 1$

$= 1 \quad \therefore \; (1,\; 1)$ is a point on the curve.



By differentiation, $f'(x) = 6x^5 - 12x^3 + 4x + 1$

This is a **GRADIENT** equation because, given a value of x, substituting that value into $f'(x)$ yields the **GRADIENT** of the curve at $(x,\; f(x))$.

For example, if $x = 1$. then $f'(1) = 6 \times 1^5 - 12 \times 1^3 + 4 \times 1 + 1$

$= -1 \quad \therefore$ The gradient at $(1,\; 1)$ is -1

By differentiating a second time, $f''(x) = 30x^4 - 36x^2 + 4$

This is a **BEND** equation because, given a value of x, substituting that value into $f''(x)$ yields the **BEND** of the curve at $(x,\; f(x))$.

For example, if $x = 1$. then $f''(1) = 30 \times 1^4 - 36 \times 1^3 + 4$

$= -2$

The fact that this is negative reveals that the bend at (1, 1) is CLOCKWISE

4.2 Turning Points

The forgoing is of most use when trying to determine the nature of turning points.

Definition: Turning Point

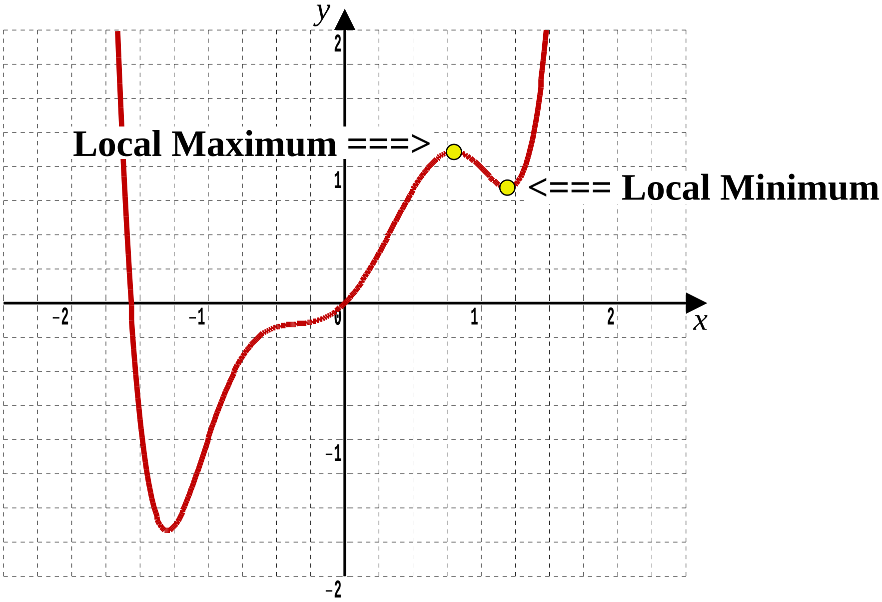
A turning point is a point where the gradient is zero and the sign of the gradient is different either side of the turning point.

They are so called because the gradient turns (BENDS) through zero.

If a function $f(x)$ has a turning point when $x = a$, then;

- if $f''(a) > 0$, the point is a local minimum i.e. if $f''(a)$ is positive
- if $f''(a) < 0$, the point is a local maximum i.e. if $f''(a)$ is negative

If $f''(a) = 0$, the bend has not been determined; the bend detector test has failed !



4.3 Example

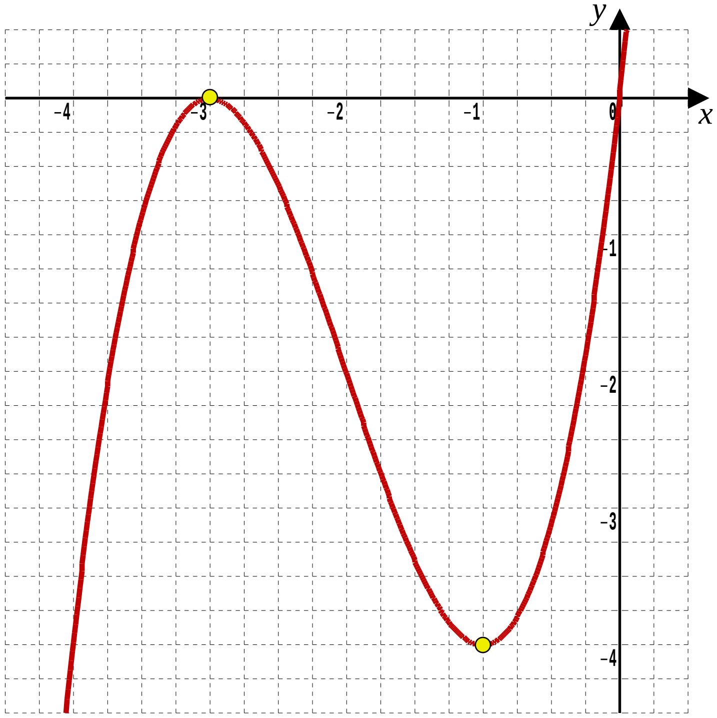
(i) Sketch an example of a curve which has a single point, P, with gradient zero, but for which P is not a turning point.
Such a point is described as a ***stationary point of inflection***.

