Lesson 4

Further A-Level Pure Mathematics : Core 1
Series and Visual Proof

4.1 Cubic Numbers

The cubic numbers are so called because the start of the sequence, which begins 1, 8, 27, 64, 125, ... can be visualised as unit cubes arranged in large cubes like so;

4.2 Sum of Cubes

The obvious question is "What is the formula for summing the first n cubes" ?
In other words, of considering the series,

$$\sum_{1}^{n} r^3 = 1 + 8 + 27 + 64 + 125 + \ldots + n^3$$

This has a formula which is worth memorising.
It can be used as a quotable result in examinations.

The Sum of Cubes Formula

$$\sum_{1}^{n} r^3 = \frac{1}{4} n^2 (n + 1)^2$$

Notice that it is the formula for a triangular number, squared.
That is,

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r \right)^2$$

4.3 Proof

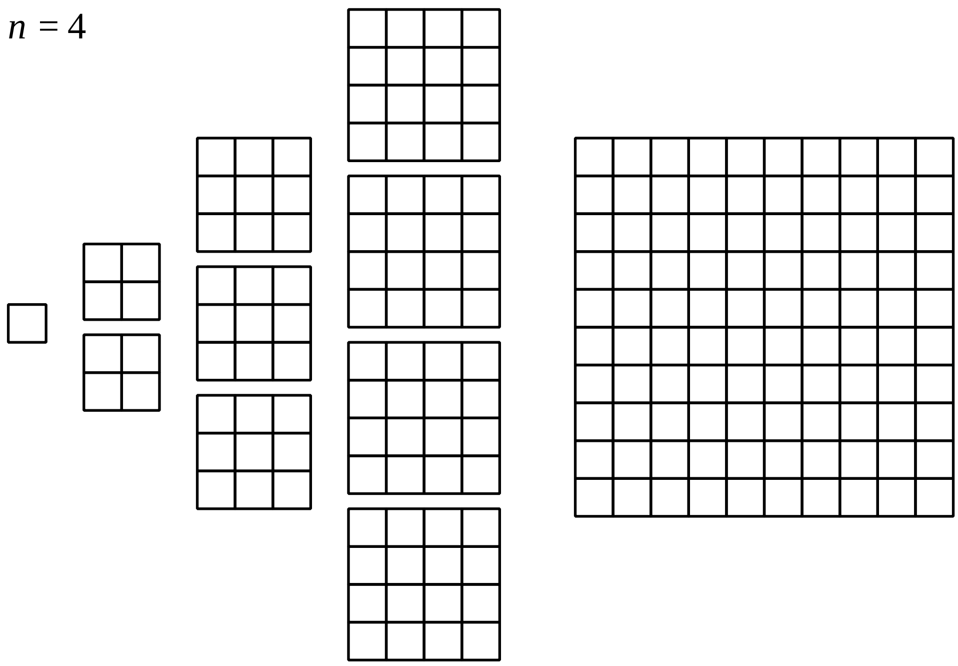
The idea behind the visual proof is to take the cubes and slice them up.
The slices are then arranged into a square.

Teaching Video : http://www.NumberWonder.co.uk/v9092/4a.mp4
http://www.NumberWonder.co.uk/v9092/4b.mp4

<= Part 1 Part 2 =>

Watch the Part 1 video, which takes the sliced up cubes and starts to transfer the slices into the right hand rectangle. Colour the diagrams below to show the transfer. You are invited to try to complete the proof before watching Part 2.

$n = 4$



[6 marks]

4.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable.

Marks available : 40

Question 1

Evaluate,

(i) $\sum_{1}^{12} r^3$ **(ii)** $\sum_{r=50}^{75} r^3$

[4 marks]

Question 2

(i) Show that $\sum_{r=n+1}^{3n} r^3 = n^2 (4n + 1) (5n + 2)$

[4 marks]

(ii) Hence evaluate $\sum_{r=11}^{30} r^3$

[3 marks]

Question 3

(i) Show that $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n+1)(3n+2)(n-1)$

[4 marks]

(ii) Hence find the sum of the series $4 + 18 + 48 + 100 + ... + 3840$

[3 marks]

Question 4

Further A-Level Examination Question from June 2012. IAL, FP1, Q4 (Edexcel)

(a) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that,

$$\sum_{r=1}^{n} \left(r^3 + 6r - 3 \right) = \frac{1}{4} n^2 \left(n^2 + 2n + 13 \right)$$

for all positive integers *n*.

[5 marks]

(b) Hence find the exact value of $\sum_{r=16}^{30} \left(r^3 + 6r - 3 \right)$

[2 marks]

Question 5

Further A-Level Examination Question from June 2014. IAL, FP1(R), Q5

(a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that,

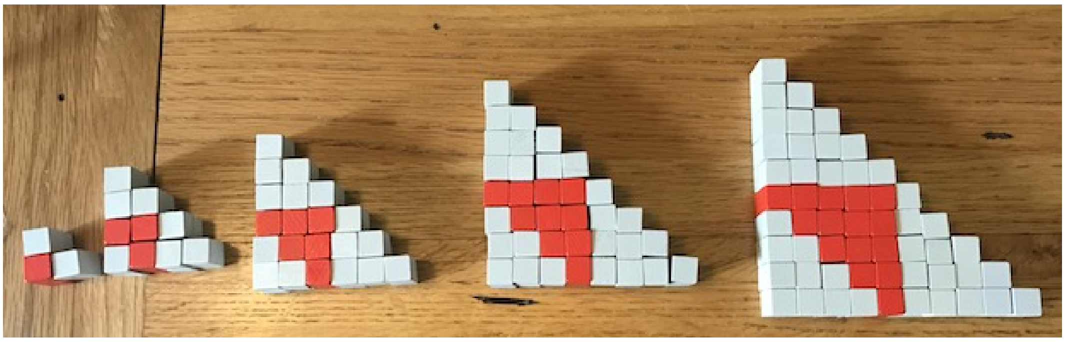
$$\sum_{r=1}^{n} r\left(r^2 - 3\right) = \frac{1}{4} n(n+1)(n+3)(n-2)$$

[5 marks]

(b) Calculate the value of $\sum_{r=10}^{50} r\left(r^2 - 3\right)$

[3 marks]

Question 6



The pattern suggests an iterative relationship between triangular numbers.

(i) Which one of the following is the relationship suggested ?

(a) $4T_n = T_{2n}$

(b) $T_{n+1} + 3T_n = T_{2n}$

(c) $3T_n + T_{n-1} = T_{2n}$

(d) $4T_n - T_{n-1} = T_{2n}$

[2 marks]

(ii) Prove the relationship using algebra and sigma notation

[5 marks]