Further Pure A-Level Mathematics
Compulsory Course Component
Core 1

SERIES
~ AND ~
VISUAL PROOF

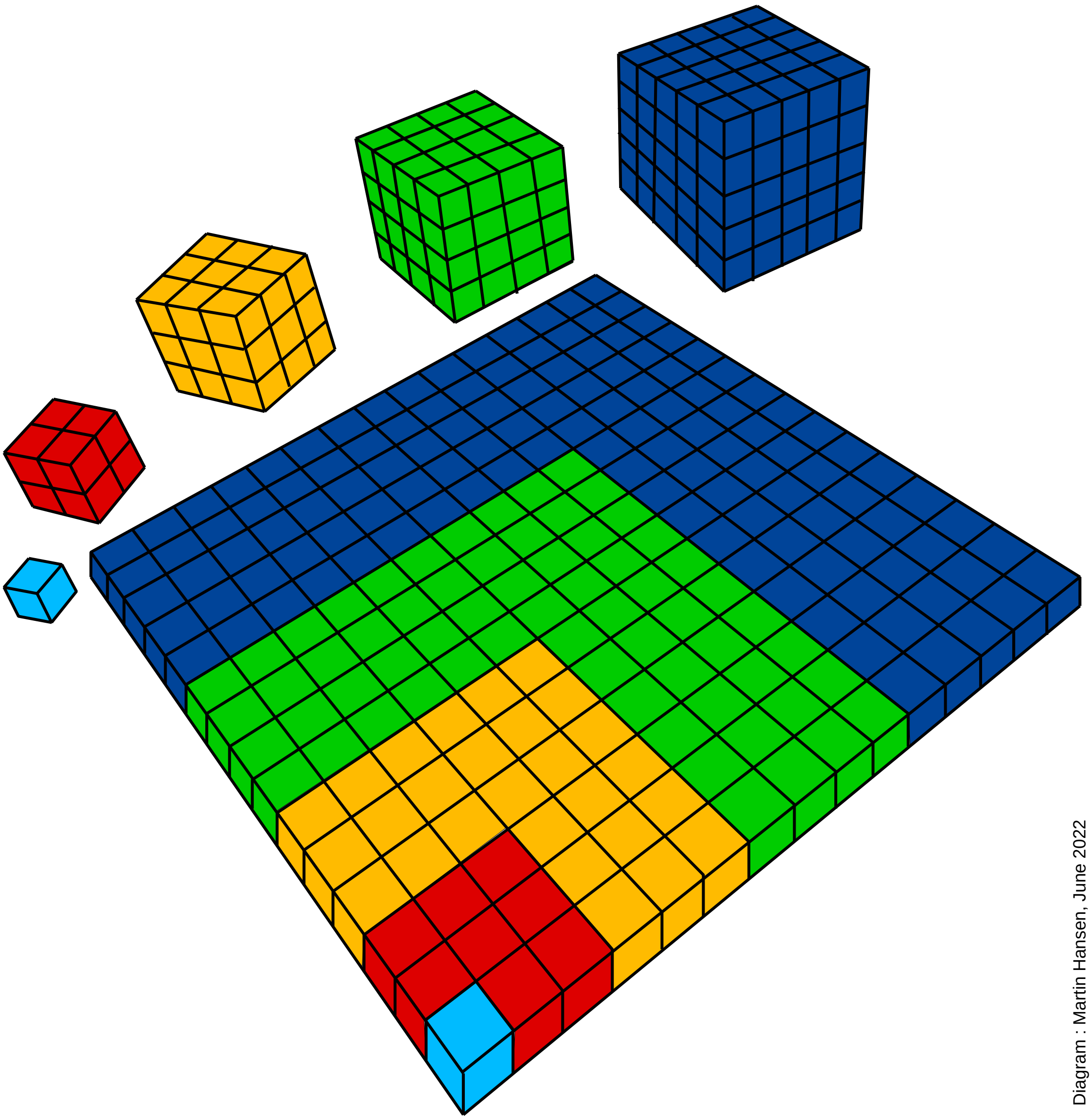


Diagram : Martin Hansen, June 2022

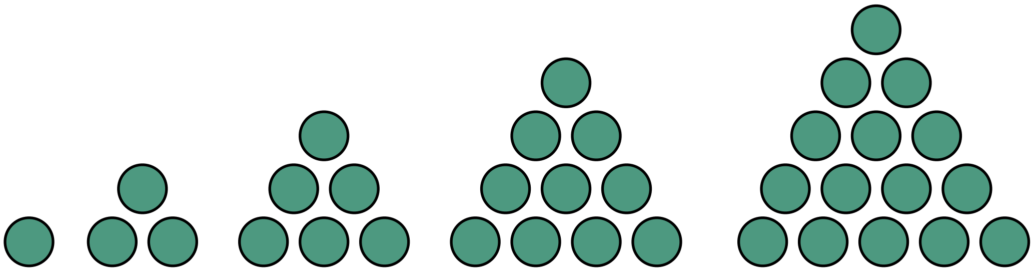
SERIES
AND
VISUAL PROOF

Lesson 1

Further A-Level Pure Mathematics : Core 1
Series and Visual Proof

1.1 Triangular Numbers

The triangular numbers are so called because the start of the sequence, which begins 1, 3, 6, 10, 15, ... can be visualised as dots arranged in equilateral triangles like so;



The sixth triangular number can be described in sigma notation as $\sum_{1}^{6} r$ and this can be found easily enough without drawing the next diagram as,

$$\sum_{1}^{6} r = 1 + 2 + 3 + 4 + 5 + 6$$

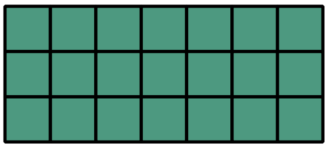
$$= 21$$

Notice that an item in a sequence has been found from a series.

