Lesson 4

A-Level Pure Mathematics : Year 2

Differentiation IV

4.1 Tricky Trig Troubles

The more awkward examination questions on parametric differentiation invariably require a good grasp of the various trigonometric identities.
The following are frequently needed and not provided in the exam formulae booklet.

$\cos^2\theta + \sin^2\theta = 1$		$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$1 + \tan^2\theta = \sec^2\theta$		$\sin 2\theta = 2\sin\theta\cos\theta$
$\cot^2\theta + 1 = \csc^2\theta$		$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

4.2 Example

For the parametric equations $x = \sin^2\theta$

and $y = \cos 2\theta$

(i) Show that $\dfrac{dy}{dx} = -2$

(ii) Sketch the graph of the parametric equations.

Teaching Video : http://www.NumberWonder.co.uk/v9081/4.mp4

[3, 3 marks]

4.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

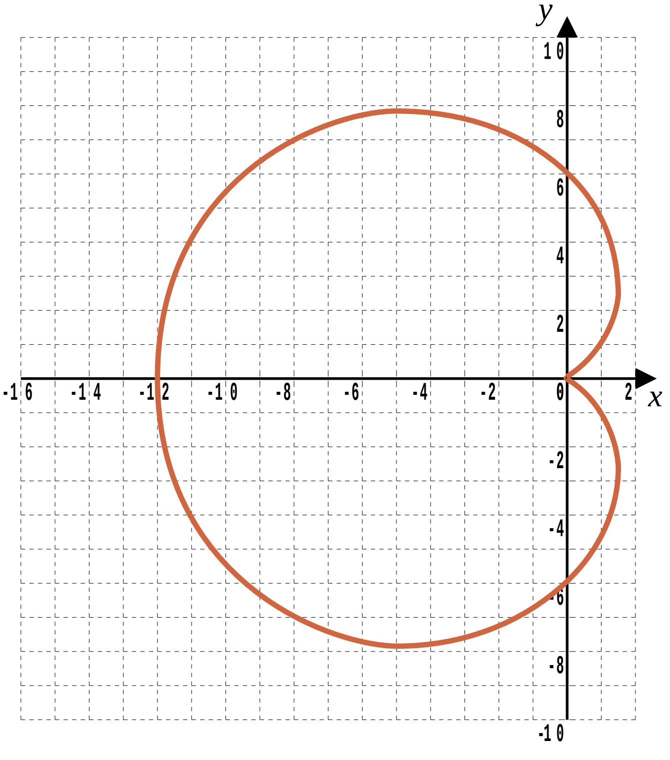
Marks Available : 60

Question 1

In the photograph, on the bottom of the empty cereal bowl, an interesting shape can be seen, made up from the light reflecting off the sides. This is an example of a caustic. A caustic is an envelope of light rays formed by reflection or refraction from a curved surface. More complicated caustics can be seen dancing about on the bottom of swimming pools, for example.

Photograph by Martin Hansen

The cereal bowl caustic is in the shape of a cardioid, from a Greek word that translates as "heart". The graph below is of a perfect mathematical cardioid,



The cardioid in the graph was plotted from the parametric equations,

$$x = 6(1 - \cos\theta)\cos\theta$$

$$y = 6(1 - \cos\theta)\sin\theta$$

(i) What point corresponds to $\theta = \pi$?
Mark this point on the graph and label it A

[2 marks]

(ii) Without making any calculations, write down the equation of the tangent to the cardioid at point A and add this tangent to the graph.

[2 mark]

(iii) What point corresponds to $\theta = \dfrac{\pi}{2}$?
Mark this point on the graph and label it B

[2 marks]

(iv) Show that $\dfrac{dy}{dx} = \dfrac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

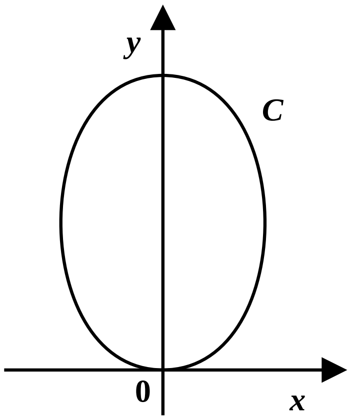
[6 marks]

(v) Find the equation of the tangent to the cardioid at point B
Draw this tangent on the graph.

[4 marks]

Question 2

A-Level Examination Question from June 2012, Paper C4, Q6 (Edexcel)



The curve C has parametric equations

$$x = \sqrt{3}\,\sin 2t, \qquad y = 4\cos^2 t, \qquad 0 \leqslant t \leqslant \pi$$

(a) Show that $\dfrac{dy}{dx} = k\sqrt{3}\,\tan 2t$ where k is a constant to be determined

[5 marks]

(b) Find the equation of the tangent to C at the point where $t = \dfrac{\pi}{3}$

Give your answer in the form $y = ax + b$, where a and b are constants

[4 marks]

