Lesson 5

A-Level Pure Mathematics : Year 2

Differential Equations I

5.1 Rates Of Change (without integration)

Differentiation is often used by physicists to model a *rate of change*. Some rates of change occur so often that they are given names. For example, when considering a moving object, the rate of change of its displacement with respect to time is termed its velocity. This fact can be written in mathematics as,

$$v = \frac{ds}{dt}$$

Economists also make use of *rates of change*. For example, the rate at which the price of goods in shops are increasing with respect to time is termed inflation. This time we can write,

$$I = \frac{dp}{dt}$$

Many rates of change are interconnected as the following example will show.

5.2 The Coffee Stain Example

The photograph shows a coffee stain on a carpet as it slowly spreads outward. (The biscuit is there to give a sense of scale)

Photograph by Martin Hansen

The biscuit has since been eaten (before you ask)

Careful measurements of the photographs reveal that the radius of a coffee stain (which will be modelled as a circle) is increasing at a steady 0.04 mm.s^{-1}.

(a) Write down the well known formulae for

(i) The circumference of a circle in terms of its radius.

[1 mark]

(ii) The area of a circle, also in terms of its radius.

[1 mark]

(b) Find the rate at which the circumference is increasing.
Give your answer correct to 3 significant figures and state the units.

[3 marks]

(c) Find the rate at which the area is increasing when the circumference is 8 mm.
Give your answer correct to 3 significant figures and state the units.

[3 marks]

(d) Identify a major underlying assumption within this model, and comment on the resulting limitations of this model.
How might the model be improved ?

[3 marks]

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 40

Question 1

The melting of an ice cube in a cool room is modelled by assuming it is melting at a constant rate of 2700 mm^3 per hour.

(i) Find the rate at which the length of one side of the cube is decreasing when the volume of the ice cube is 8000 mm^3.
Give an exact answer and state the units of your answer.

HINT: $$\frac{dL}{dt} = \frac{dL}{dV} \times \frac{dV}{dt}$$

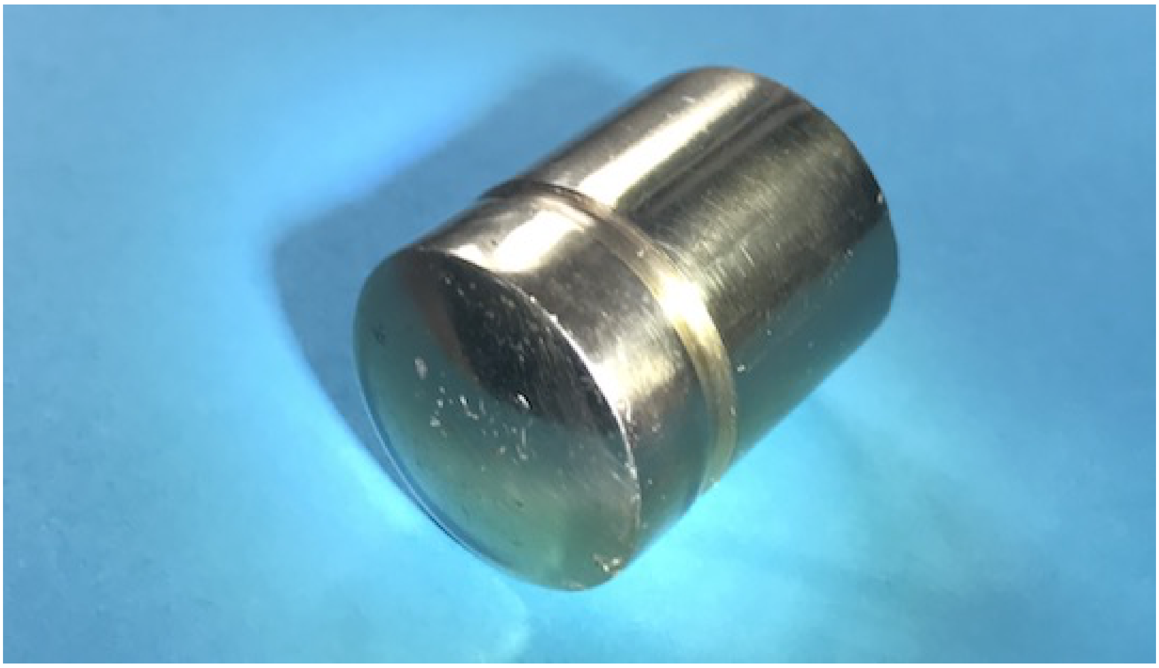
where L is the length of a side, V is the volume and t is time.

