Lesson 8

Further A-Level Pure Mathematics : Core 1
Matrix Transformations

8.1 Singularities: The Implications

In Lesson 1, the definition of the determinant for a 2×2 matrix was given,

---

Definition of the Determinant

Given the generalised 2×2 square matrix, **M**, where,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant **M** is given by,

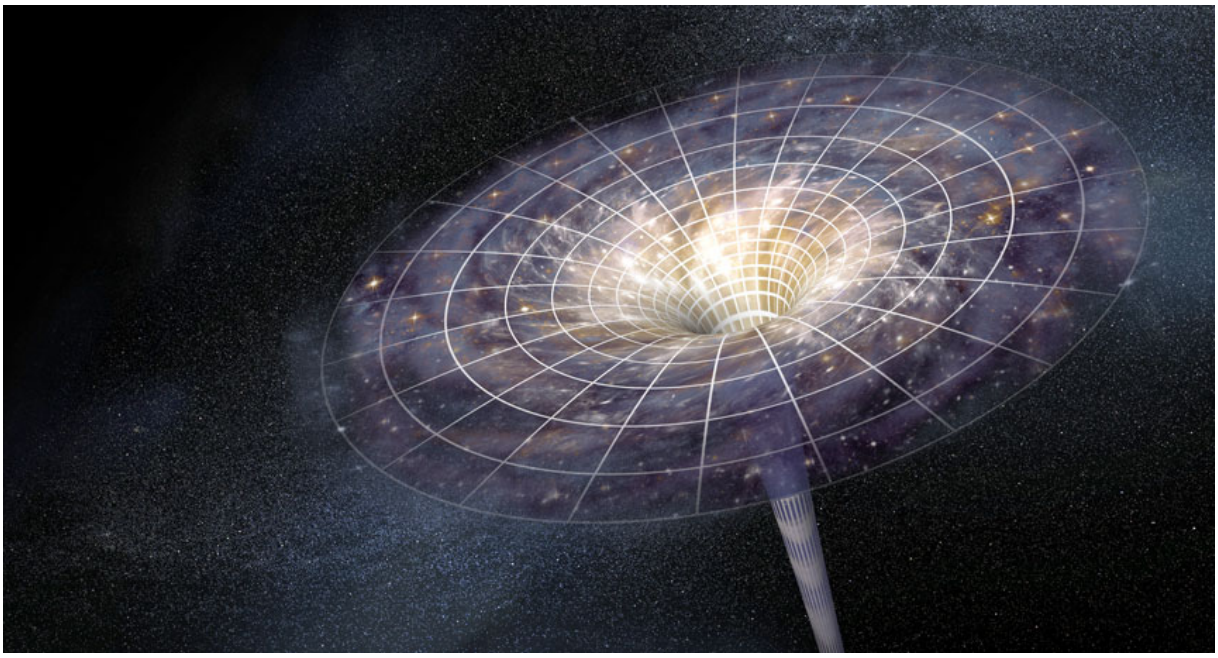
$$|\mathbf{M}| = ad - bc$$

This can also be written *det* **M** or $\Delta\mathbf{M}$

- If $|\mathbf{M}| = 0$ then **M** is a singular matrix
- If $|\mathbf{M}| \neq 0$ then **M** is a non-singular matrix

---

The magnitude of a matrix's determinant gives the Area Scale Factor of the transformation represented by the matrix.



A singularity is generally seen as a place to avoid. Mathematically they typically correspond to a division by zero. As a singularity is approached savage distortions occur and the singularity itself corresponds to a collapse of rules and structure, a loss of information, and an explosion of contradictions.
Singularities occur in nature; a black hole being the obvious example. The extreme curvature of time and space around a black hole mean that the standard rules of physics cease to apply.

8.2 Exploring A Singularity

Consider the matrix $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$

$$\begin{aligned} det\,\mathbf{M} &= 1 \times 2 - (-1) \times (-2) \\ &= 0 \end{aligned}$$

This shows that **M** is a singular matrix.

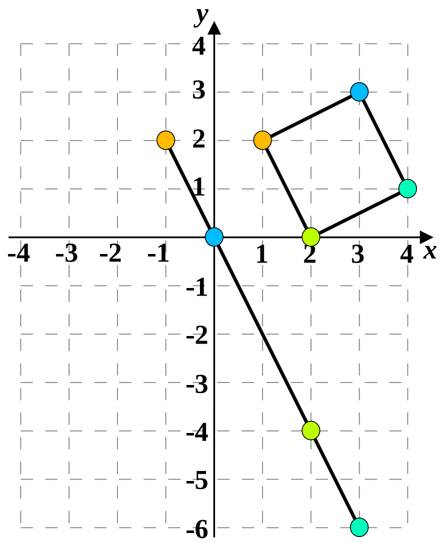
To explore what this means geometrically it will be applied to a test shape, the square represented by the multipoint matrix,

$$\mathbf{S} = \begin{pmatrix} 2 & 4 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

MS	$\begin{pmatrix} 2 & 4 & 3 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$
$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 0 & -1 \\ -4 & -6 & 0 & 2 \end{pmatrix}$

$$\therefore\ \mathbf{MS} = \begin{pmatrix} 2 & 3 & 0 & -1 \\ -4 & -6 & 0 & 2 \end{pmatrix}$$

When the pre-image square is plotted and its image, it's clear that all the points of the image are on a straight line. In one respect this is not unexpected for the determinant has indeed given the Area Scale Factor of the transformation of zero.



