Lesson 4

A-Level Pure Mathematics : Year 2

Differential Equations I

4.1 Type One, Two and Three Mash Up

In this Lesson, the three types of differential equation, previously considered separately, will occur in random order.
Additionally, the letters in use will not always be x and y.

Example

During the decay of a radioactive substance, the rate at which mass is lost is proportional to the mass present at that instant.
A radioactive chemical has a rate of decay given by;

$$\frac{dm}{dt} = -3m$$

and $m = 12$ when $t = 0$.
Show that this has a solution of the form

$$m = Ae^{-kt}$$

where A and k are numbers that you should determine.

[6 marks]

4.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1
Solve the following differential equation,

$$\frac{dy}{dx} = \frac{e^{2x}}{y} \qquad \text{given that } y = 3 \text{ when } x = 0$$

Present your solution as elegantly as possible and in the form $y^2 = f(x)$

[6 marks]

Question 2

Solve the following differential equation,

$$\frac{dy}{dx} + csc^2 3x = 0 \qquad \text{given that} \quad y = 1 \quad \text{when} \quad x = \frac{\pi}{4}$$

Present your solution as elegantly as possible and in the form $y = f(x)$

[6 marks]

Question 3

The population of a colony of rabbits increases at a rate proportional to the population. When first observed, the population is 50 rabbits, increasing at a rate of 5 rabbits per month.

(i) The differential equation that models this situation is of the form

$$\frac{dP}{dt} = K\,P$$

Determine the value of the constant of the proportionality, *K*.

[2 marks]

(ii) Solve the part (i) differential equation to obtain an expression of the form;

$$P = A\,e^{k\,t}$$

where *A* and *k* are numbers you have determined.

[6 marks]

(iii) What will be the population two years after first observed ?

[1 mark]

Question 4

A-Level Examination Question from January 2012, Paper C4, Q8 (Edexcel)

(a) Express in partial fractions;

$$\frac{1}{P(5-P)}$$

[3 marks]

A team of conservationists is studying the population of meerkats on a nature reserve. The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{15}P(5-P), \qquad t \geqslant 0$$

where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that when $t = 0$, $P = 1$,

(b) solve the differential equation, giving your answer in the form,

$$P = \frac{a}{b + c\,e^{-\frac{1}{3}t}}$$

where a, b and c are integers.

[8 marks]

(c) Hence show that the population cannot exceed 5000

[1 mark]

Question 5

(i) Using partial fractions, find

$$\int \frac{1}{4 - x^2} \, dx$$

[4 marks]

(ii) The amount of chemical present in a particular reaction at time t is x.

$$\frac{dx}{dt} = 10\left(4 - x^2\right) \qquad \text{with } x = 0 \text{ when } t = 0$$

Given that $0 \leqslant x < 2$, solve this to express x in terms of t.

[6 marks]

Question 6

A-Level Examination Question from June 2019, Paper 2, Q14 (Edexcel)

(a) Use the substitution $u = 4 - \sqrt{h}$ to show that,

$$\int \frac{dh}{4 - \sqrt{h}} = -8\, ln \left| 4 - \sqrt{h} \right| - 2\sqrt{h} + k$$

where k is a constant.

[6 marks]

A team of scientists is studying a species of slow growing tree. The rate of change in height of a tree in this species is modelled by the differential equation,

$$\frac{dh}{dt} = \frac{t^{0.25}\left(4 - \sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, in years, after the tree is planted.

(b) Find, according to the model, the range in height of trees in this species.

[2 marks]

One of these trees is one metre high when it is first planted.
According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

[7 marks]