

## Lesson 8

### Further A-Level Pure Mathematics : Core 1 Roots of Polynomials

#### 8.1 Extension Material

In order to solve a general cubic equation, Cardano first reduced it to depressed cubic form,

$$t^3 + pt + q = 0$$

This provides a motivation to look at the algebra of the roots of this equation which has the special property of  $\alpha + \beta + \gamma = 0$

The general result that,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

becomes,

$$\alpha^2 + \beta^2 + \gamma^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

and another general result that,

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 &= (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma \end{aligned}$$

becomes

$$\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

In this extension lesson a further couple of formulae are derived and then a question given in which applying them may be practiced.

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#### A Cubic's Sum of Squares of Product Pairs of Roots

If  $\alpha, \beta$  and  $\gamma$  are roots of  $ax^3 + bx^2 + cx + d = 0$        $a, b, c, d \in \mathbb{C}$

then,  $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

For a depressed cubic,

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

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#### Proof

$$\begin{aligned} P^2 + Q^2 + R^2 &= (P + Q + R)^2 - 2(PQ + QR + RP) \\ (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta) \\ &= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \quad \square \end{aligned}$$

For a depressed cubic,  $\alpha + \beta + \gamma = 0$

$$\therefore \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 \quad \square$$

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### A Cubic's Sum of Cubes of Product Pairs of Roots

If  $\alpha, \beta$  and  $\gamma$  are roots of  $ax^3 + bx^2 + cx + d = 0$        $a, b, c, d \in \mathbb{C}$

then,

$$\begin{aligned} \alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 \\ = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2 \end{aligned}$$

For a depressed cubic,

$$\alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^3$$

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#### Proof

$$P^3 + Q^3 + R^3 = (P + Q + R)^3 - 3(P + Q + R)(PQ + QR + RP) + 3PQR$$

$$\begin{aligned} (\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 \\ &\quad - 3(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha\beta^2\gamma + \beta\gamma^2\alpha + \gamma\alpha^2\beta) + 3(\alpha\beta)(\beta\gamma)(\gamma\alpha) \end{aligned}$$

$$\begin{aligned} \alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 &= (\alpha\beta + \beta\gamma + \gamma\alpha)^3 \\ &\quad - 3\alpha\beta\gamma(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) + 3(\alpha\beta\gamma)^2 \quad \square \end{aligned}$$

For a depressed cubic,  $\alpha + \beta + \gamma = 0$

$$\therefore \alpha^3 \beta^3 + \beta^3 \gamma^3 + \gamma^3 \alpha^3 = (\alpha\beta + \beta\gamma + \gamma\alpha)^3 + 3(\alpha\beta\gamma)^2 \quad \square$$

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### A Cubic's Sum of Quartics of Roots

If  $\alpha, \beta$  and  $\gamma$  are roots of  $ax^3 + bx^2 + cx + d = 0$        $a, b, c, d \in \mathbb{C}$

then,  $\alpha^4 + \beta^4 + \gamma^4$

$$= \left( (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2 \\ - 2 \left( (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \right)$$

For a depressed cubic,

$$\alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

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### Proof

$$P^2 + Q^2 + R^2 = (P + Q + R)^2 - 2(PQ + QR + RP)$$

$$(\alpha^2)^2 + (\beta^2)^2 + (\gamma^2)^2 = (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$$

$$\alpha^4 + \beta^4 + \gamma^4$$

$$= \left( (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2 - 2(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)$$

$$= \left( (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) \right)^2$$

$$- 2 \left( (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) \right) \quad \square$$

For a depressed cubic,  $\alpha + \beta + \gamma = 0$

$$\therefore \alpha^4 + \beta^4 + \gamma^4 = 2(\alpha\beta + \beta\gamma + \gamma\alpha)^2 \quad \square$$

## 8.2 Exercise

### Question 1

If  $\alpha$ ,  $\beta$  and  $\gamma$  are the roots of  $x^3 + x + 1 = 0$ , then, without solving the equation, find the equation whose roots are  $(\alpha - \beta)^2$ ,  $(\beta - \gamma)^2$  and  $(\gamma - \alpha)^2$

[ 8 marks ]

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Teachers may obtain detailed worked solutions to the exercises by email from [mhh@shrewsbury.org.uk](mailto:mhh@shrewsbury.org.uk)