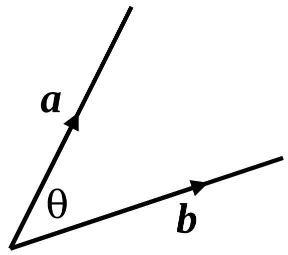
3.1 Mutually Perpendicular Vectors

The Scalar Product (Also called **The Dot Product**)

By definition : $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$



Where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the magnitudes of vectors ***a*** and ***b***

θ is the angle between vectors ***a*** and ***b***

Whenever angles are involved in a problem involving vectors, use the scalar product with the **DIRECTION PARTS** of the vector equations of the lines.

A consequence of the definition is that The Scalar Product is a very good detector of 90° angles, because if $\theta = 90°$, then $\cos\theta = 0$.

Mutually Perpendicular Vectors

- If two vectors are mutually perpendicular, their scalar product will be zero.
- If the scalar product is zero, vectors ***a*** and ***b*** are mutually perpendicular.

3.2 Example

Prove that the following two lines are mutually perpendicular;

$$\boldsymbol{r}_1 = \begin{pmatrix} -6 \\ 5 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 7 \\ -14 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 12 \\ 3 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ 12 \\ 8 \end{pmatrix}$$

[3 marks]

3.3 Proof of The Scalar Product Formula In Two Dimensions

Prove that for any two vectors;

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

it follows that;

$$a_1 b_1 + a_2 b_2 = \sqrt{a_1{}^2 + a_2{}^2}\,\sqrt{b_1{}^2 + b_2{}^2}\,\cos\theta$$

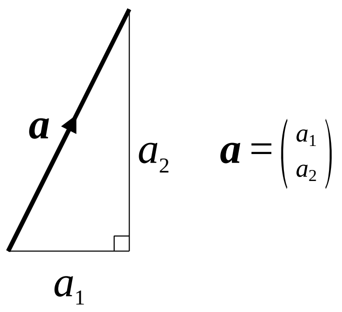
The Proof

By definition;

$$\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

Let

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ in which case } |\boldsymbol{a}| = \sqrt{a_1{}^2 + a_2{}^2}$$



Similarly

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ in which case } |\boldsymbol{b}| = \sqrt{b_1{}^2 + b_2{}^2}$$

Thus, from the definition

$$\begin{aligned} \boldsymbol{a} \bullet \boldsymbol{b} &= |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta \\ &= \sqrt{a_1{}^2 + a_2{}^2}\,\sqrt{b_1{}^2 + b_2{}^2}\,\cos\theta \end{aligned}$$

However,

$$\begin{aligned} \boldsymbol{a} \bullet \boldsymbol{b} &= (a_1\boldsymbol{i} + a_2\boldsymbol{j}) \bullet (b_1\boldsymbol{i} + b_2\boldsymbol{j}) \\ &= a_1b_1\,\boldsymbol{i} \bullet \boldsymbol{i} + a_1b_2\,\boldsymbol{i} \bullet \boldsymbol{j} + a_2b_1\,\boldsymbol{j} \bullet \boldsymbol{i} + a_2b_2\,\boldsymbol{j} \bullet \boldsymbol{j} \\ &= a_1b_1 + a_2b_2 \end{aligned}$$

because $\boldsymbol{i} \bullet \boldsymbol{i} = \boldsymbol{j} \bullet \boldsymbol{j} = 1$, and $\boldsymbol{i} \bullet \boldsymbol{j} = \boldsymbol{j} \bullet \boldsymbol{i} = 0$

Thus,

$$a_1 b_1 + a_2 b_2 = \sqrt{a_1{}^2 + a_2{}^2}\,\sqrt{b_1{}^2 + b_2{}^2}\,\cos\theta$$

□

3.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 32

Question 1

Show that the following two lines are mutually perpendicular;

$$\boldsymbol{r}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \\ -4 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 4 \\ 7 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 7 \\ -9 \\ 2 \end{pmatrix}$$

[3 marks]

Question 2

Given that the following two lines are mutually perpendicular, find the value of a

$$\boldsymbol{r}_1 = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ 4 \\ -3 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2a \\ 5 \end{pmatrix}$$

[3 marks]

Question 3

$$\boldsymbol{a} = p\,\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{k}$$

$$\boldsymbol{b} = 5\,\boldsymbol{i} - 3\boldsymbol{j} - 7\boldsymbol{k}$$

Find the value of p if $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular

[3 marks]

Question 4

A, B and C are three points in three dimensional space with coordinates;

$$A(6,\ 3,\ 7) \qquad B(-3,\ 8,\ 5) \qquad C(-4,\ -13,\ 12)$$

(a) Determine;

(i) $\overrightarrow{AB}$

[1 mark]

(ii) $\overrightarrow{AC}$

[1 mark]

(b) Show that $\angle BAC$ is a right angle

[2 marks]

(c) Hence, or otherwise, find the area of $\triangle ABC$

[3 marks]

Question 5

Determine;

$$\begin{pmatrix} 1 \\ m \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix}$$

[1 mark]

What well known GCSE result have you proved ?

[1 mark]

Question 6

Write down an equation of the vector line that passes through the points;

$$A(3,\ 2,\ -1) \qquad \text{and} \qquad B(5,\ 7,\ 3)$$

[2 marks]

Question 7

Write down a vector line equation for a line that passes through the point $4\,\boldsymbol{i} + \boldsymbol{j} - 2\,\boldsymbol{k}$ and which is parallel to the line with equation

$$\boldsymbol{r} = 3\,\boldsymbol{i} + 3\boldsymbol{j} - 5\,\boldsymbol{k} + \lambda(-2\,\boldsymbol{i} + 8\boldsymbol{j} + \boldsymbol{k})$$

[2 marks]

Question 8
Given that the vectors

$$\boldsymbol{u} = \begin{pmatrix} \lambda + 1 \\ 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{pmatrix} \lambda - 5 \\ \lambda \\ -11 \end{pmatrix}$$

are perpendicular, find the possible values of λ

[2 marks]

Question 9

This question is about finding a vector that is perpendicular to both of the lines

$$\boldsymbol{r}_1 = \begin{pmatrix} 10 \\ 2 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -2 \\ 7 \end{pmatrix}$$

Without loss of generality, assume the direction vector sought is of the form;

$$\boldsymbol{d} = \begin{pmatrix} 1 \\ P \\ Q \end{pmatrix}$$

(a) Obtain an equation, in P and Q, from the fact that;

$$\begin{pmatrix} 1 \\ P \\ Q \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 0$$

[1 mark]

(b) Obtain a second equation, in P and Q, from the fact that;

$$\begin{pmatrix} 1 \\ P \\ Q \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -2 \\ -7 \end{pmatrix} = 0$$

[1 mark]

(c) Solve the two equations simultaneously to determine the value of P and the value of Q and hence give the required direction vector which is perpendicular to both of the lines $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

[2 marks]

Question 10

Find a vector that is perpendicular to both of the lines

$$\boldsymbol{r}_1 = \begin{pmatrix} -4 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_2 = \begin{pmatrix} 8 \\ -7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -9 \\ 7 \end{pmatrix}$$

[4 marks]