Lesson 2

Further A-Level Pure Mathematics : Core 1

Series and Visual Proof

2.1 Formalising Series Manipulations

Here is a formal statement of the rules that were used without being explicitly stated in Lesson 1,

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Non Start From 1

To find the sum of a series that does not start at $r = 1$,

$$\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$$

Factor Extractor

$$\sum_{r=1}^{n} k\,f(r) = k \sum_{r=1}^{n} f(r) \qquad \text{where } k \text{ is a constant}$$

The Linearity Property

$$\sum_{r=1}^{n} \big(f(r) + g(r) \big) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

2.2 Example

The following question makes use of all of the above rules,

Given that $\displaystyle\sum_{r=n}^{2n} (15 - 2r) = 0$, find n

Teaching Video : http://www.NumberWonder.co.uk/v9092/2a.mp4
http://www.NumberWonder.co.uk/v9092/2b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, writing out your own version of the initial steps below.
The part 1 video invites you to complete the solution before watching Part 2.

Given that $\sum_{r=n}^{2n} (15 - 2r) = 0$, find n

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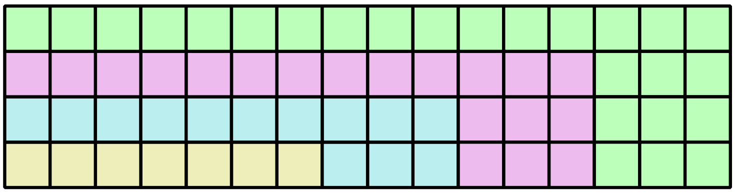
[6 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1



The diagram suggests the pattern,

$$7 = 1 \times 7 \quad : \quad n = 1$$

$$7 + 13 = 2 \times 10 \quad : \quad n = 2$$

$$7 + 13 + 19 = 3 \times 13 \quad : \quad n = 3$$

$$7 + 13 + 19 + 25 = 4 \times 16 \quad : \quad n = 4$$

(i) Write down the next line of the pattern corresponding to $n = 5$

[1 mark]

The pattern suggests that,

$$\sum_{r=1}^{n} (6r + 1) = n(3n + 4)$$

(ii) Prove the suggested result by using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

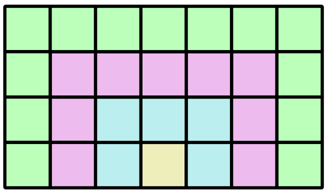
The start of the proof is given below,

$$\text{LHS} = \sum_{r=1}^{n} (6r + 1)$$

$$=$$

[4 marks]

Question 2



The diagram suggests the pattern,

$$1 = 1 \times 1 \qquad : \qquad n = 1$$

$$1 + 5 = 2 \times 3 \qquad : \qquad n = 2$$

$$1 + 5 + 9 = 3 \times 5 \qquad : \qquad n = 3$$

$$1 + 5 + 9 + 13 = 4 \times 7 \qquad : \qquad n = 4$$

(i) Write down the next line of the pattern corresponding to $n = 5$

[1 mark]

(ii) The pattern suggests a relationship of the form,

$$\sum_{r=1}^{n} (ar + b) = n (cn + d)$$

Write down the suggested relationship with the integer constants a, b, c and d replaced with their numerical values.

[2 marks]

(iii) Prove the suggested result by using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

[4 marks]

Question 3

Prove $\sum_{r=1}^{2n}(5r-4) = n(10n-3)$ using the standard results for $\sum_{1}^{n} 1$ and $\sum_{1}^{n} r$

[4 marks]

Question 4

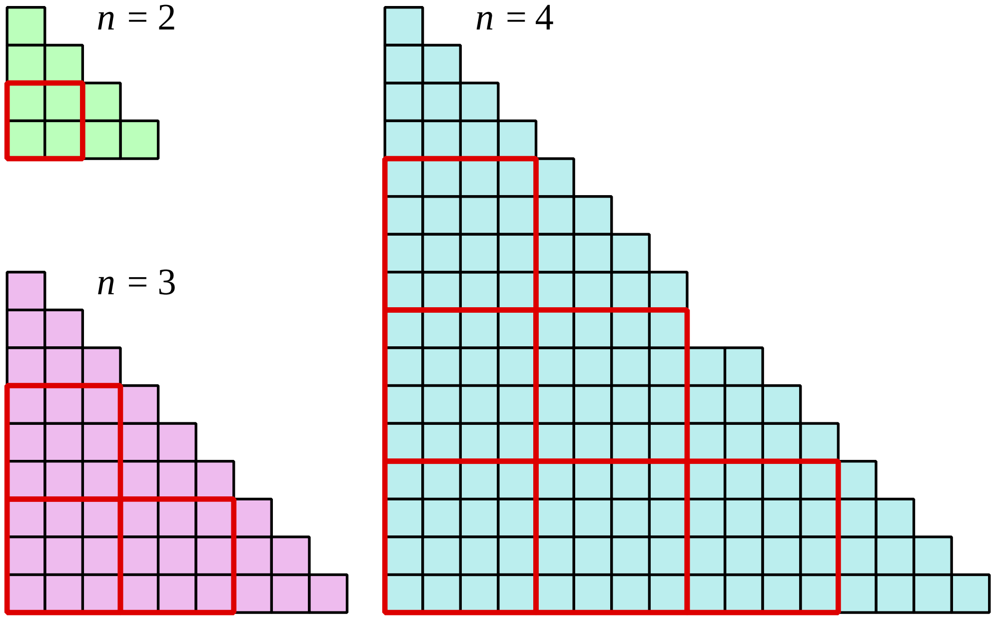
(i) Find an expression for $\sum_{r=1}^{2n-1} r$

[3 marks]

(ii) Hence show that $\sum_{r=n+1}^{2n-1} r = \frac{3}{2}n(n-1)$, for $n \geqslant 2$

[3 marks]

Question 5



The diagrams suggest the following relationship between triangular numbers,

$$T_{n^2} = n^2 T_{n-1} + n T_n \qquad n \geqslant 2$$

The first two lines of a proof are given below. Complete the proof.

$$\begin{aligned} \text{RHS} &= n^2 T_{n-1} + n T_n \\ &= n^2 \sum_{1}^{n-1} r + n \sum_{1}^{n} r \\ &= \end{aligned}$$

[5 marks]

Question 6

Arithmetic Progressions
Algebraically, an Arithmetic Progression is a number sequence of the form;

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots, \quad a+(n-1)\,d$$

where a is the initial term,
d is the common difference,
and n is number of terms.

The n^{th} term, L_n, is given by $L_n = a + (n-1)\,d \qquad n \geqslant 1$

The sum of a AP is given by, $S_n = \dfrac{n}{2}\{a + L\} \qquad n \geqslant 1$

In words this can be remembered as :

"n times the average of the first and last terms"

From substituting the first formula into the second, another formula is obtained for the sum of an AP. It is, $S_n = \dfrac{n}{2}\{2a + (n-1)\,d\}, \qquad n \geqslant 1$

Observe that $\displaystyle\sum_{r=1}^{n} r$ is an Arithmetic Progression.

For this arithmetic progression,

(i) What is the the first term ?

[1 mark]

(ii) What is the common difference

[1 mark]

(iii) Hence show how the formula for the sum of an Arithmetic Progression can be used to derive a formula for $\displaystyle\sum_{r=1}^{n} r$

[3 marks]

Question 7

Given that $\sum_{r=n}^{3n} (80 - 3r) = 54$, find n

[8 marks]