Were we just unlucky in our choice of test shape ?
Will all points be moved onto this line ?
A more general argument is needed to answer these questions.

For a general pre-image point (u, v), and image point (x, y)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$= \begin{pmatrix} u - v \\ -2u + 2v \end{pmatrix}$$

$$= \begin{pmatrix} u - v \\ -2(u - v) \end{pmatrix}$$

Let $p = u - v$ to get the result that,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -2p \end{pmatrix}$$

Which are points on the line $y = -2x$

All points under the transformation represented by this matrix, have images on the line $y = -2x$. The two dimensional surface collapses into a one dimensional line !

The situation is even worse than has so far been revealed.

Consider any point on the line $y = x$. Such a point could be written as (k, k).

Applying **M** to this point,

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} k - k \\ -2k + 2k \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This is revealing that all of the points on the line $y = x$ map to the origin.

This in turn means that any hope of undoing the transformation has been extinguished for it could not be decided where on the line $y = x$ a point with image (0, 0) had come from. The singularity has caused information to be irretrievably lost.

Naked Singularity by Mark Tedin

8.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1

(i) Consider any point on the line $y = x + 1$

Such a generalised point could be written as (k, k + 1)

Apply matrix **M** to this generalised point where **M** is the singularity matrix investigated in 8.2 , namely,

$$\mathbf{M} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$

[2 marks]

(ii) To what single point are all points on the line $y = x + 1$ mapped under the transformation associated with matrix **M** ?

[1 mark]

(iii) Prove that all points on the line $y = x - 2$ are mapped to a single point under the transformation associated with matrix **M**.

Also, state the coordinates of that single point.

[3 marks]

Question 2

The two matrices, **M** and **N**, are,

$$\mathbf{M} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} -1 & k \\ 4 & 3 \end{pmatrix} \text{ where } k \text{ is a constant}$$

(a) Evaluate the determinant of **M**

[1 mark]

(b) Given that the determinant of **N** is 7, find the value of *k*

[1 mark]

(c) Using the value of *k* found in part (b), find **MN**

[1 mark]

(d) Verify that

$$det\,\mathbf{MN} = det\,\mathbf{M} \times det\,\mathbf{N}$$

[1 mark]

(e) Does,

$$det\,(\,\mathbf{M} + \mathbf{N}\,) = det\,\mathbf{M} + det\,\mathbf{N} \qquad ?$$

Justify your answer.

[2 marks]

Question 3

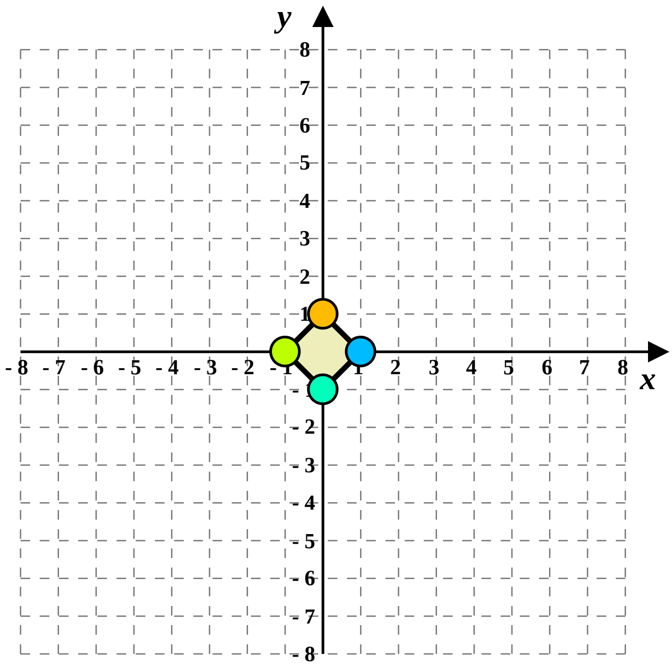
(i) Which one of the following matrices is singular ?

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 6 & 4 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 8 & 4 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 3 & 4 \\ 7 & 8 \end{pmatrix}$$

Justify your answer.

[2 marks]

(ii) Apply the singular matrix from part **(i)** to transform the square below, and plot the resulting points and the line through them.



[3 marks]

(iii) Write down the equation of the line along which all image points lie.

[1 mark]

(iv) Prove that a general point in the plane, (u, v) when transformed using your part (i) matrix will lie on the straight line of your your part (ii) answer.

[3 marks]

(v) Prove that under the transformation represented by your part (i) matrix all points on the line $y = -2x + 3$ map onto the same point, and state the coordinates of that point.

[3 marks]

(vi) Write down the equation of another line along which all points will have the same image when transformed using the part (i) matrix.
For the example you give, state the coordinates of the image point to which all pre-images points will be mapped.

[3 marks]

Question 4

Using only 1s and 0s find as many 2×2 singular matrices as you can.
There are ten to be found.

[5 marks]

Question 5

Given that k is a real number and that,

$$\mathbf{M} = \begin{pmatrix} -2 & 1 - k \\ k - 1 & k \end{pmatrix}$$

find the exact values of k for which M is a singular matrix.

[3 marks]

Question 6

By using the two general 2×2 matrices,

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{N} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Prove that,

$$det\,\mathbf{MN} = det\,\mathbf{M} \times det\,\mathbf{N}$$

[5 marks]