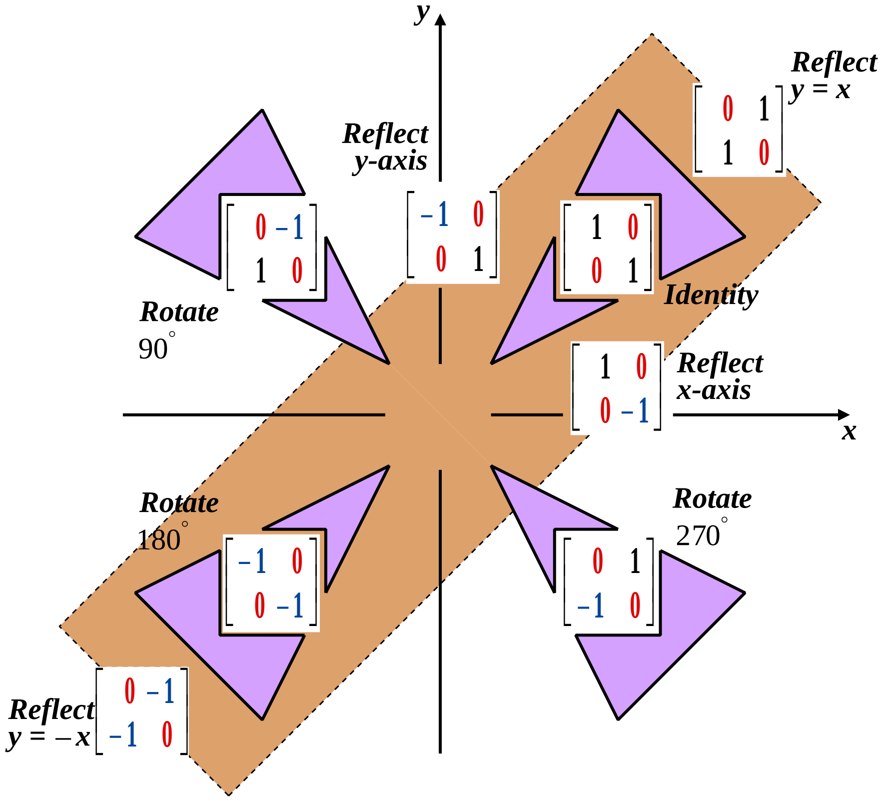
Lesson 7

7.1 Congruent Shapes

In general a matrix can represent a transformation that moves points around. Of more interest is the movement of a set of connected points; a shape. Sometimes a shape is savagely distorted by the transformation represented by the matrix. Thus far, of particular interest are matrices that give rise to transformations that preserve both size and shape. In other words, transformations in which the image is congruent with the pre-image.



A useful summary of some matrices which represent transformations that preserve both the size and shape of a pre-image and its image. The idea is that the zeros of the matrices lie along zero-lines and so allow the diagram to be memorised.

7.2 A Calculation Technique

The rotation matrices above, $\mathbf{R}_0$, $\mathbf{R}_{90}$, $\mathbf{R}_{180}$ and $\mathbf{R}_{270}$ are special cases of,

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

With an angle such as 45° the $\sqrt{2}$ involved can be factorised out to give,

$$\mathrm{R}_{45} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which may help to keep subsequent multiplications compact and easier to follow. The angle θ can, and often is, expressed in radians.

7.3 Stretches and Enlargements

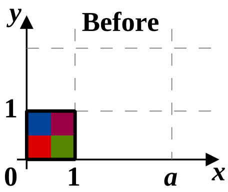
In starting to probe the distortions that a linear transformation can do to a shape the world of the stretch is entered.
The image is no longer congruent with the pre-image.

