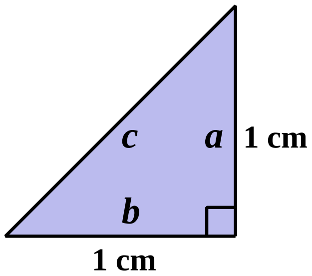
3.1 Discovery Of Irrationals

Pythagoras was an Ancient Greek philosopher who was born around 570 BC. He founded a political and religious movement, The Pythagoreans, who's motto was “number is all”. They believed that through an understanding of mathematics they would have mastery of all workings of the Universe. They loved integers, and embraced rational numbers which they thought of as ratios of integers.

They viewed the fraction $\frac{p}{q}$ as the ratio $p : q$ and believed (mistakenly) that all numbers could be written in this way. Although “The Theorem of Pythagoras” was actually known well before Pythagoras was born by the Babylonians and the Indians it is attributed to Pythagoras, partly because it presented The Pythagoreans with a major problem that challenged a fundamental belief; that all numbers were rational.

The problem that profoundly challenged their view of mathematics was the following; Determine the length of the hypotenuse, c, of a right angled triangle in which each of the shorter sides, a and b, were of length 1.



$$
\begin{aligned}
c^2 &= a^2 + b^2 \qquad \text{(By the theorem of Pythagoras)}\\
&= 1^2 + 1^2\\
&= 2\\
\therefore\ c &= \sqrt{2}
\end{aligned}
$$

Try as they might, none of the Pythagorean brotherhood could write this troublesome answer as an exact ratio of integers. Initially, they assumed they had just not been clever enough to work out the ratio but one of them, Hippasus, found a logical argument that proved that no matter how long they searched, $\sqrt{2}$ could never be written as a ratio of integers. He showed that $\sqrt{2}$ was a fundamentally different type of number. Hippasus was murdered, drowned at sea, in a vain attempt to keep his discovery secret for it undermined the Pythagorean's core belief in a precise and orderly Universe.

Hippasus is now thought by scholars to have been the person who discovered a new type of number, the ***irrationals***.

3.2 How To Suspect A Number Is Irrational

An irrational person is someone who is not predictable. When an irrational number starts to be written out as a decimal, it's not obvious what digit will be in the next decimal place.

$$\sqrt{2} = 1.\,?$$
$$\sqrt{2} = 1.\,4\,?$$
$$\sqrt{2} = 1.\,4\,1\,?$$
$$\sqrt{2} = 1.\,4\,1\,4\,?$$
$$\sqrt{2} = 1.\,4\,1\,4\,2\,?$$
$$\ldots$$
$$\sqrt{2} = 1.\,4\,1\,4\,2\,1\,3\,5\,6\,2\,?$$
$$\ldots$$
$$\sqrt{2} = 1.\,4\,1\,4\,2\,1\,3\,5\,6\,2\,3\,7\,3\,0\,?$$

So, when an irrational number is written out as a decimal, the decimal never terminates and there is no pattern that continuously repeats. This is the hallmark of an irrational number, the fingerprint that gives it away.

3.3 Example

Consider the following sequence of square rooted integers;

$$\sqrt{1} \quad \sqrt{2} \quad \sqrt{3} \quad \sqrt{4} \quad \sqrt{5} \quad \sqrt{6} \quad \sqrt{7} \quad \sqrt{8}$$

$$\sqrt{9} \quad \sqrt{10} \quad \sqrt{11} \quad \sqrt{12} \quad \sqrt{13} \quad \sqrt{14} \quad \sqrt{15} \quad \sqrt{16}$$

Under each member of this sequence write down $\mathbb{Q}$ if the member is *rational*, and $\mathbb{P}$ if the member is *irrational*.

3.4 A Useful Table Of squares

So that the next exercise can be done without a calculator, here is a table of integers and their squares.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
256	289	324	361	400	441	484	529	576	625	676	729	784	841	900

3.5 Exercise

Do ***NOT*** use a calculator

Marks Available : 64

Question 1

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational*.

(i) $\sqrt{8}$ **(ii)** $\sqrt{144}$ **(iii)** $\sqrt{14}$

(iv) $\sqrt{225}$ **(v)** $\sqrt{324}$ **(vi)** $\sqrt{37}$

(vii) $\sqrt{361}$ **(viii)** $\sqrt{305}$ **(ix)** $\sqrt{\sqrt{625}}$

[9 marks]

Question 2

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational*.

