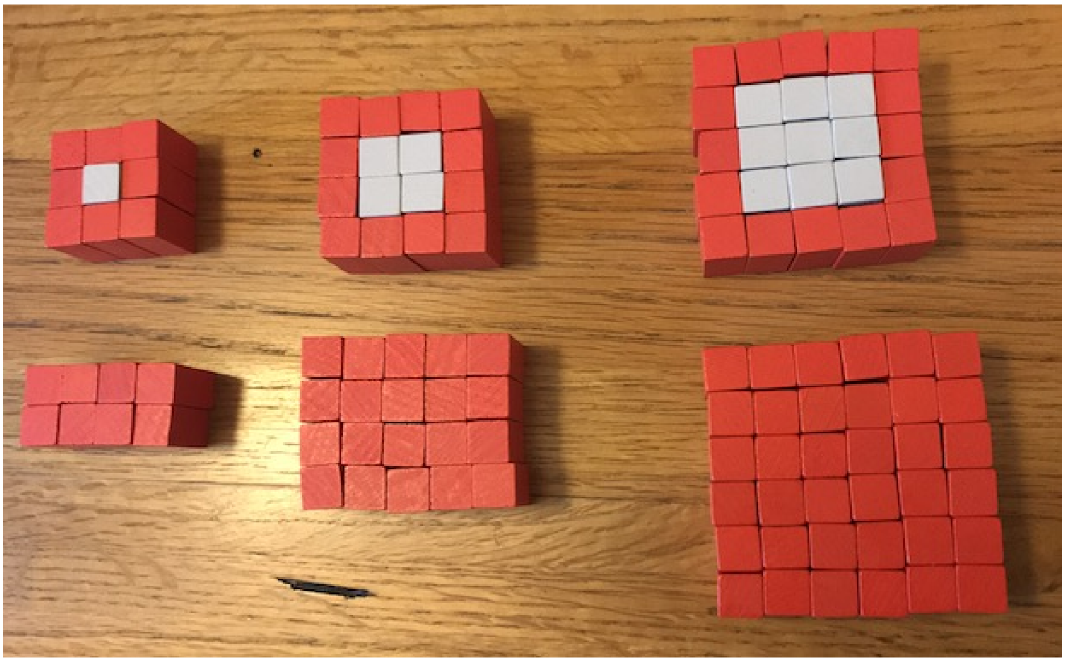
5.1 Pattern Spotting

Knowledge of the standard results for $\sum_1^n r^3$, $\sum_1^n r^2$, $\sum_1^n r$ and $\sum_1^n 1$ allows an efficient means of proving suspected results involving series.



For example, in the above photograph the framed square sequence is depicted.
The upper portion of the photograph shows 3×3, 4×4 and 5×5 framed squares.
The individual squares are actually small unit cubes because it's not possible in the physical world to have an area without depth. As the interested is in the area of the uppermost face of each cube, the cubes will frequently be called squares !
The squares forming the frames are red, those not forming the frame, white.

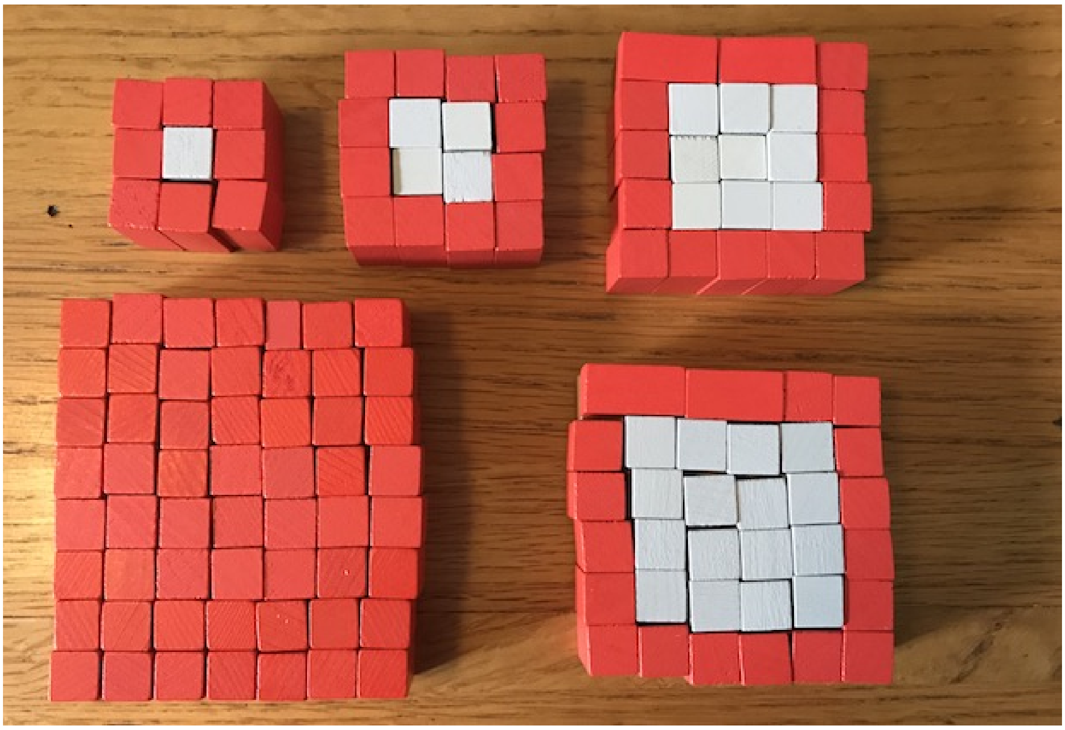
The upper left 3×3 framed square has a frame of eight squares.
These eight squares have been arranged into a 2×4 rectangle, lower left.

