Lesson 9

A-Level Pure Mathematics : Year 2

Differentiation IV

9.1 Exponentials Implicitly

Implicit differentiation questions can involve the more "fancy functions" such as the exponentials, logarithms or trigonometric. The focus in this lesson is upon the exponential function.

It's already known that if $y = e^x$ then $\frac{dy}{dx} = e^x$

In other word the exponential function $y = e^x$ is its own derivative.

However, this is a special result of the following more general rule:

The Derivative of an Exponential

Given that a is a constant,

$$\text{if } y = a^x \text{ then } \frac{dy}{dx} = a^x \ln a$$

Proof

$$\begin{aligned} y &= a^x \\ &= e^{\ln a^x} && \text{as } e^x \text{ and } \ln x \text{ are each the inverse function of the other} \\ &= e^{x \ln a} && \text{using a rule of logs} \\ \frac{dy}{dx} &= e^{x \ln a} \ln a && \text{by the Chain Rule : } \ln a \text{ is "just a number"} \\ &= e^{\ln a^x} \ln a && \text{using a rule of logs} \\ &= a^x \ln a && \text{as } e^x \text{ and } \ln x \text{ are each the inverse function of the other} \end{aligned}$$

□

Notes : 1) This rule is not given in the examination formulae booklet

2) The rule that if $y = e^x$ then $\frac{dy}{dx} = e^x$ is a special case of this more general rule with $a = e$, because $\ln e = 1$

9.2 Example

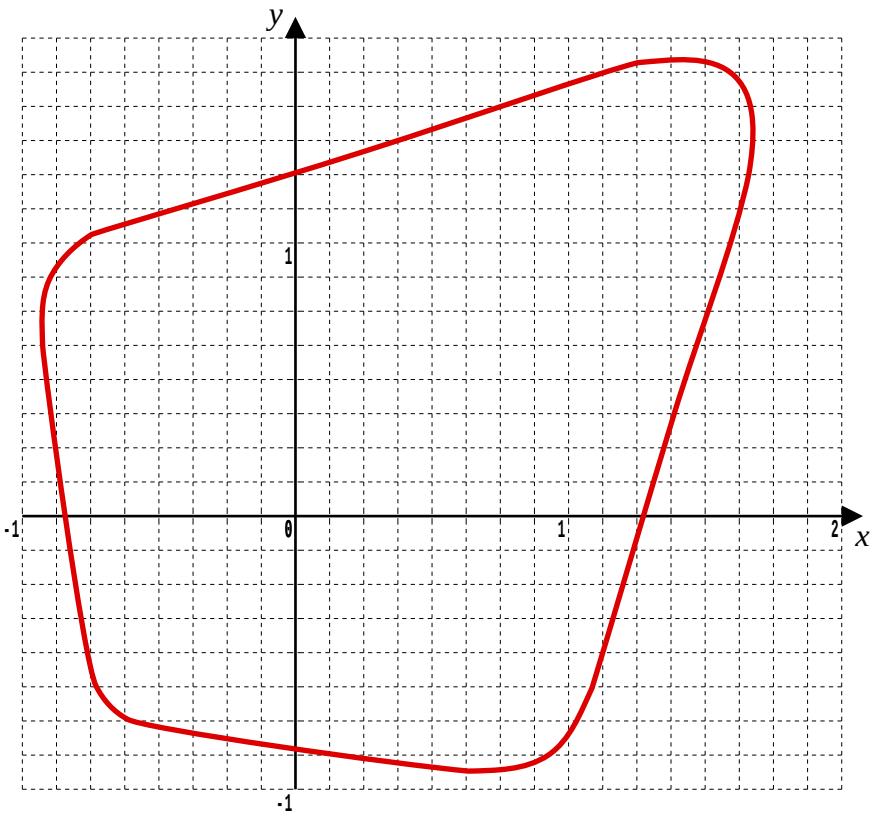
Differentiate $y = 2^x$

Solution : Using the rule for The Derivative of an Exponential with $a = 2$

$$\frac{dy}{dx} = 2^x \ln 2$$ And that's it done !

