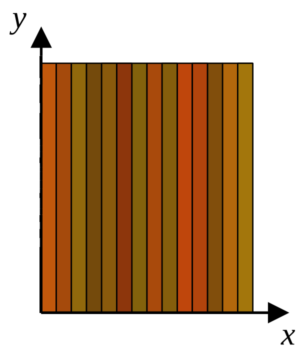
8.1 In The Corner Of A House

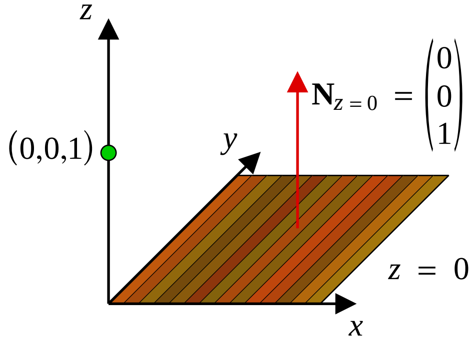
Traditionally, in moving from two dimensions into three, the x-axis stays where is is and the old wall made by the x-axis and the y-axis falls backward to become the floor, with the newly introduced z-axis pointing skyward.

The floor is a surface, and points on that surface have no height.

In other words, the floor is a plane with equation $z = 0$ and the z-axis gives the direction of that plane's normal. The point (0, 0, 1) is on the z-axis and the displacement vector from the origin to that point is a handy version of the (floor) plane's normal. That is,

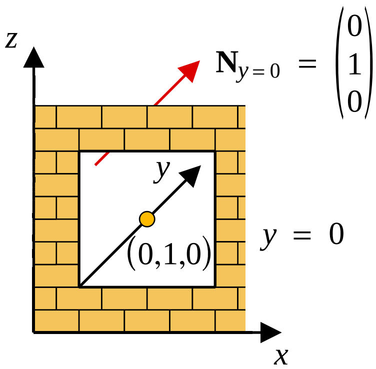
$$\mathbf{N}_{z=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

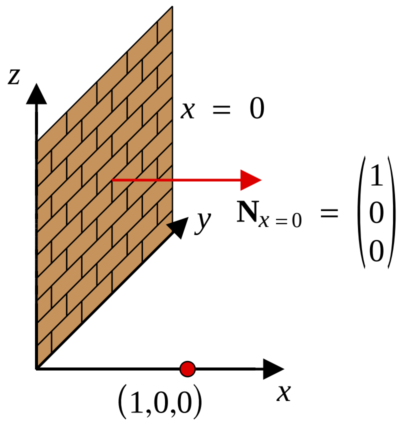




Likewise, the front wall (with a window in it) is the plane with equation $y = 0$. It has normal, $\mathbf{N}_{y=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Finally, the left side wall has equation $x = 0$ and normal $\mathbf{N}_{x=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.





8.2 The Matrix Of Normals

The three normals can be used as the columns of a 3×3 matrix,

$$\mathbf{N}_{x=0} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \mathbf{N}_{y=0} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{N}_{z=0} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

The matrix formed is the 3×3 identity matrix. If any three dimensional point (x, y, z) is multiplied by this matrix it will remain where it is. Now for the clever bit: for a tranformation of interest, T, ask "what will T do to the standard three normals"? Then, write down the corresponding matrix of (transformed) normals. You will then have a matrix that will T transform any points you feed it.

8.3 Reflection in z = 0 (Example)

Suppose that it is desired to reflect points in the floor, the plane $z = 0$. To work out the matrix that will do this note that the side wall and the front (with a window) walls normal vectors need to be left alone, but the floor's normal vector, instead of pointing upward, will (after the reflection) point downward.

The matrix to reflect in the plane $z = 0$ is now formed like so;

$$\mathbf{N}_{x=0} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \mathbf{N}_{y=0} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \mathbf{Ref}_{z=0} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

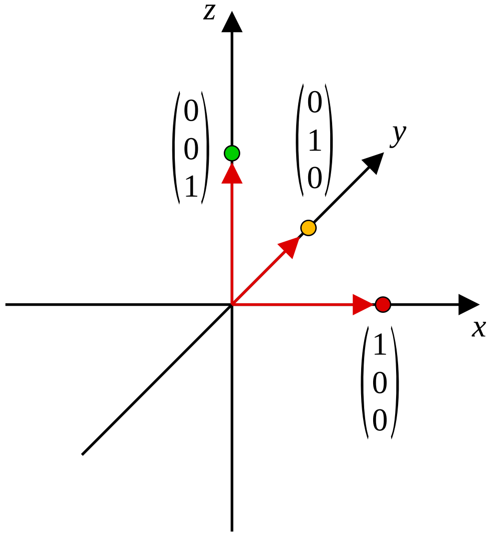
$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix}$$

