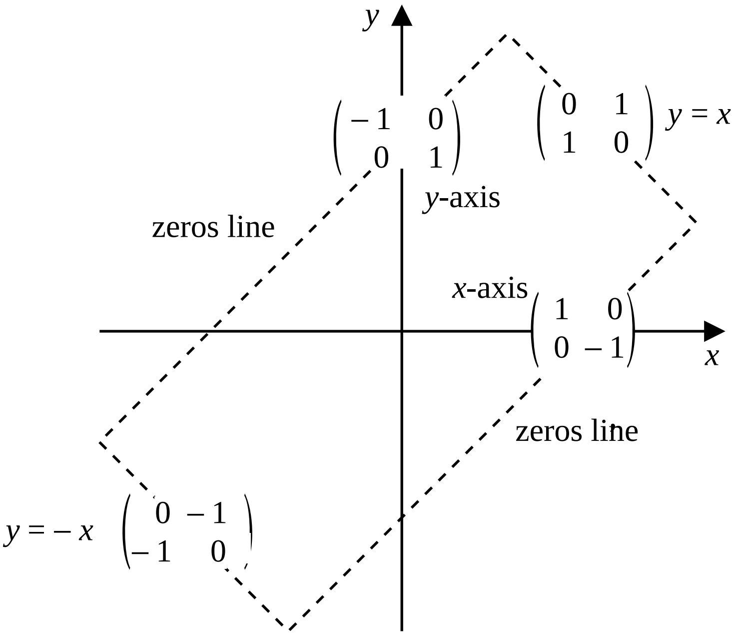
Lesson 10

Further A-Level Pure Mathematics : Core 1
Matrix Systems of Equations

10.1 Reflections In Two and Three Dimensions

Consider the following diagram. It is an aid to remembering the two dimensional reflection matrices for reflections in the lines $y = x$, $y = -x$, the x-axis and y-axis.



$$\mathbf{M}_{y=x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{M}_{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{M}_{y=-x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{M}_{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These matrices are special cases of the following general result,

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Reflection in the mirror line through (0, 0) at angle θ

$$\mathbf{M}_{\theta} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

The following video briefly shows one way of proving this result, leaving it to the interested viewer to properly work through the steps shown.

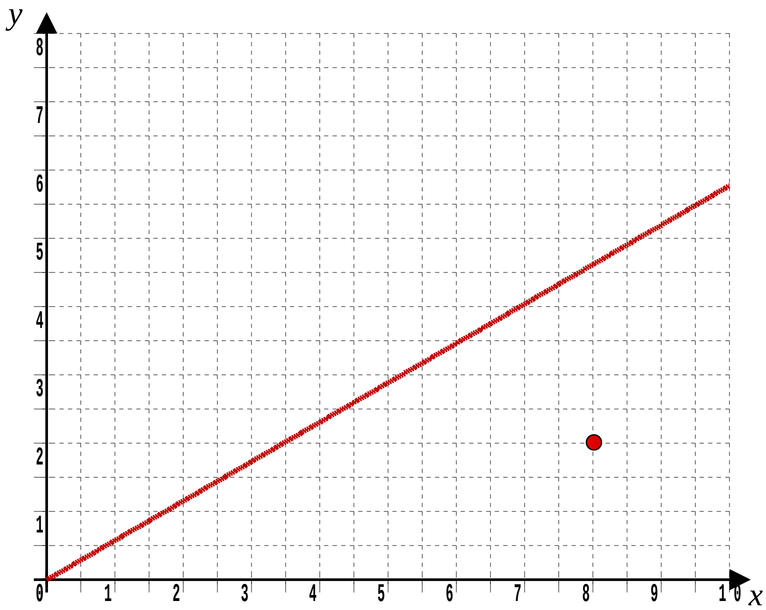
https://www.NumberWonder.co.uk/v9095/10.mp4

10.2 Example

(a) Use matrix methods to work out the exact coordinates of the image when the point (8, 2) is reflected in the line with equation $y = \dfrac{1}{\sqrt{3}}x$

[4 marks]

(b) Mark the image of *P* on the diagram below.



[1 mark]

10.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 58

Question 1

$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(i) Describe fully the transformations represented by the matrices **A** and **B**

[4 marks]

(ii) The point (p, q) is transformed by the matrix product **AB**
Give the coordinates of the image of this point in terms of p and q.

[2 marks]

Question 2

(i) Calculate $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}^2$

[2 marks]

(ii) Give a geometric interpretation of your part (i) answer.

[1 mark]

(iii) Hence, or otherwise, determine, $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}^{11}$

[2 marks]

Question 3

Further AS-Level Specimen Examination Paper, 2017, Q4 (AQA)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) Find the value of k for which matrix **A** is singular.

[1 mark]

(b) Describe the transformation represented by matrix **B**

[1 mark]

(c) **(i)** Given that **A** and **B** are both non-singular, verify that,

$$\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$$

[4 marks]

(ii) Prove the result $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$ for all non-singular square matrices **M** and **N** of the same size.

[4 marks]

Question 4

Advanced Higher Examination Question from May 2015, Q11 (SQA)

(i) Write down the 2×2 matrix, $\mathbf{M}_1$, associated with a reflection in the y-axis.

[1 mark]

(ii) Write down a second 2×2 matrix, $\mathbf{M}_2$, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

[1 mark]

(iii) Find the 2×2 matrix, $\mathbf{M}_3$, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

[1 mark]

(iv) What single transformation is associated with $\mathbf{M}_3$?

[1 mark]

Question 5

Find the 3×3 matrix representing the single transformation that is equivalent to a reflection in the plane $x = 0$, followed by a rotation of $270°$ about the y-axis, followed by a reflection in the plane $y = 0$

[4 marks]

Question 6

Further AS-Level Examination Question from October 2020, Q8 (MEI)

(a) The matrix **M** is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(i) Find $\mathbf{M}^2$

[1 mark]

(ii) Write down the transformation represented by **M**

[1 mark]

(iii) Hence state the geometrical significance of the result of part (i)

[1 mark]

(b) The matrix **N** is $\begin{pmatrix} k+1 & 0 \\ k & k+2 \end{pmatrix}$, where k is a constant.

Using determinants, investigate whether **N** can represent a reflection.

[4 marks]

Question 7

Further AS-Level Examination Question from May 2018, Q5 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

(a) Describe fully the single geometric transformation U represented by $\mathbf{A}$

[3 marks]

The transformation V, represented by the 2×2 matrix $\mathbf{B}$, is a reflection in $y = -x$

(b) Write down the matrix $\mathbf{B}$

[1 mark]

Given that U followed by V is the transformation T, represented by the matrix $\mathbf{C}$,

(c) find the matrix C

[2 marks]

(d) Show that there is a real number k for which $(1, k)$ is invariant under T

[4 marks]

Question 8

Further A-Level Examination Question from June 2019, Q11 (MEI)

(a) Specify fully the transformation represented by the following matrices,

$$\mathbf{M}_1 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \qquad \mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[4 marks]

(b) Find the equation of the mirror line of the reflection *R* represented by the matrix $\mathbf{M}_3 = \mathbf{M}_1\,\mathbf{M}_2$

[5 marks]

(c) It is claimed the reflection represented by the matrix $\mathbf{M}_4 = \mathbf{M}_2\,\mathbf{M}_1$ has the same mirror line as *R.* Explain whether or not this claim is correct.

[3 marks]