(ii) Sketch an example of a curve with a point of inflection, but which has no point where the gradient is zero.
Such a point is described as a ***non-stationary point of inflection***.
Is your sketch of an ***increasing*** or ***decreasing*** function ?

[2 marks]

4.4 Example

Consider the curve, graphed below, with equation, $y = x^3 + 6x^2 + 9x$



(i) Find $\dfrac{dy}{dx}$

[2 marks]

(ii) Find $\dfrac{d^2y}{dx^2}$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

4.5 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable

Marks Available : 40

Question 1

Consider the curve with equation $y = 5x^2 - 30x$

(i) Find $\dfrac{dy}{dx}$

[1 mark]

(ii) Find $\dfrac{d^2y}{dx^2}$

[1 mark]

The curve with has one turning point.

(iii) Work out the coordinates of the turning point and determine its nature.

[3 marks]

Question 2

A curve has equation $y = 4x^2 + 16x + 21$

(i) Find $\dfrac{dy}{dx}$

[2 marks]

(ii) Find the coordinates of the turning point by solving the equation;

$$\frac{dy}{dx} = 0$$

Show your working clearly.

[4 marks]

(iii) Find $\dfrac{d^2y}{dx^2}$ and explain what this is telling you about the turning point.

[2 marks]

Question 3

A cubic curve has equation $y = x^3 + 9x^2 + 15x$

(i) Find $\frac{dy}{dx}$

[2 marks]

(ii) Find $\frac{d^2y}{dx^2}$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

Question 4

A function is described by $f(x) = x^3 + 6x^2 + 5$

(i) Find $f'(x)$

[2 marks]

(ii) Find $f''(x)$

[2 marks]

The curve has two turning points.

(iii) Work out the coordinates of both turning points and determine their nature.

[4 marks]

Question 5

A-Level Examination Question from January 2019, Paper C12, Q3 (Edexcel)

A curve has equation $y = \sqrt{2}\,x^2 - 6\sqrt{x} + 4\sqrt{2}, \quad x > 0$

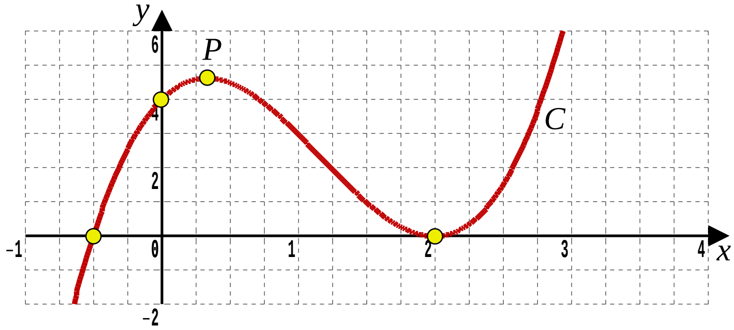
Find the gradient of the curve at the point $\left(2,\ 2\sqrt{2}\right)$

Write your answer in the form $a\sqrt{2}$, where a is a constant.

[4 marks]

Question 6

A-Level Examination Question from June 2018, Paper C12, Q14 (Edexcel)



The graph shows a curve, C, with equation $y = f(x)$ where

$$f(x) = (x - 2)^2(2x + 1) \qquad x \in \mathbb{R}$$

The curve crosses the x-axis at $\left(-\frac{1}{2},\ 0\right)$, touches it at $(2,\ 0)$ and crosses the y-axis at $(0,\ 4)$. There is a maximum turning point marked P.

Use $f'(x)$ to find the exact coordinates of the turning point P.

[7 marks]