8.4 Asleep On The Floor

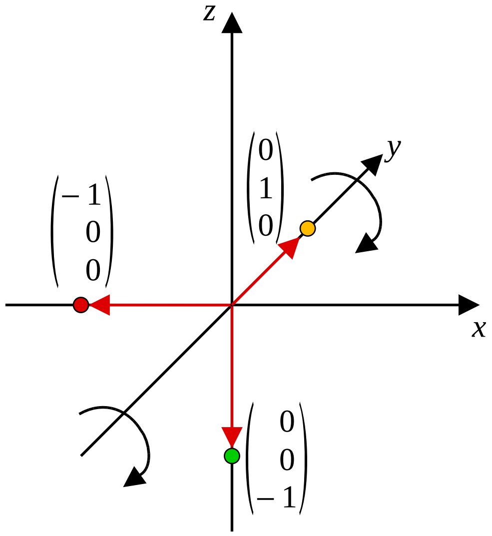
Sleep on the floor : Sleep is $z\ z\ z\ Z\ Z$: The floor is $z = 0$...

8.5 Rotation of 180° about the y-axis

To reduce clutter in diagrams, regard the axes as the normal vectors and view the points $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$ and $(0,\ 0,\ 1)$ as displacement vectors that are the normals to the planes $x = 0$, $y = 0$ and $z = 0$ respectively.



Suppose that it is required that the matrix representing a 180° rotation about the y-axis is determined. Think about where this would send the three normal vectors associated with the red, amber, and green points in the above diagram. After a pause for thought, a diagram like the one below will be in mind.



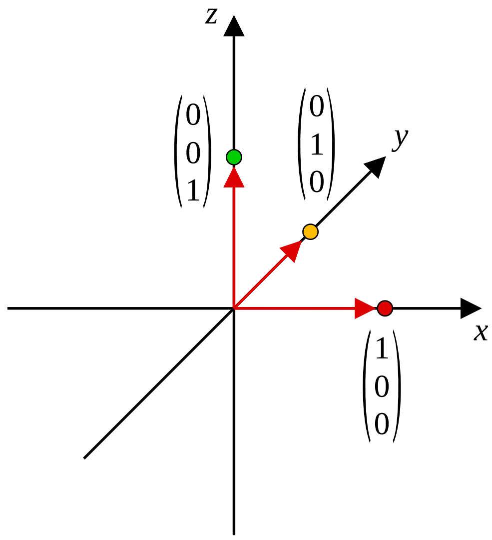
Writing down the matrix of normals in red, amber, green order: $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Multiplying any points by this matrix will now rotate them by 180° about the y-axis. Note that the rotation is anticlockwise when looking down the positive y-axis towards the origin. For 180° is does not matter if you went the wrong way but for other angles, say 90°, it would be important to get that correct.

8.6 Example

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(i) With the aid of the following diagram, or otherwise, determine the single transformation represented by the matrix **M**.



[3 marks]

(ii) The point $A(3, -1, 4)$ is transformed using this matrix.
Find the coordinates of the image of A.

[1 mark]

(iii) The point $B(a, -a, 2a - 1)$ is transformed to the point with coordinates $(a, a - 5, -a)$ using matrix **M**.
Find the value of a.

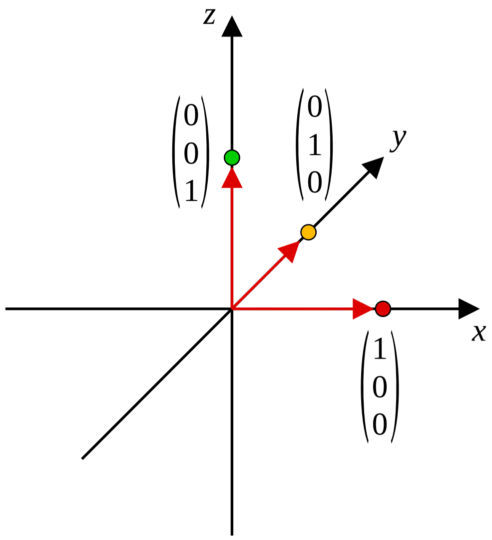
[3 marks]

