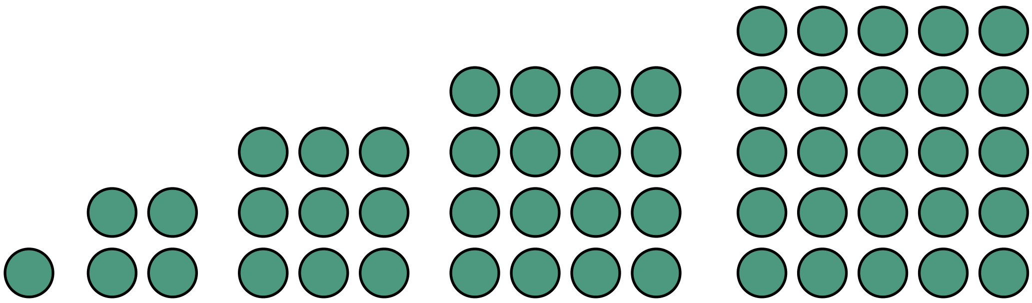
3.1 Square Numbers

The square numbers are so called because the start of the sequence, which begins 1, 4, 9, 16, 25, ... can be visualised as dots arranged in squares like so;

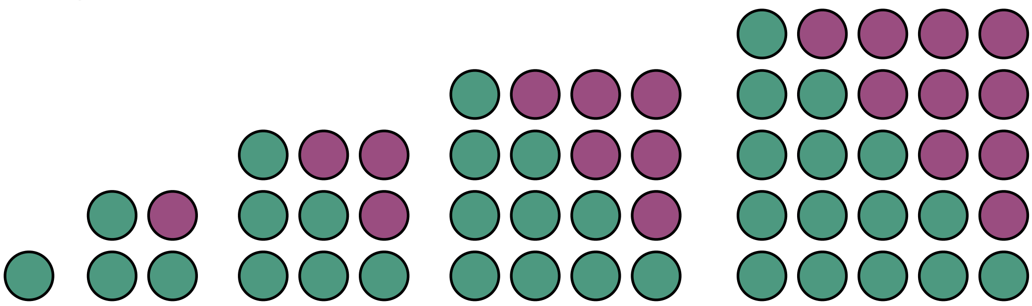


The sixth square number, $\square_6$, can be described in sigma notation as $\left(\sum_1^6 r + \sum_1^5 r \right)$

Here is the calculation,

$$\begin{aligned}
\square_6 &= T_6 + T_5 \\
&= \sum_1^6 r + \sum_1^5 r \\
&= 1 + 2 + 3 + 4 + 5 + 6 + 1 + 2 + 3 + 4 + 5 \\
&= 36
\end{aligned}$$

As a method of working out that 6^2 is 36, it's unnecessarily complicated, although the real interest is that it suggests that any square number, $\square_n$ is the sum of the two triangular numbers, $T_n + T_{n-1}$



The diagram shows visually how any given square number can be constructed from two triangular numbers.

$$\square_n = T_n + T_{n-1}$$

To absolutely confirm the relationship, most mathematicians would also like an algebraic proof.

Theorem

$$\square_n = T_n + T_{n-1}$$

Proof

$$\begin{aligned}
\text{RHS} &= T_n + T_{n-1} \\
&= \sum_{1}^{n} r + \sum_{1}^{n-1} r \\
&= \frac{1}{2} n (n + 1) + \frac{1}{2} (n - 1) n \\
&= \left(\frac{1}{2} n \right) \left(n + 1 + n - 1 \right) \\
&= \frac{1}{2} n (2n) \\
&= n^2 \\
&= \text{LHS} \qquad \square
\end{aligned}$$

3.2 Sum of Squares

Of more interest than the squares themselves, is the idea of summing the squares.
In other words, of considering the series,

$$\sum_{1}^{n} r^2 = 1 + 4 + 9 + 16 + 25 + \dots + n^2$$

This has a formula which is worth memorising.
It can be used as a quotable result in examinations.

