Lesson 5

Further A-Level Pure Mathematics : Core 1

Matrix Transformations

5.1 Composite Transformations

Let $\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ This is the matrix for reflection in the *x*-axis

Let $\mathbf{D} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ This is the matrix, reflection in the diagonal line $y = -x$

Next is determined **DX** and, separately, **XD.**

XD	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

DX	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

The Commutative Law

Notice that $\mathbf{XD} \neq \mathbf{DX}$: These two matrices do not obey The Commutative Law.
In general, matrices do not commute, although there are special cases where they do.

Also notice that the answers to the multiplications are recognizable.

$\mathbf{XD} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ This is the matrix for rotation of 90° about the origin.

$\mathbf{DX} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ This is the matrix for rotation of 270° about the origin.

Getting The Order of Operations Correct

By way of working towards a concrete example, suppose now that a parallelogram, **P**, is defined by the points (5, 0), (5, 2), (3, 3) and (3, 1).

$$\mathbf{P} = \begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

From what is already know of composite functions, it should not be a surprise that **XDP** means to **P**, first do **D**, then do **X**.
This is like $fg(x)$ meaning put x into function g first, then that result into function f.

The Associative Law

With matrices, The Associative Law holds; it does not matter how the matrices in **XDP** are grouped. In other words, (**XD**)**P** = **X**(**DP**)

Advantage

The advantage of working out **XD** first rather than **DP** is that a general result was obtained; that reflection in the line $y = -x$ followed by reflection in the *x*-axis is equivalent to a rotation of 90° about the origin.

Disadvantage

The disadvantage of working out **XD** first rather than **DP** is that the interim position of the parallelogram is not found. So, next, this alternative method will be worked through for both **XDP** and **DXP** and diagrams obtained showing, step-by-step, the composite transformations.

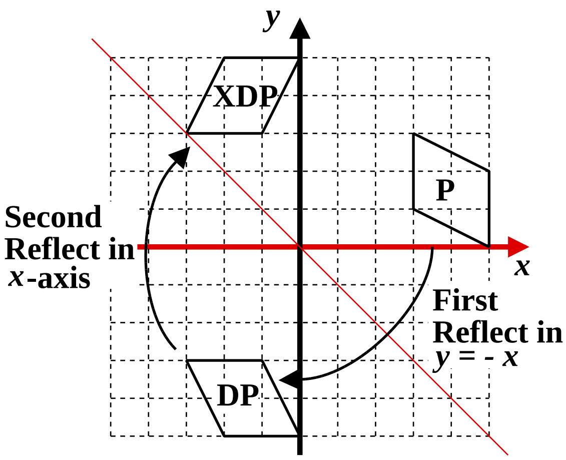
X(DP)

First : Reflect in $y = -x$

DP	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$

Second : Reflect in the x-axis

X(DP)	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 & -3 & -1 \\ 5 & 5 & 3 & 3 \end{pmatrix}$



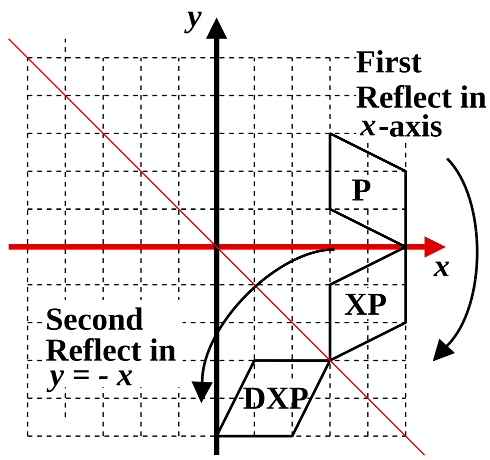
D(XP)

First : Reflect in the x-axis

XP	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & -2 & -3 & -1 \end{pmatrix}$

Second : Reflect in $y = -x$

D(XP)	$\begin{pmatrix} 5 & 5 & 3 & 3 \\ 0 & -2 & -3 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 3 & 1 \\ -5 & -5 & -3 & -3 \end{pmatrix}$