8.7 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 40

Question 1



With the aid of the above diagram, or otherwise, write down the matrix that will represent,

(i) reflection in the plane $x = 0$

[2 marks]

(ii) rotation of 180° about the x-axis

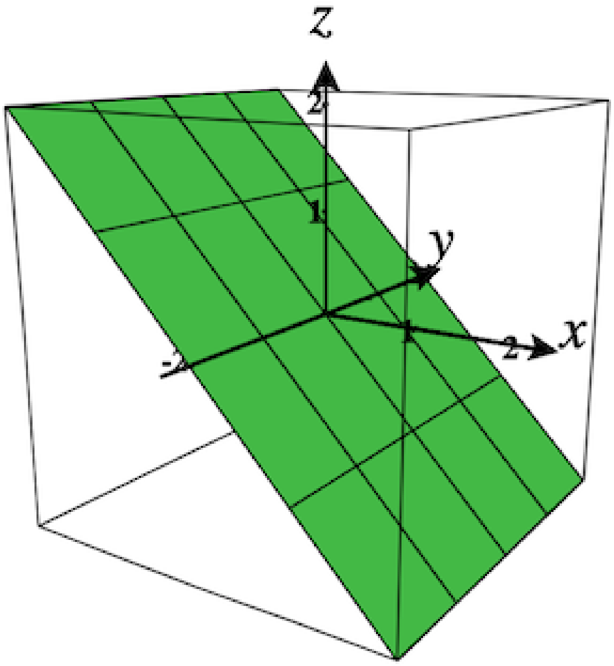
[2 marks]

(iii) rotation of 90° about the y-axis

[2 marks]

Question 2

The three dimensional plot is of the plane with equation $z = -y$



Write down the matrix that will reflect points in the plane $z = -y$

[2 marks]

Question 3

Describe the transformations represented by the following matrices,

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

(ii) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

[2 marks]

(iii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[2 marks]

Question 4

Further A-Level Examination Question from October 2020, Paper 1, Q3 (OCR)

Your are given the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

(a) Find $\mathbf{A}^4$

[1 mark]

(b) Describe the transformation that **A** represents.

[2 marks]

The matrix **B** represents a reflection in the plane $x = 0$

(c) Write down the matrix **B**

[1 mark]

The point P has coordinates $(2,\ 3,\ 4)$.

The point P' is the image of P under the transformation represented by **B**

(d) Find the coordinates of P'

[1 mark]

Question 5

Further A-Level Examination Question from Practice Paper Set 1, Q5 (OCR)

(a) Write down the 3×3 matrix $\mathbf{M}_1$ that represents a reflection in the plane $y = 0$

[1 mark]

(b) Write down the single transformation represented by the matrix $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

[1 mark]

(c) **(i)** Find the determinants of $\mathbf{M}_1$ and $\mathbf{M}_2$

[2 marks]

(ii) Explain how the signs and magnitudes of these determinants relate to the transformations represented by $\mathbf{M}_1$ and $\mathbf{M}_2$

[2 marks]

(d) **(i)** Find the matrix $\mathbf{M}_3$ where $\mathbf{M}_3 = \mathbf{M}_1 \mathbf{M}_2$

[1 mark]

(ii) Describe the single transformation represented by $\mathbf{M}_3$

[2 marks]

Question 6

A is the matrix representing a reflection in the plane $x = 0$ and **B** is the matrix representing a reflection in the plane $y = 0$

(i) Write down the matrices **A** and **B**

[2 marks]

(ii) The point $P(a,\ b,\ c)$ is transformed using matrix **A**.
Find the coordinates of P' in terms of a, b and c

[2 marks]

(iii) P' is transformed using matrix **B**.
Find the coordinates of the image of P' in terms of a, b and c.

[2 marks]

Question 7

Further A-Level Examination Question from May 2020, Paper 1, Q3 (AQA)

Which one of the matrices below represents a rotation of 90° about the x-axis ?
Circle your answer.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

[2 marks]

Question 8

Further AS-Level examination Question from October 2020, Q4, (OCR)

The matrix **M** is $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) **(i)** Calculate det **M**

[1 mark]

(ii) State two geometrical consequences of this value for the transformation associated with **M**.

[2 marks]

(b) Describe fully the transformation associated with **M**.

[1 mark]