Lesson 6

6.1 Rates Of Change (with integration)

Situations involving *rates of change* often result in a differential equation. There is a skill in setting up the differential equation that effectively models the physical situation, and another skill in solving it (if it's solvable!).

6.2 Will The Sink Overflow ?

Photograph by Martin Hansen

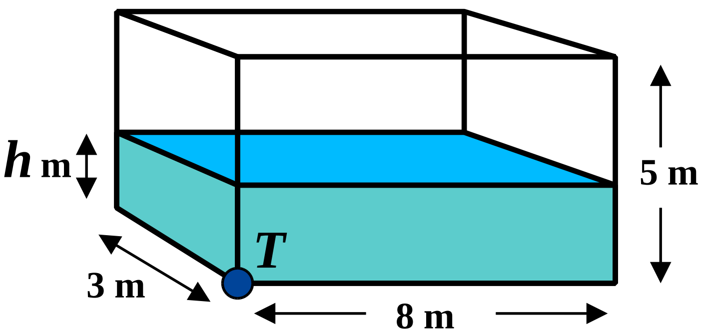
A bathroom sink has a maximum capacity of 11 litres.
A small child has left a tap running and water is entering the sink at a constant rate of 3 litres per minute. Fortunately the plug has been left out.
Given a volume of water, V, in the sink, the rate at which water can exit is $0.25V$.
Form a differential equation and obtain its general solution.
Use the general solution to determine if the sink will overflow or not.

[8 marks]

6.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 65

Question 1
A-Level Examination Question from October 2021, Paper 2, Q14 (Edexcel)



Water flows at a constant rate into a large tank.
The tank is a cuboid, with all sides of negligible thickness.
The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.
There is a tap at a point T at the bottom of the tank, as shown.
At time t minutes after the tap has been opened,

- the depth of the water in the tank is h metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

[4 marks]

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + Be^{-kt}$$

where A, B and k are constants to be found.

[6 marks]

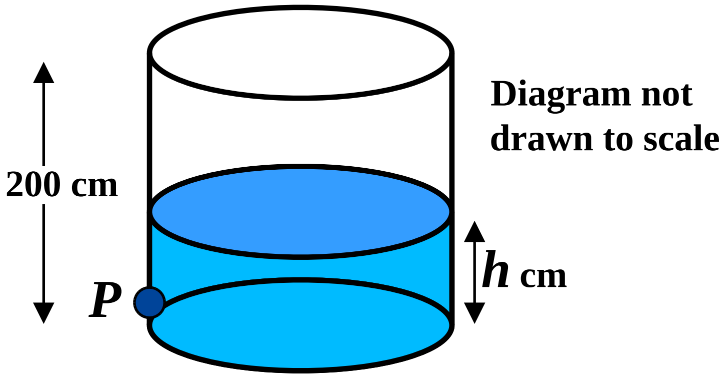
Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

[2 marks]

Question 2

A-Level Examination Question from June 2017, Paper C4, Q7 (Edexcel)



The diagram shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.
At time t minutes after the leaking starts, the height of water in the tank is h cm.
The height h cm of the water in the tank satisfies the differential equation,

$$\frac{dh}{dt} = k(h-9)^{\frac{1}{2}}, \quad 9 < h \leqslant 200 \text{ where } k \text{ is a constant.}$$

When $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute.

(a) Find the value of k

[2 marks]

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

[6 marks]

Question 3

A-Level Examination Question from January 2017, Paper C34, Q12

In freezing temperatures, ice forms on the surface of the water in a barrel.
At time t hours after the start of freezing, the thickness of the ice formed is x mm.
You may assume the thickness of the ice is uniform across the surface of the water.
At 4 pm there is no ice on the surface, and freezing begins.
At 6 pm, after two hours of freezing, the ice is 1.5 mm thick.
In a simple model, the rate of increase of x, in mm per hour, is assumed to be constant for a period of 20 hours.
Using this simple model,

(a) express t in terms of x,

[2 marks]

(b) find the value of t when $x = 3$

[1 mark]

In a second model, the rate of increase of x, in mm per hour, is given by,

$$\frac{dx}{dt} = \frac{\lambda}{(2x + 1)} \text{ where } \lambda \text{ is a constant and } 0 \leqslant t \leqslant 20$$

Using this second model,

(c) solve the differential equation and express t in terms of x and λ

[3 marks]

(d) find the exact value for λ,

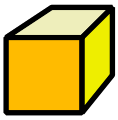
[1 mark]

(e) find at what time the ice is predicted to be 3 mm thick.

[2 marks]

Question 4

A-Level Examination Question from June 2006, Paper C4, Q7 (Edexcel)



At time t seconds the length of the side of a cube is x cm, the surface area of the cube is S cm^2, and the volume of the cube is V cm^3.

The surface area of the cube is increasing at a constant rate of 8 cm^2 s^{-1}

Show that,

(a) $\dfrac{dx}{dt} = \dfrac{k}{x}$, where k is a constant to be found,

[4 marks]

(b) $\dfrac{dV}{dt} = 2V^{\frac{1}{3}}$

[4 marks]

(c) Given that $V = 8$ when $t = 0$ solve the differential equation in part (b), and find the value of t when $V = 16\sqrt{2}$

[7 marks]

Question 5

A-Level Examination Question from January 2013, Paper C4, Q8 (Edexcel)

A bottle of water is put into a refrigerator. The temperature inside the refrigerator remains constant at 3 °C and t minutes after the bottle is placed in the refrigerator the temperature of the water in the bottle is θ °C

The rate of change of the temperature of the water in the bottle is modelled by the differential equation, $\frac{d\theta}{dt} = \frac{(3 - \theta)}{125}$

(a) By solving the differential equation show that, $\theta = A e^{-0.008t} + 3$ where A is a constant.

[4 marks]

Given that the temperature of the water in the bottle when it was put in the refrigerator was 16 °C,

(b) find the time taken for the temperature of the water in the bottle to fall to 10 °C, giving your answer to the nearest minute.

[5 marks]

Question 6

A-Level Examination Question from January 2018, Paper C34, Q14 (Edexcel)

The volume of a spherical balloon of radius r cm is V cm^3, where $V = \frac{4}{3}\pi r^3$

(a) Find $\frac{dV}{dr}$

[1 mark]

The volume of the balloon increases with time t seconds according to the formula,

$$\frac{dV}{dt} = \frac{9000\pi}{(t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

(b) Using the chain rule, or otherwise, show that

$$\frac{dr}{dt} = \frac{k}{r^n (t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

where k and n are constants to be found.

[2 marks]

Initially, the radius of the balloon is 3 cm.

(c) Using the values of k and n found in part (b), solve the differential equation

$$\frac{dr}{dt} = \frac{k}{r^n (t + 81)^{\frac{5}{4}}} \qquad t \geqslant 0$$

to obtain a formula for r in terms of t.

[6 marks]

(d) Hence find the radius of the balloon when $t = 175$, giving your answer to 3 significant figures.

[1 mark]

(e) Find the rate of increase of the radius of the balloon when $t = 175$
Give your answer to 3 significant figures.

[2 marks]