6.1 Test Preparation

Gerolamo Cardano's first step in solving the general cubic equation was to make a substitution that turned it into a depressed cubic equation.

Depressed Cubic Formation Rule

Faced with a general cubic,

$$a\,x^3 + b\,x^2 + cx + d = 0 \qquad\qquad a,\ b,\ c,\ d \in \mathbb{C}$$

initiate a change of variable by replacing x with $t - \dfrac{b}{3\,a}$

This will always result in what is termed a depressed cubic, one of the form,

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

The depressed cubic equation has some useful properties one of which shall now be studied, as it has a beautiful proof.

6.2 Sum of Cubes for a Depressed Cubic

Sum of Cubes for a Depressed Cubic

For a depressed cubic.

$$t^3 + pt + q = 0 \qquad\qquad p,\ q \in \mathbb{C}$$

with roots α, β and γ

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

Proof

As there is no term in t^2, $\alpha + \beta + \gamma = 0$

$$\therefore\ \alpha + \beta = -\gamma$$

Cube both sides

$$(\alpha + \beta)^3 = -\gamma^3$$

$$\alpha^3 + 3\,\alpha^2\beta + 3\alpha\,\beta^2 + \beta^3 = -\gamma^3$$

$$\alpha^3 + \beta^3 + \gamma^3 = -3\alpha\beta(\alpha + \beta)$$

$$\alpha^3 + \beta^3 + \gamma^3 = -3\alpha\beta(-\gamma)$$

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma \qquad \square$$

6.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable. Make the method used clear.

Marks available : 50

Question 1

The roots of the quadratic equation $a x^2 + bx + c = 0$ are $\dfrac{-3 \pm 4\mathrm{i}}{2}$

Find integer values for *a*, *b* and *c*

[3 marks]

Question 2

The roots of the equation $a x^4 + 7x^3 + 5x^2 + 3x - 4 = 0$ are α, β, γ and δ

(a) Given that $\alpha\beta\gamma\delta = -1$, write down the value of a

[1 mark]

(b) $\sum \alpha\beta$ is (sloppy) shorthand for "sum of the product pairs of roots".

For a quartic $\sum \alpha\beta = \alpha(\beta + \gamma + \delta) + \beta(\gamma + \delta) + \gamma(\delta)$

$= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

Write down the values of $\sum \alpha$, $\sum \alpha\beta$ and $\sum \alpha\beta\gamma$

[3 marks]

(c) Hence find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

[3 marks]

Question 3

Further A-Level Examination Question from May 2018, IAL, F1, Q7 (Edexcel)

It is given that α and β are roots of the equation $5x^2 - 4x + 3 = 0$

Without solving the equation,

(a) find the exact value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

[5 marks]

(b) find a quadratic equation which has roots $\frac{3}{\alpha^2}$ and $\frac{3}{\beta^2}$ giving your answer in the form $ax^2 + bx + c = 0$, where a, b and c are integers.

[4 marks]

Question 4

(i) For the general cubic, $a\,x^3 + b\,x^2 + cx + d = 0$, write down from memory the formula for $\alpha^3 + \beta^3 + \gamma^3$

[2 marks]

(ii) Show that your part (i) formula yields the same expression as proven in section "6.2 Sum of Cubes for a Depressed Cubic" when $b = 0$ in the general cubic equation.

[2 marks]

(iii) Consider the cubic equation,

$$x^3 + 3\,x^2 + 50 = 0$$

Using a suitable substitution, transform this into a depressed cubic.

[2 marks]

(iv) By thinking of $\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$ as $P^3 + Q^3 + R^3$ and using the same formula as in part (i) derive a formula for $\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3$ with component parts that are either $\alpha + \beta + \gamma$ or $\alpha\beta + \beta\gamma + \gamma\alpha$ or $\alpha\beta\gamma$.

[4 marks]

(v) Let α, β and γ be the roots of your part (iii) depressed cubic. Without solving the equation find a cubic with roots, α^3, β^3 and γ^3 in the form $w^3 + pw^2 + qw + r$ where p, q and r are integers.

[6 marks]

Question 5

$$f(x) = mx^3 + 12x^2 + 4x + 16 \qquad \text{where } m \text{ is a real constant}$$

Given that,

- f has at least one root on the imaginary axis

Solve completely,

$$f(x) = 0$$

[7 marks]

Question 6

Prove that the roots of $x^3 + px + qx + r = 0$ form an Arithmetic Progression if and only if $2p^3 + 27r = 9pq$

[8 marks]