Lesson 2

2.1 The 3×3 Matrix

To solve n equations with n unknowns, the working is with an $n \times n$ matrix. In this course, the focus is upon 3 equations with 3 unknowns, and on working with a 3×3 general matrix.

The general 3×3 matrix can be expressed in two ways.

Firstly, and the method preferred by the A-Level course,

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

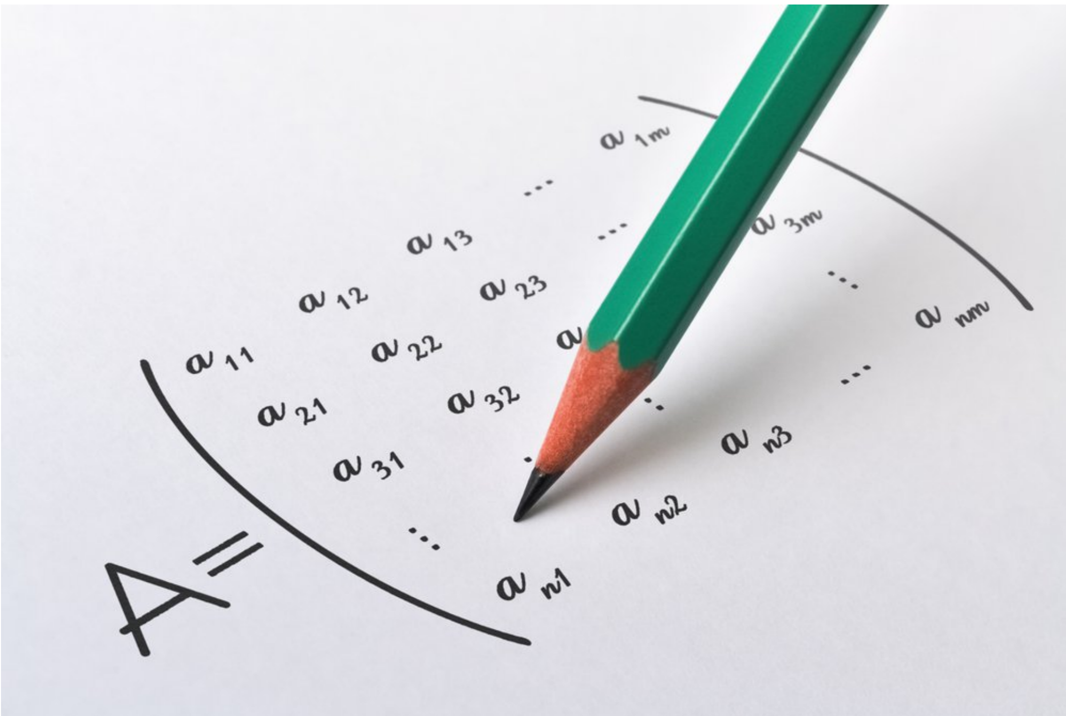
Secondly, and more easily extendable to larger matrices,

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbf{A} = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix} \end{array}$$

However it is expressed, for the 3×3 matrix, two questions immediately arise.

- How is its determinant worked out ?
- How is its inverse, assuming one exists, found ?

The first will be addressed in this lesson, the second in the next.



2.2 The Determinant of a 3×3 Matrix

Before looking at the method of finding the determinant of a 3×3 matrix it is useful to be able to find what is termed the minor of such a matrix.

The Minor of a 3×3 Matrix

The minor of an element in a 3×3 matrix is the determinant of the 2×2 matrix that remains after the row and column containing that element are crossed out.

Example #1

Find the minor of the element 6 in the matrix $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2a.mp4

Watch the video and then write out a full solution here:

☞

[3 marks]

Spot Check

Show that, for $\mathbf{A} = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 7 & -4 \\ 8 & 4 & 9 \end{pmatrix}$, $M_{21} = -51$

[3 marks]

The determinant of a 3 × 3 matrix is found by "expanding along a row or column". As beginners, always expand along the top row. Using other rows or columns will be explored in the exercise. All, of course, arrive at the same answer !

The Determinant of a 3 × 3 Matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Notice that this could be written using the minors of the matrix as,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\,M_{11} - b\,M_{12} + c\,M_{13}$$

which is easier to remember.

Example #2

Find the value of $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{vmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/2b.mp4

Watch the video and
then write out a full
solution here:

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[4 marks]

2.3 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 25

Question 1

The matrix **H** is $\begin{pmatrix} -3 & 1 & 9 \\ 2 & 5 & 3 \\ -1 & 6 & 8 \end{pmatrix}$

Without using a calculator, show that **H** has a determinant of 68

[4 marks]

Question 2

Further A-Level Examination Question from June 2017, FP3, Q6 (a) (Edexcel)

The matrix **M** is given by,

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ 2 & -2 & 1 \\ -4 & 1 & -1 \end{pmatrix}, \quad k \in \mathbb{R}, \quad k \neq \frac{1}{2}$$

Show that $det\,\mathrm{M} = 1 - 2k$

[2 marks]

Question 3

If a matrix is described as being singular, what does this tell you about its determinant ?

[1 mark]

Question 4

Further A-Level Examination Question from June 2016, FP3, Q1 (Edexcel)

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -3 \\ k & 1 & 3 \\ 2 & -1 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that the matrix **A** is singular, find the possible values of k

[4 marks]

Question 5

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$, where k is a constant

Given that the determinant of **A** is 8, find the possible values of k

[5 marks]

Question 6

Every element a_{ij} in a matrix has associated with it a cofactor.

The cofactor c_{ij} is minor M_{ij} which may or may not have its sign changed.

If $i + j$ is even the cofactor is the unchanged minor, otherwise its sign is changed. In practice, this means that the cofactors in a 3×3 matrix are the minors adjusted in accordance with the following pattern,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

When expanding the matrix by the top row, this is where the signs in front of *a*, *b* and *c* came from; they were the top row of of this pattern matrix, +, –, +

That is,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = +a\,M_{11} - b\,M_{12} + c\,M_{13}$$

Let G be the matrix, $G = \begin{pmatrix} -7 & 3 & 5 \\ 0 & 2 & 0 \\ 5 & 6 & 4 \end{pmatrix}$

Show that the determinant obtained by expansion along the top row is the same as that obtained by expansion along the middle row.

[5 marks]

Question 7

Show that, for all real values of x, the matrix $\begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$ is non singular.

[4 marks]