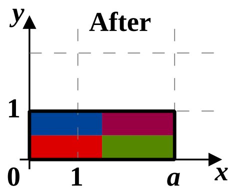
Parallel to *x*-axis

A stretch, (length) scale factor a parallel to the x-axis (only) will have matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$



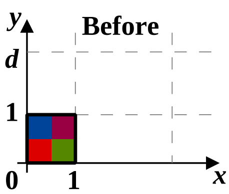


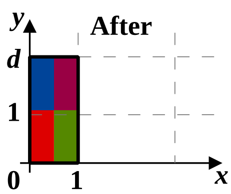
Parallel to *y*-axis

A stretch. (length) scale factor d parallel to the y-axis (only) will have matrix $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & d & d \end{pmatrix} \end{array}$$



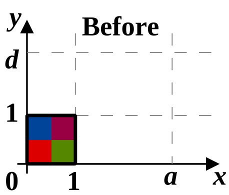


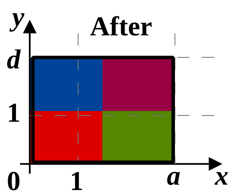
Parallel to both *x*-axis and *y*-axis

A stretch, (length scale factor a parallel to the x-axis, and (length) scale factor d parallel to the y-axis will have matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$

Applying this to the unit square, **U**, gives,

$$\begin{array}{c|c} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\ \hline \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} & \begin{pmatrix} 0 & a & a & 0 \\ 0 & 0 & d & d \end{pmatrix} \end{array}$$





Enlargement : A Special Case of Parallel to both *x*-axis and *y*-axis

In the special case where $a = d = k$, $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ becomes $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ which represents an enlargement with (length) scale factor k centre (0, 0)

7.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) Describe fully the transformation represented by $\mathbf{M} = \begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$

[3 marks]

(ii) A triangle T has vertices at (2, 0), (0, 3) and (7, 0)
Find the area of the triangle.

[3 marks]

(iii) The triangle T is transformed using the matrix **M**.
Show how to use the determinant of **M** to find the area of the image of T.

[3 marks]

Question 2

(i) Describe fully the transformation represented by $\mathbf{A} = \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix}$

[3 marks]

(ii) A triangle T has coordinates (a, 1), (4, 1) and (4, 3).
Given that T is transformed using matrix A, and the area of the resulting triangle is 60, find the two possible values of a

[4 marks]

Question 3

Further A-Level Examination Question from June 2011, FP1, Q3 (Edexcel)

(a) Given that $\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$

(i) find $\mathbf{A}^2$

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$

[4 marks]

(b) Given that $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ describe fully the geometrical transformation represented by $\mathbf{B}$

[2 marks]

(c) Given that $\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$ where k is a constant, find the value of k for which the matrix $\mathbf{C}$ is singular.

[3 marks]

Question 4

Further A-Level Examination Question from May 2018, F1, Q2 (Edexcel)

The transformation represented by the 2×2 matrix **P** is an anticlockwise rotation about the origin through 45 degrees.

(a) Write down the matrix **P**, giving the exact numerical value of each element.

[1 mark]

$$\mathbf{Q} = \begin{pmatrix} k\sqrt{2} & 0 \\ 0 & k\sqrt{2} \end{pmatrix} \text{ where } k \text{ is a constant and } k > 0$$

(b) Describe fully the single geometrical transformation represented by the matrix **Q**

[2 marks]

The combined transformation represented by the matrix **PQ** transforms the rhombus R_1 onto the rhombus R_2

The area of the rhombus R_1 is 6 and the area of the rhombus R_2 is 147

(c) Find the value of the constant k

[4 marks]

Question 5

Further A-Level Examination Question from June 2010, FP1, Q6 (Edexcel)

Write down the 2×2 matrix that represents,

(a) an enlargement with centre (0, 0) and scale factor 8

[1 mark]

(b) a reflection in the x-axis

[1 mark]

Hence, or otherwise,

(c) find the matrix **T** that represents an enlargement with centre (0, 0) and scale factor 8 followed by reflection in the x-axis

[2 marks]

$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix}$ where k and c are constants

(d) Find **AB**

[3 marks]

Given that **AB** represents the same transformation as **T**

(e) find the value of k and the value of c

[2 marks]

Question 6

Further A-Level Examination Question from January 2019, F1, Q7 (Edexcel)

$$\mathbf{P} = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

(a) Show that $\mathbf{P}^3 = 8\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix

[3 marks]

(b) Describe fully the transformation represented by the matrix **P** as a combination of two simple geometrical transformations,

[4 marks]

(c) Find the matrix $\mathbf{P}^{35}$, giving your answer in the form

$$\mathbf{P}^{35} = 2^k \begin{pmatrix} -1 & a \\ b & -1 \end{pmatrix}$$

where k is an integer and a and b are surds to be found.

[2 marks]