The upper left 3×3 plus the upper middle 4×4 frame squares contain a total of twenty red framing squares, these arranged as a 4×5 rectangle, lower middle.

Finally, the frames of all three framed squares provides thirty-six red squares and the lower right arrangement shows these as a 6×6 square.

Two questions; **(a)** What is the suspected pattern ?
(b) How might that pattern's existence be confirmed ?

5.2 Enter the Mathematics



It's often wise to obtain one or two more terms in the emerging pattern to check that what is suspected continues to hold. The above photograph shows the next "all red" rectangle in the sequence and the four framed squares it came from.

Tables helps clarify observations. This first records the number of red squares in each term. The formula for the n^{th} term arises from noticing that the number of red squares in any given term is given by subtracting a (large white) square number from the corresponding (large red plus white) square number. The red+white large square has a side length two more than the corresponding large white square.

Term	1	2	3	4	...	n
Boundary	8	12	16	20	...	$(n+2)^2 - n^2$

This "cumulative" table gives the dimensions of the "all red" rectangles,

Term	1	2	3	4	...	n
Cumulative	8	20	36	56	...	
Rectangle	2×4	4×5	6×6	8×7	...	$(2n)(n+3)$

After this considerable amount of setting up, a conjecture can be put forward,

$$\sum_{1}^{n} \left((r+2)^2 - r^2 \right) = 2n(n+3)$$

And now, at last, a proof can be worked on.

5.3 The Proof

$$\sum_{1}^{n}\left((r+2)^2 - r^2 \right) = 2n\,(n+3)$$

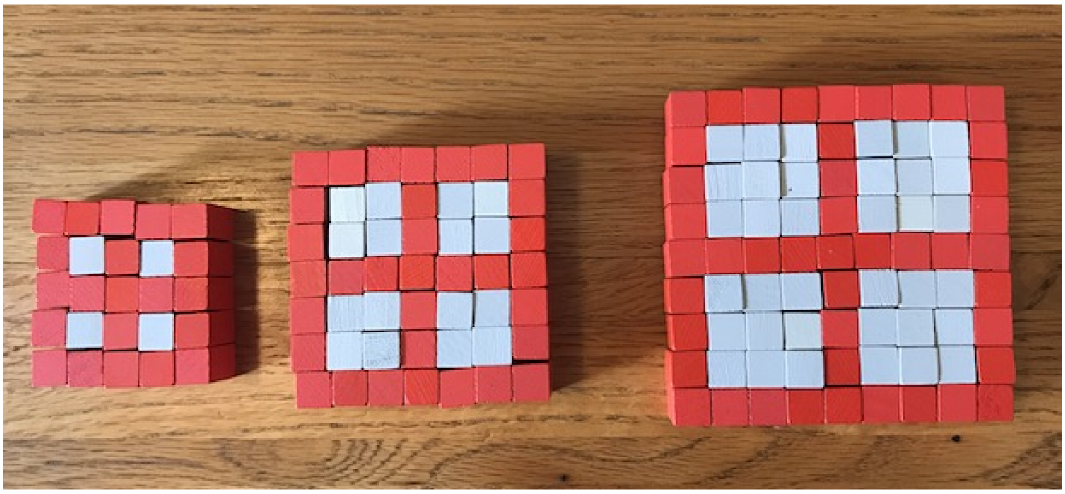
$$\begin{aligned}
\text{LHS} &= \sum_{1}^{n}\left((r+2)^2 - r^2 \right) \\
&= \sum_{1}^{n}\left(r^2 + 4r + 4 - r^2 \right) \\
&= \sum_{1}^{n}\left(4r + 4 \right) \\
&= 4\sum_{1}^{n} r + 4\sum_{1}^{n} 1 \\
&= 4 \times \frac{1}{2}\, n\,(n+1) + 4n \\
&= \left(2n\right)\left(n + 1 + 2\right) \\
&= 2n\,(n+3) \\
&= \text{RHS} \qquad \square
\end{aligned}$$

This shows that an “all red” rectangle can always be formed measuring (2*n*) by (*n* + 3).