(i) $\sqrt{3}$ **(ii)** $\dfrac{\sqrt{3}}{\sqrt{3}}$ **(iii)** $\sqrt{3} \times \sqrt{3}$

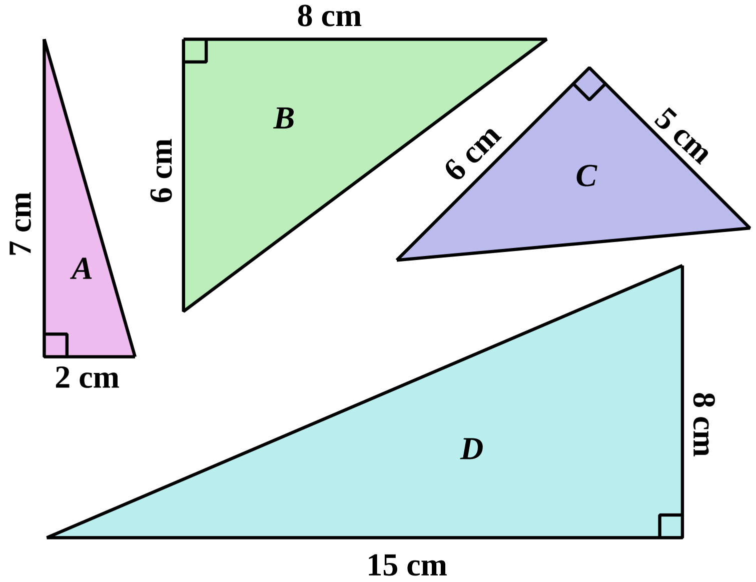
(iv) $\left(\sqrt{3}\right)^0$ **(v)** $\sqrt{3} \times \sqrt{12}$ **(vi)** $\dfrac{\sqrt{12}}{\sqrt{3}}$

[6 marks]

Question 3

Work out the hypotenuse of each of the following triangles.

In each case, state if the answer is ***Rational,*** $\mathbb{Q}$ or ***Irrational,*** $\mathbb{P}$



[8 marks]

Question 4

Show that each of the following is rational, $\mathbb{Q}$, by writing them in the form $\dfrac{p}{q}$

where p and q are integers and $q \neq 0$.

(i) $\dfrac{3}{7} \times \dfrac{5}{2}$ **(ii)** $4\dfrac{1}{5}$ **(iii)** $\dfrac{2}{3} + \dfrac{5}{7}$

(iv) $\left(\dfrac{5}{6}\right)^2$ **(v)** $\left(\dfrac{1}{9}\right)^2$ **(vi)** $\left(\dfrac{3}{2}\right)^3$

(vii) $\left(\dfrac{100}{121}\right)^{\frac{1}{2}}$ **(viii)** 13^2 **(ix)** $\left(2 + \dfrac{1}{2}\right)^2$

(x) $\sqrt{\dfrac{9}{16}}$ **(xi)** $\sqrt{\dfrac{3^2 + 4^2}{4}}$ **(xii)** $8^{\frac{1}{3}}$

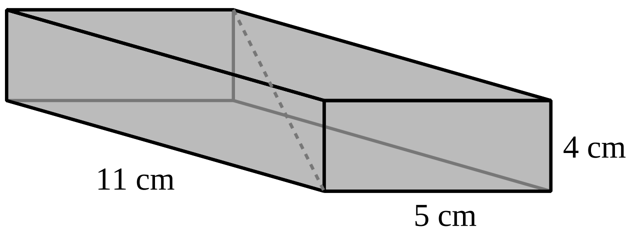
(xiii) $\sqrt[3]{\dfrac{1}{27}}$ **(xiv)** $\sqrt{\dfrac{169}{196}}$ **(xv)** $\sqrt{\left(6 + \dfrac{1}{4}\right)}$

[15 marks]

Question 5

In a cuboid with sides of lengths a, b and c, the longest diagonal in the box, d, is given by a three dimensional version of Pythagoras' Theorem.

$$d^2 = a^2 + b^2 + c^2$$



(i) Work out the longest diagonal's length in an 11 cm by 5 cm by 4 cm cuboid.

[2 marks]

(ii) Is your part **(i)** answer a rational number, $\mathbb{Q}$, or an irrational number, $\mathbb{P}$?

[1 mark]

Question 6

Under each expression write $\mathbb{Q}$ if it's *rational*, or $\mathbb{P}$ if it's *irrational.*

(i) 0.425 **(ii)** $0.\dot{3}4\ddot{8}$ **(iii)** 3.14159

(iv) $\left(\sqrt{5}\right)^3$ **(v)** 72 **(vi)** 3^{-2}

(vii) $0.1762222222...$ **(viii)** $1^{\sqrt{2}}$ **(ix)** $\dfrac{1}{\sqrt{2}}$

[9 marks]

Question 7

(i) Write down a ***Rational*** number that lies between 1 and 2

[1 mark]

(ii) Write down an ***Irrational*** number that lies between the integers 1 and 2

[1 mark]

Question 8

Show that each of the following is a member of the set of rational numbers, $\mathbb{Q}$.

(i) $\sqrt{2\frac{1}{4}}$ **(ii)** $\sqrt{36}$ **(iii)** $\sqrt{7\frac{1}{9}}$

(iv) $\sqrt{3600}$ **(v)** $\sqrt{14\,\frac{1}{16}}$ **(vi)** $\sqrt{0.36}$

[12 marks]