[4 marks]

(ii) According to the model, how long will it take for the ice cube to melt completely ? Give your answer in hours and minutes the nearest minute.

[1 mark]

Question 2



Photograph by Martin Hansen

The solid brass right circular cylindrical rod shown in the photograph is to be heated.
After t seconds, the radius of the rod is x cm and the length of the rod is $1.5x$ cm.
The cross-sectional area of the rod is increasing at the constant rate of $0.032\ \text{cm}^2\ \text{s}^{-1}$

(i) Find $\dfrac{dx}{dt}$ when the radius of the rod is 3 cm.
Give your answer to 3 significant figures and state the units of your answer.

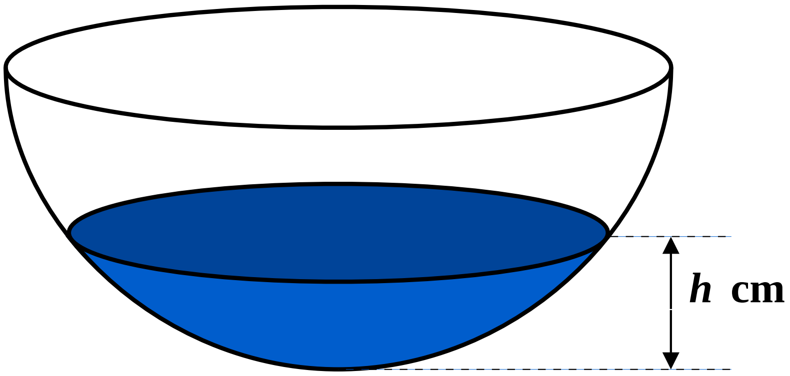
[4 marks]

(ii) Find the rate of increase of the volume of the rod when x is 3 cm.
Give your answer to 3 significant figures and state the units of your answer.

[4 marks]

Question 3

A-Level Examination Question from June 2018, Paper C34, Q7 (Edexcel)



The diagram is of a hemispherical bowl.

Water is flowing into the bowl at a constant rate of 180 $cm^3 s^{-1}$

When the height of the water is h cm, the volume of water V cm^3 is given by

$$V = \frac{1}{3}\pi h^2 (90 - h), \qquad 0 \leqslant h \leqslant 30$$

Find the rate of change of the height of the water, in cm s^{-1}, when $h = 15$

Give your answer to 2 significant figures.

[5 marks]

Question 4

A-Level Examination Question from June 2014, Paper C4(R), Q5 (Edexcel)

At time t seconds the radius of a sphere is r cm, its volume is V cm^3 and its surface area is S cm^2.

You are given that $V = \frac{4}{3}\pi r^3$ and that $S = 4\pi r^2$

The volume of the sphere is increasing uniformly at a constant rate of 3 cm^3 s^{-1}

(a) Find $\frac{dr}{dt}$ when the radius of the sphere is 4 cm.

Give your answer to 3 significant figures.

[4 marks]

(b) Find the rate at which the surface area of the sphere is increasing when the radius is 4 cm.

[2 marks]

Question 5

A-Level Examination Question from January 2008, Q4 (OCR)

Earth is being added to a pile so that, when the height of the pile is h metres, its volume is V cubic metres, where

$$V = \left(h^6 + 16\right)^{\frac{1}{2}} - 4$$

(i) Find the value of $\frac{dV}{dh}$ when $h = 2$

[3 marks]

(ii) The volume of the pile is increasing at a constant rate of 8 cubic metres per hour. Find the rate, in metres per hour, at which the height of the pile is increasing at the instant when $h = 2$.
Give your answer correct to 2 significant figures.

[3 marks]

Question 6

In a tennis ball manufacturing process, rubber spheres are produced in such a way that the volume, V, of a sphere increases at a constant rate of 10 cm^3 per second. Find the rate of change of the surface area, A, of a sphere at the moment when the surface area is equal to 32π cm^2.

You are given that $V = \frac{4}{3}\pi r^3$ and that $S = 4\pi r^2$

[5 marks]

Question 7

The surface area A, of a metallic cube of side length x, is increasing at the constant rate of 0.45 $cm^2\ s^{-1}$

Find the rate at which the volume of the cube is increasing, when the cube's side length is 8 cm.

[5 marks]