Sequence and Series

Sequence : A list of numbers, repetitions allowed, where order matters.
Series : The summation of a sequence.
To remember which is which : "S, e, q, u, e, n, c, e and S + e + r + i + e + s"

When working with the Natural Numbers† it is useful to associate abutted unit squares with the numbers, such that, for example, 3×7, is associated with a rectangle of height 3 and width 7. The area of the rectangle, 21, is then both the answer to the multiplication, and the number of squares in the rectangle.



† The natural numbers, $\mathbb{N}$, are the set { 1, 2, 3, 4, 5, 6, 7, ... }
Notice that in the A-Level course $\mathbb{N}$ does not include zero, by definition.

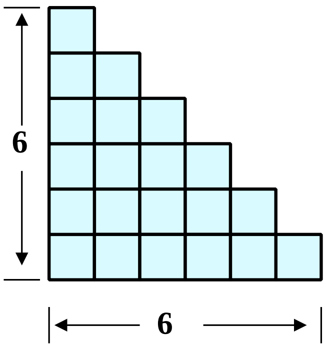
Formula for the n^{th} Triangular Number

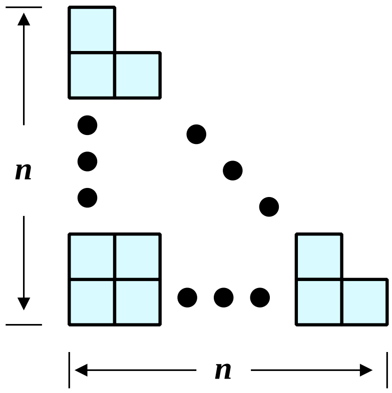
$$\sum_{1}^{n} r = \frac{1}{2} n (n + 1)$$

Proof

Watch the following teaching video then, in the space below, make use of the diagrams provided to write out the proof described.

Teaching Video : http://www.NumberWonder.co.uk/v9092/1.mp4





[6 marks]

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 54

Question 1

Use the appropriate formula to determine the 100^{th} triangular number, $\sum_{1}^{100} r$

[2 marks]

Question 2

Evaluate, $\sum_{1}^{200} r$

[2 marks]

Question 3

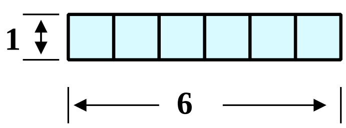
Find the sum of the first 500 natural numbers.

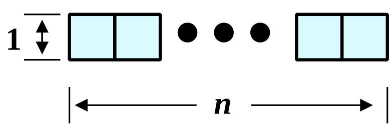
[2 marks]

Question 4

Use the diagram below to visually prove that,

(i) $\sum_{1}^{6} 1 = 6$ **(ii)** $\sum_{1}^{n} 1 = n$





[4 marks]

Question 5

Evaluate **(i)** $\sum_{1}^{50} 1$ **(ii)** $5 \sum_{1}^{23} 1$

[2 marks]

Question 6

(i) By first writing out all six terms in the series, determine the value of,

$$\sum_{1}^{6} (3r - 2)$$

[2 marks]

(ii) By using the appropriate formula determine,

$$3 \sum_{1}^{6} r - 2 \sum_{1}^{6} 1$$

[2 marks]

Question 7

By expanding $\sum_{1}^{14} (4r + 5)$ as $4 \sum_{1}^{14} r + 5 \sum_{1}^{14} 1$ and using appropriate formula, evaluate,

$$\sum_{1}^{14} (4r + 5)$$

[4 marks]

Question 8

Determine the value of $\sum_{1}^{30} (2r + 7)$

[4 marks]

Question 9

By expanding $\sum_{4}^{7} r$ as $\sum_{1}^{7} r - \sum_{1}^{3} r$ and using appropriate formula, evaluate,

$$\sum_{4}^{7} r$$

[4 marks]

Question 10

Determine the value of $\frac{1}{100} \sum_{80}^{120} r$

[4 marks]

Question 11

(i) Show that,

$$\sum_{r=1}^{n} (7r - 4) = \frac{1}{2} n (7n - 1)$$

[4 marks]

(ii) Hence evaluate,

$$\sum_{r=20}^{50} (7r - 4)$$

[4 marks]

Question 12

Given that $\sum_{r=1}^{n} r = 528$, find the value of n

[4 marks]

Question 13

Let the n^{th} triangular number be $T_n = \sum_{1}^{n} r$

(i) Using algebra, prove that $T_n - T_{n-1} = n$

[4 marks]

(ii) By drawing triangles, find a visual proof that $T_n - T_{n-1} = n$

[6 marks]