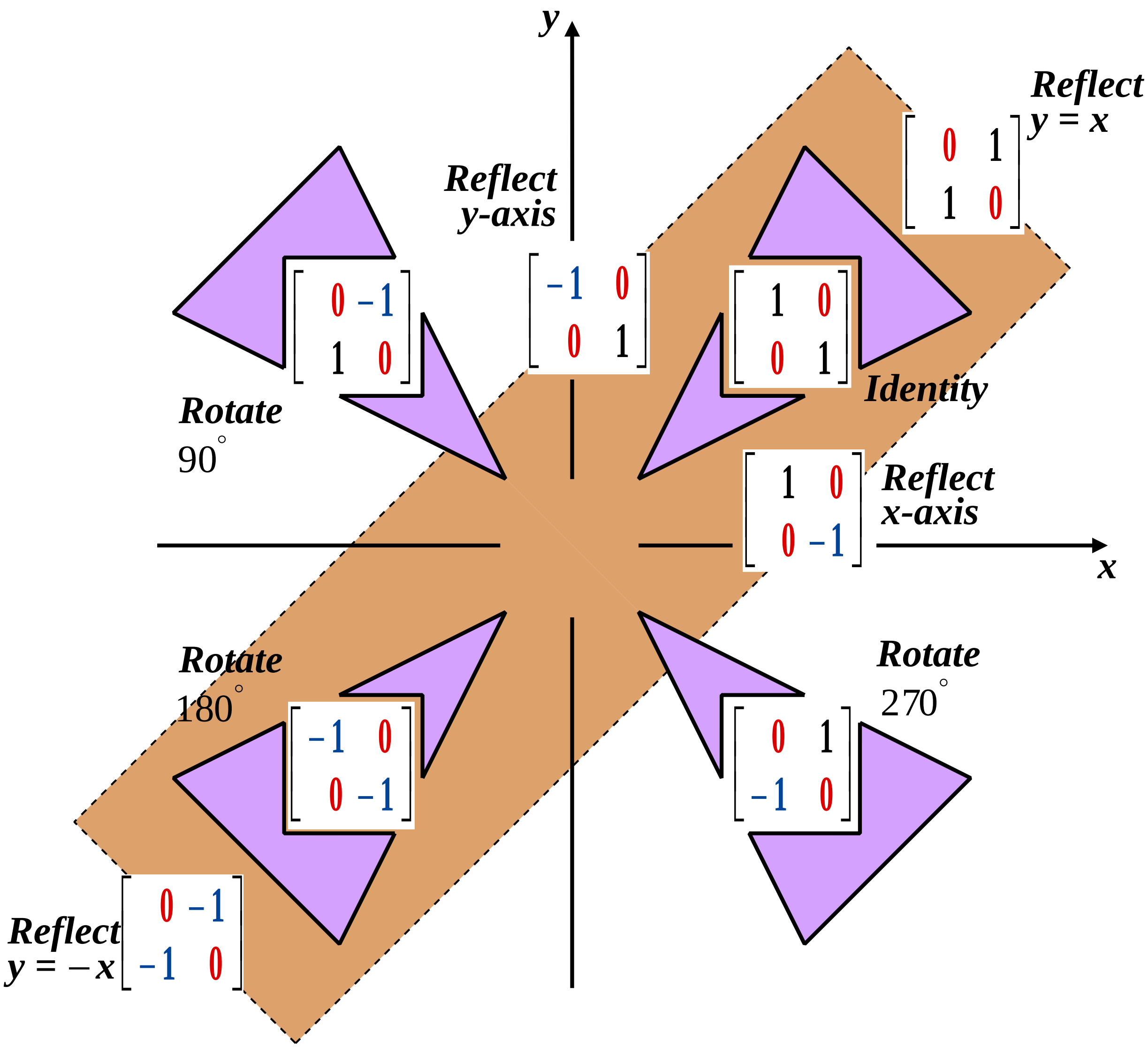
Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

MATRIX TRANSFORMATIONS



MATRIX TRANSFORMATIONS

Lesson 1

Further A-Level Pure Mathematics : Core 1
Matrix Transformations

1.1 Introduction

A matrix is an array of elements (typically numbers) arranged in rows and columns.
Here, for example, is a 3×4 matrix, **A**,

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 & 6 \\ -6 & 1 & 12 & 4 \\ 5 & 0 & -7 & 9 \end{pmatrix}$$

The 3×4 references that **A** has three rows and 4 columns.

An individual element within a matrix can be referenced by by specifying which row and which column it is in.
For matrix **A**,

$$\mathbf{A} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{23} & a_{33} & a_{34} \end{pmatrix} \end{array}$$

Thus, for example,

$$a_{23} = 12$$

If the matrix is small, the individual entries may instead be described with a single lower case letter, bold upper case letters usually being reserved for the name of an entire matrix.
For example, a generalised 2×2 square matrix, **B**, could be described as,

$$\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

One major use of matrices is in the description and analysis of networks; in other words, “connectivity”. This is a vast and thriving area of mathematics called Graph Theory. Applications include modelling how a virus might spread, for example.

