Lesson 3

Further A-Level Pure Mathematics : Core 1

Matrix Systems of Equations

3.1 Inverting a 3×3 Matrix

Finding the inverse of a 3×3 matrix makes use of a "Cookbook Recipe".
Before listing the five steps in the recipe, there is one matrix manipulation that has not been previously mentioned:

The Transpose of a 3×3 Matrix

Given, for example, the matrix $\mathbf{G} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the transpose of matrix $\mathbf{G}$

is denoted $\mathbf{G}^{\mathrm{T}}$ and is formed by an interchange of rows and columns.

Thus, $$\mathbf{G}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Starting with a matrix, $\mathbf{A}$, the recipe cooks up the matrix, $\mathbf{A}^{-1}$

Inverse Matrix "Cookbook Recipe"

Step 1 : Find the determinant of of $\mathbf{A}$, $det\ \mathbf{A}$

Step 2 : Form, $\mathbf{M}$, the matrix of minors of $\mathbf{A}$ by replacing each of the nine elements of the matrix $\mathbf{A}$ with that element's minor.

Step 3 : Form, $\mathbf{C}$, the matrix of cofactors by reversing the sign of some elements of the matrix of minors according to the pattern matrix,

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$+$ indicates no change whereas $-$ indicate change

Step 4 : Write down, $\mathbf{C}^{\mathrm{T}}$, the transpose of the matrix of cofactors.

Step 5 : $\mathbf{A}^{-1} = \dfrac{1}{det\ \mathbf{A}} \mathbf{C}^{\mathrm{T}}$

3.2 Example

Find the inverse of the matrix, $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

Teaching Video : http://www.NumberWonder.co.uk/v9095/3.mp4

Watch the video and
then write out a full
solution here:

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[6 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 25

Question 1

Calculate the product of $\mathbf{A}^{-1}$, in the above example, with $\mathbf{A}$. That is, $\mathbf{A}^{-1} \times \mathbf{A}$
Explain why the answer is not a surprise.

[3 marks]

Question 2

By use of the "Cookbook Recipe", find the inverse of $\mathbf{W} = \begin{pmatrix} -4 & 5 & 2 \\ -5 & 6 & 2 \\ 8 & -9 & -3 \end{pmatrix}$

In your solution, label each of the five steps.

[6 marks]

Question 3

By use of the "Cookbook Recipe", find the inverse of $\mathbf{R} = \begin{pmatrix} 3 & 2 & -2 \\ -2 & k & 0 \\ -1 & -3 & 3 \end{pmatrix}$

In this matrix k is a constant, $k \neq 0$.

Your answer will, of course, be in terms of k

[6 marks]

Question 4

In examinations, if a matrix contains only numbers and no unknown constants, you may use your calculator to obtain the inverse matrix.
Use your calculator to find the inverse of the following matrix,

$$\mathbf{S} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

[2 marks]

Question 5

(i) Prove that if $\mathbf{A} = \mathbf{A}^{-1}$ then $\mathbf{A}^2 = \mathrm{I}$

[2 marks]

(ii) The matrix $\mathbf{A} = \begin{pmatrix} 5 & a & 4 \\ b & -7 & 8 \\ 2 & -2 & c \end{pmatrix}$

Given that $\mathbf{A} = \mathbf{A}^{-1}$, find the values of the constants a, b and c

[6 marks]