5.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable.
Marks Available : 40

Question 1



The first three members of the “windows” sequence are shown in the photograph.

(i) Complete the following table so show the number of red squares in each diagram. (The squares are actually unit cubes)

(ii) Add a formula for the n^{th} term.

Term	1	2	3	4	...	n
Red Frame		33		57	...	

[6 marks]

(iii) In the manner of the introductory example, complete the following “Cumulative Reds” table for the sequence of “all red” rectangles that could be formed along with the dimensions of those rectangles, chosen to form a sequence.

(iv) Complete the formula for the dimensions of the n^{th} “all red ”rectangle

Term	1	2	3	4	...	n
Cumulative		54			...	
Rectangle		6×9			...	$(3n)\ (\quad\quad)$

[6 marks]

(v) Write down a conjecture for the relationship between the two tables.

[3 marks]

(vi) Prove your conjecture.

[4 marks]

Question 2

(i) Show that,

$$\sum_{1}^{n} (4r - 3)^2 = \frac{1}{3} n \left(16 n^2 - 12n - 1 \right)$$

[4 marks]

(ii) Show that,

$$\sum_{2}^{n} (4r - 5)^2 = \frac{1}{3} n \left(16 n^2 - 36n + 23 \right) - 1$$

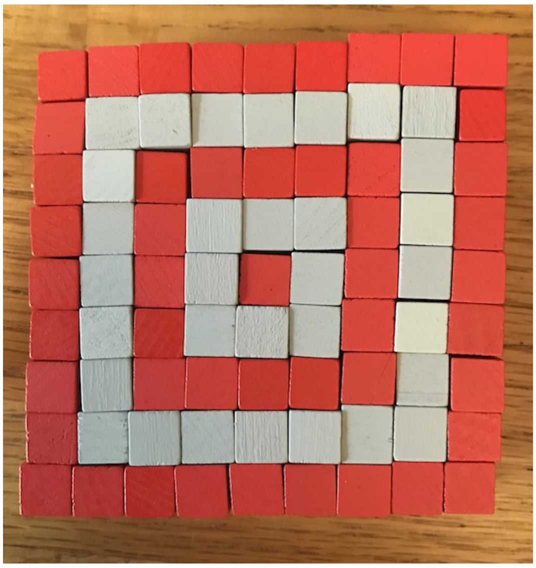
[4 marks]

(iii) Evaluate.

$$\sum_{1}^{3}(4r-3)^2-\sum_{2}^{3}(4r-5)^2$$

[2 marks]

(iv)



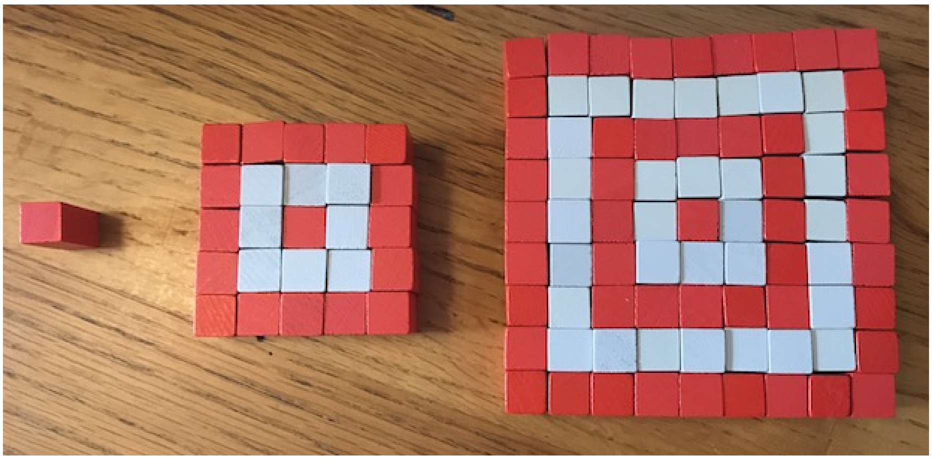
Explain why your part (iii) answer gives the number of red squares in the "concentric square frames" photograph.

[3 marks]

(v) Use your part **(i)** and **(ii)** answers to show that,

$$\sum_{1}^{n} (4r - 3)^2 - \sum_{2}^{n} (4r - 5)^2 = 8n^2 - 8n + 1$$

[3 marks]



(vi) Find an expression for the cumulative number of red squares in all the terms in the "concentric square frames" sequence up to term n and hence show that when $n = 5$ the red squares, actually unit cubes, could build a cuboid measuring $5 \times 5 \times 13$.

[5 marks]