Most computer languages can do arithmetic with matrices. This is because matrices are used extensively in computer graphics. They are used to move points about a computer screen and to transform shapes; rotations, reflections, enlargements and sheers can all be handled by matrices. These manipulations can be in both two or three dimensions. Matrices can also handle the mathematics of projecting a three dimensional shape onto a two dimensional screen.

1.2 Matrix Arithmetic

1.2.1 Matrix Addition

To add two matrices together, add corresponding elements.
For example,

$$\begin{pmatrix} 4 & -6 \\ 7 & 3 \\ -1 & -4 \end{pmatrix} + \begin{pmatrix} 5 & -2 \\ 2 & -9 \\ 1 & 8 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ 9 & -6 \\ 0 & 4 \end{pmatrix}$$

Only matrices of the same size can be added.

1.2.2 Matrix Subtraction

Subtraction problems can be turned into addition problems.
The idea is to reverse all signs in the second matrix.
For example,

$$\begin{pmatrix} 5 & -8 & -9 \\ -7 & 4 & 8 \end{pmatrix} - \begin{pmatrix} -6 & 3 & -9 \\ 2 & -1 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -8 & -9 \\ -7 & 4 & 8 \end{pmatrix} + \begin{pmatrix} 6 & -3 & 9 \\ -2 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -11 & 0 \\ -9 & 5 & 16 \end{pmatrix}$$

Only matrices of the same size can be subtracted.

1.2.3 Scalar Multiplication

To multiply a matrix by a scalar, multiply every element by the scalar.
For example,

$$4\begin{pmatrix} 2 & -1 & -7 \\ 0 & 0.5 & -3 \\ -8 & 6 & 9 \end{pmatrix} = \begin{pmatrix} 8 & -4 & -28 \\ 0 & 2 & -12 \\ -32 & 24 & 36 \end{pmatrix}$$

1.2.4 Scalar Division

Scalar division problems can be turned into scalar multiplication problems.

To divide by s, multiply by $\frac{1}{s}$

For example,

to divide the matrix $\begin{pmatrix} 18 & -9 \\ 0 & 42 \end{pmatrix}$ by 3, instead multiply by $\frac{1}{3}$

$$\frac{1}{3}\begin{pmatrix} 18 & -9 \\ 0 & 42 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 0 & 14 \end{pmatrix}$$

1.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

The matrices **A** and **B** are defined as,

$$\mathbf{A} = \begin{pmatrix} 4 & -8 \\ 2 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$$

Find,

(i) $\mathbf{A} + \mathbf{B}$ **(ii)** $\mathbf{A} - \mathbf{B}$

(iii) $4\mathbf{A} - 3\mathbf{B}$ **(iv)** $\frac{1}{2}\mathbf{A} + \frac{2}{3}\mathbf{B}$

[8 marks]

Question 2

Given that,

$$\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 2 & c \\ d & -2 \end{pmatrix} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

find the values of the constants a, b, c and d

[4 marks]

Question 3
Given that,

$$\begin{pmatrix} 4 & b \\ a & c \end{pmatrix} - \begin{pmatrix} a & 6 \\ a & d \end{pmatrix} = \begin{pmatrix} 11 & a \\ c & b \end{pmatrix}$$

find the values of the constants a, b, c and d

[4 marks]

Question 4
Find the value of k such that,

$$2k^2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 4k \end{pmatrix} = \begin{pmatrix} k \\ -9 \end{pmatrix}$$

[4 marks]

Question 5
Given that,

$$\begin{pmatrix} p & 0 & 0 \\ 0 & q^2 & r \\ 0 & 0 & 5 \end{pmatrix} - k\begin{pmatrix} 2q & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

find the value of k and the positive constants, p, q and r

[4 marks]

Did you know ?

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ and is called the 3×3 Identity matrix.

It's the matrix equivalent of the number 1. The 2×2 version is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Definition of the Determinant

Given the generalised 2×2 square matrix, **M**, where,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant **M** is given by,

$$|\mathbf{M}| = ad - bc$$

This can also be written *det* **M** or Δ**M**

- If $|\mathbf{M}| = 0$ then **M** is a singular matrix
- If $|\mathbf{M}| \neq 0$ then **M** is a non-singular matrix

Question 6

Given that $\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$, find *det* **A**

[1 mark]

Question 7

The matrix $\mathbf{A} = \begin{pmatrix} -2 & k \\ 5 & 3 \end{pmatrix}$ has $|\mathbf{A}| = 24$. Find the value of *k*.

[2 marks]

Question 8

Given that the 2×2 matrix $\mathbf{A} = \begin{pmatrix} 3 & (k+3) \\ (k-2) & 8 \end{pmatrix}$ is singular, find the two possible values of the real number *k*.

[3 marks]