The Sum of Squares Formula

$$\sum_{1}^{n} r^2 = \frac{1}{6} n (n + 1) (2n + 1)$$

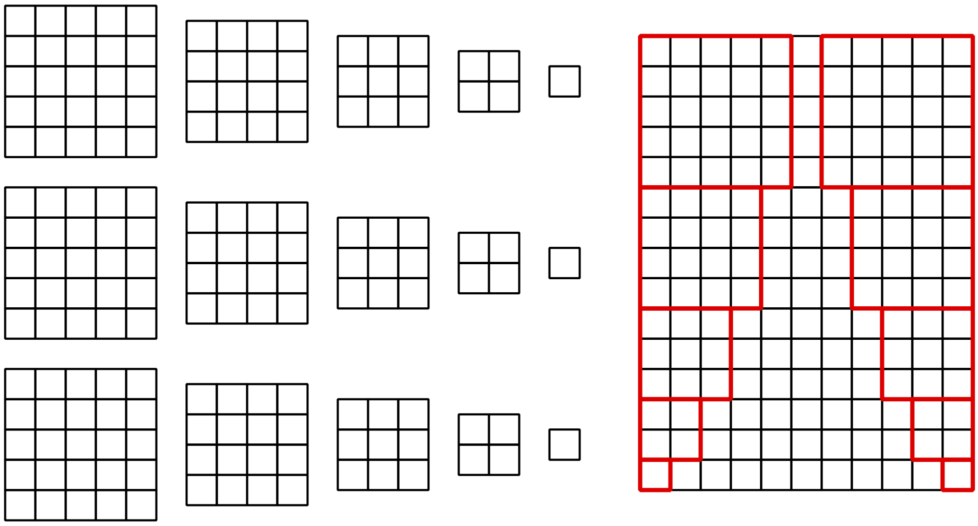
The formula is has a beautiful and clever visual proof.

Teaching video : http://www.NumberWonder.co.uk/v9092/3a.mp4
http://www.NumberWonder.co.uk/v9092/3b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, which sets up the system of partitioning one of the three sets of “sum of squares” and transfers the parts into the right hand rectangle. Colour the diagrams below to show the partitioning and transfer.

$n = 5$



The Part 1 video invites you to complete the proof before watching Part 2.

☞

[6 marks]

3.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1

Evaluate,

(i) $\sum_{1}^{10} r^2$ **(ii)** $\sum_{r=25}^{50} r^2$

[4 marks]

Question 2

(i) Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6} n\ (2n + 1)\ (7n + 1)$

[4 marks]

(ii) Hence evaluate $\sum_{r=11}^{20} r^2$

[3 marks]

Question 3

(i) Show that $\sum_{r=1}^{n}\left(r^2 + r - 2\right) = \frac{1}{3}n\,(n+4)\,(n-1)$

[4 marks]

(ii) Hence find the sum of the series $4 + 10 + 18 + 28 + 40 + ... + 418$

[3 marks]

Question 4

Further A-Level Examination Question from January 2014, F1, Q5 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that,

$$\sum_{r=1}^{n} \left(9r^2 - 4r \right) = \frac{1}{2} n (n + 1) (6n - 1)$$

for all positive integers n

[4 marks]

Given that $\sum_{r=1}^{12} \left(9r^2 - 4r + k \left(2^r \right) \right) = 6630$

(b) find the exact value of the constant k

[4 marks]

Question 5

Further A-Level Examination Question from January 2019, F1, Q3 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n,

$$\sum_{r=1}^{n} (2r + 5)^2 = \frac{n}{3}\left[(an + b)^2 + c \right]$$

where a, b and c are integers to be found.

[5 marks]

(b) Use the answer to part (a) to evaluate $\sum_{r=0}^{100} (2r + 5)^2$

[2 marks]

Question 6

(i) Find and prove a formula for the sum of the first n triangular numbers.

[5 marks]

(ii) Explain the connection between part (i) and the following photograph,

[2 marks]