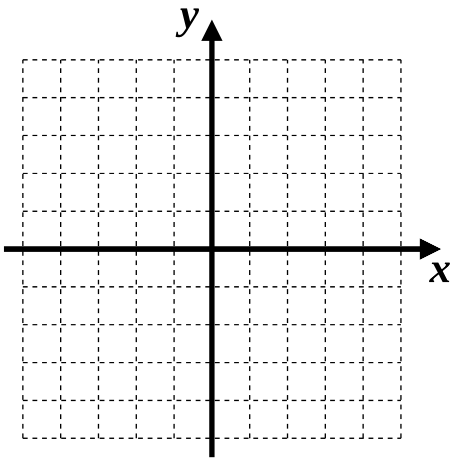
5.2 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(i) Plot the arrow shape **A** described by the multipoint matrix,

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & -2 & -4 \\ 0 & -4 & -3 & -4 \end{pmatrix}$$



[2 marks]

(ii) Transform **A** using the matrix $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and plot the result, **RA**.

[3 marks]

(iii) Transform **RA** using the matrix $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and plot the result, **MRA**.

[3 marks]

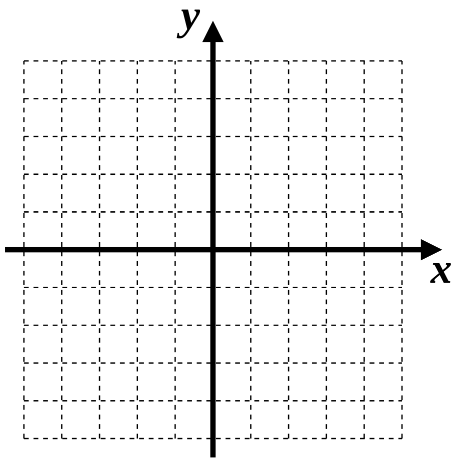
(iv) Calculate **MR** and hence or otherwise state what single transformation would have transformed **A** to **MRA**.

[4 marks]

Question 2

(i) Plot the shape described by the multipoint matrix,

$$\mathbf{P} = \begin{pmatrix} 0 & 2 & 0 & -2 \\ 0 & 0 & -4 & -4 \end{pmatrix}$$



[2 marks]

(ii) Transform **P** using the matrix $\mathbf{S} = \begin{pmatrix} -1 & 1 \\ 1 & -0.5 \end{pmatrix}$ and plot the result, **SP**

[3 marks]

(iii) Transform **SP** using the matrix $\mathbf{T} = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$ and plot the result, **TSP**

[3 marks]

(iv) Calculate **TS**, and hence or otherwise state what single transformation would have transformed **P** to **TSP**.

[4 marks]

Question 3

Further A-Level Examination Question from January 2013, FP1, Q4 (Edexcel)

The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation of $90°$ anticlockwise about the origin.

(a) Write down the matrix **P**

[1 mark]

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(b) Write down the matrix **Q**

[1 mark]

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

(c) express **R** in terms of **P** and **Q**

[1 mark]

(d) find the matrix **R**

[2 marks]

(e) give a full geometric description of T as a single transformation

[2 marks]

Question 4

Further A-Level Examination Question from January 2012, FP1, Q4 (Edexcel)

A right angled triangle T has vertices $A(1, 1)$, $B(2, 1)$ and $C(2, 4)$.

When T is transformed by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ the image is T'

(a) Find the coordinates of the vertices of T'

[2 marks]

(b) Describe fully the transformation represented by **P**

[2 marks]

The matrices $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ represent two transformations

When T is transformed by the matrix **QR**, the image is T''

(c) Find **QR**

[2 marks]

(d) Find the determinant of **QR**

[2 marks]

(e) Using your answer to part (d), find the area of T''

[3 marks]

Question 5

Further A-Level Examination Question from June 2011, FP1, Q5 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix} \text{ where } a \text{ and } b \text{ are constants}$$

Given that the matrix **A** maps the point with coordinates (4, 6) onto the point with coordinates (2, – 8),

(a) find the value of a and the value of b

[4 marks]

A quadrilateral R has area 30 square units

It is transformed into another quadrilateral S by the matrix **A**

Using your values of a and b

(b) find the area of quadrilateral S

[4 marks]