9.3 A Beautiful Curve

The graph is of the equation $3^y 3^x = x^6 + y^6$



Obtain an equation of the form $\frac{dy}{dx} = f(x, y)$ for the curve $3^y 3^x = x^6 + y^6$

Teaching Video: http://www.NumberWonder.co.uk/v9081/9.mp4

[6 marks]

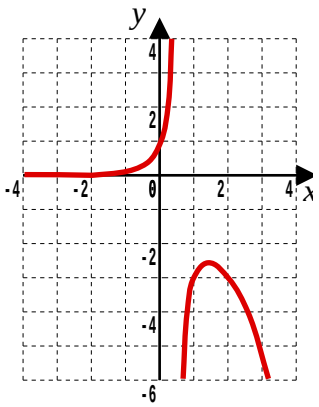
9.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

The graph is of the equation $3^x = y - 2xy$



(i) Use implicit differentiation with respect to x to obtain an equation of the form $\frac{dy}{dx} = f(x, y)$ for the curve $3^x + 2xy = y$

[7 marks]

(ii) Verify that the point $(2, -3)$ is on the curve

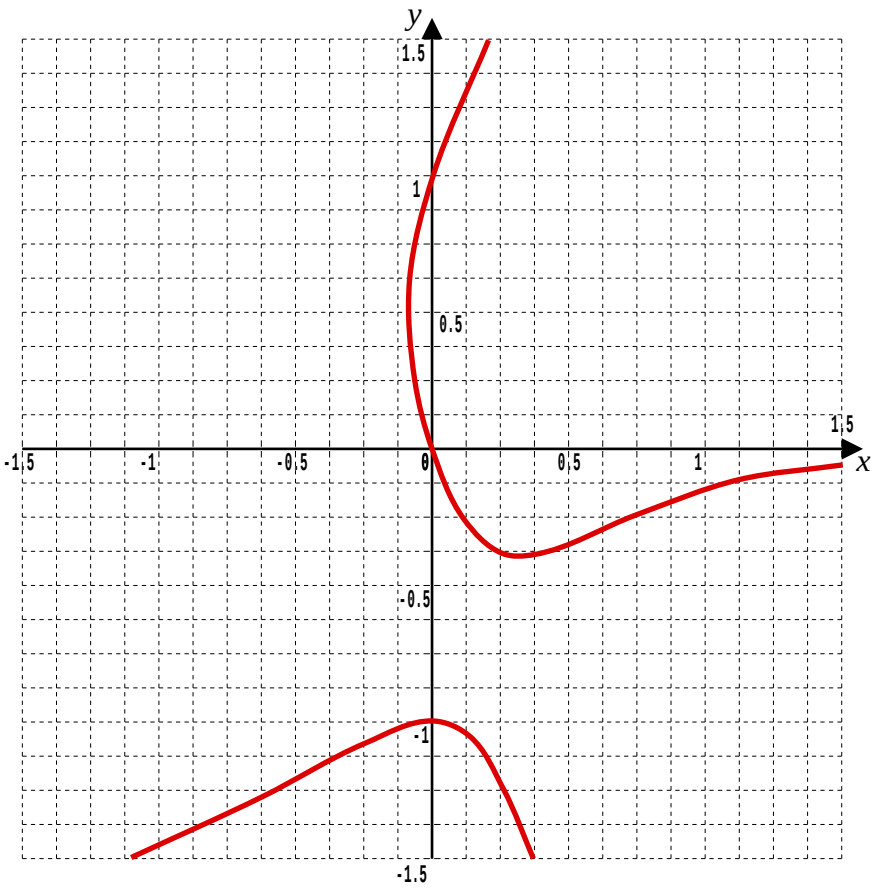
[2 marks]

(iii) Determine the exact value of the gradient at the point $(2, -3)$

[2 marks]

Question 2

The graph is of the equation $y\,e^{3x} + 3x = y^3$



(i) Use implicit differentiation with respect to x to obtain an equation of the form $\frac{dy}{dx} = f(x, y)$ for the curve $y\,e^{3x} + 3x = y^3$

[7 marks]

(ii) Find the equation of the normal to the curve at the origin.
Add this normal to the graph.

[3 marks]

(iii) Find the equation of the normal to the curve at the point $(0,\ 1)$
Add this normal to the graph.

[3 marks]

(iv) Use algebra to determine the coordinates of where the normal of part (ii) intersects the normal of part (iii).
Mark this point on the graph.

[3 marks]

Question 3

A-Level Examination Question from January 2018, IAL, Paper C34, Q1 (Edexcel)

A curve C has equation

$$3^x + xy = x + y^2 \qquad y > 1$$

The point P with coordinates $(4, 11)$ lies on C

Find the exact value of $\frac{dy}{dx}$ at the point P

Give your answer in the form $a + b \ln 3$, where a and b are rational numbers

[6 marks]

Question 4

A-Level Examination Question from June 2010, Paper C4, Q3 (Edexcel)

The curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates $(3, 2)$

[7 marks]