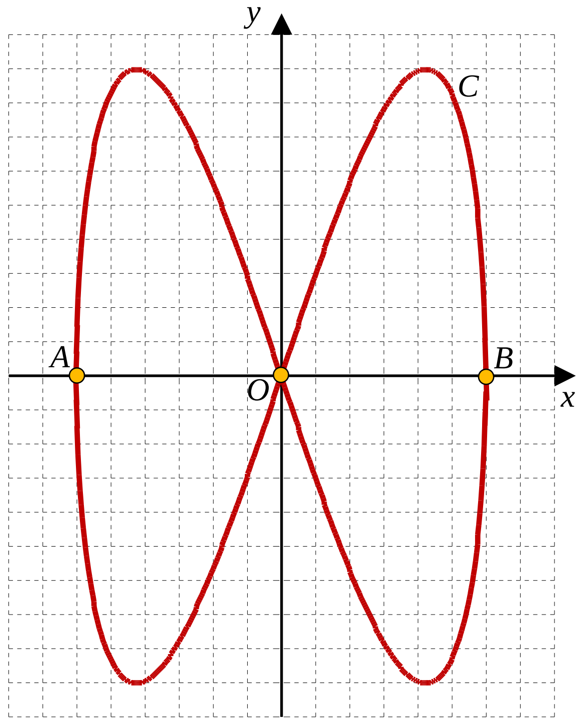
(c) Find a Cartesian equation of C

(Note : Any algebra that combines the parametric equations and then leads to an answer that has x and y in it, and no t, gets full marks)

[3 marks]

Question 3

A-Level Examination Question from January 2018, Paper C34, Q11 (Edexcel)



The curve C has parametric equations,

$$x = 3\cos t \qquad y = 9\sin 2t \qquad 0 \leqslant t \leqslant 2\pi$$

The curve C meets the x-axis at the origin and at the points A and B, as shown.

(a) Write down the coordinates of A and B

[2 marks]

(b) Find the values of t at which the curve passes through the origin.

[2 marks]

(c) Find an expression for $\dfrac{dy}{dx}$ in terms of t, and hence find the gradient of the curve when $t = \dfrac{\pi}{6}$

[4 marks]

(d) Show that the Cartesian equation for the curve C can be written in the form

$$y^2 = a\,x^2\,(\,b - x^2\,)$$

where a and b are integers to be determined

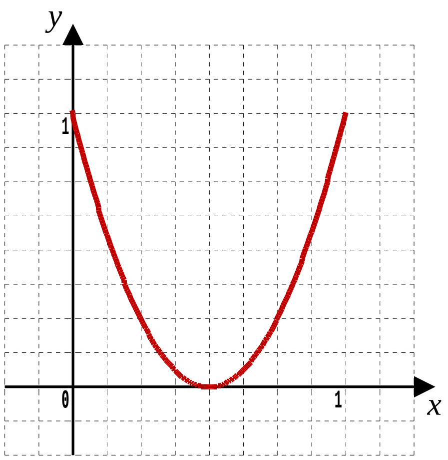
[4 marks]

Question 4

The graph is of the parametric equations,

$$x = \sin^2 \theta$$

$$y = \cos^2 2\theta$$



When $\theta = 0$, the corresponding point on the graph is (0, 1)

As θ increases the point moves along the path and is at (0.5, 0) when $\theta = \dfrac{\pi}{4}$

Further increase in θ continues the journey toward (1, 1) reached when $\theta = \dfrac{\pi}{2}$

The direction of travel now reverses and the point heads back the way it came.

(i) Determine the Cartesian equation of the curve.

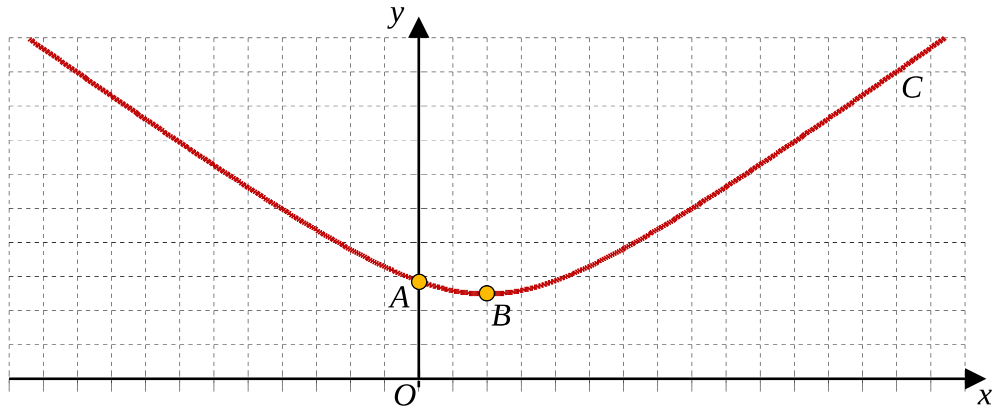
[4 marks]

(ii) Show that $\dfrac{dy}{dx} = -\dfrac{2 \sin 4\theta}{\sin 2\theta} \qquad \theta \neq \dfrac{n\pi}{2}, \; n \in \mathbb{Z}$

[4 marks]

Question 5

A-Level Examination Question from January 2017, Paper C34, Q13 (Edexcel)



The curve shown has parametric equations,

$$x = 1 + \sqrt{3}\tan\theta, \qquad y = 5\sec\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

The curve C crosses the y-axis at A and has a minimum turning point at B, as shown.

(a) Find the exact coordinates of A

[3 marks]

(b) Show that $\dfrac{dy}{dx} = \lambda \sin\theta$, giving the exact value of the constant λ

[4 marks]

(c) Find the coordinates of B

[2 marks]

(d) Show that the Cartesian equation for the curve C can be written in the form

$$y = k\sqrt{x^2 - 2x + 4}$$

where k is a simplified surd to be found

[3 marks]