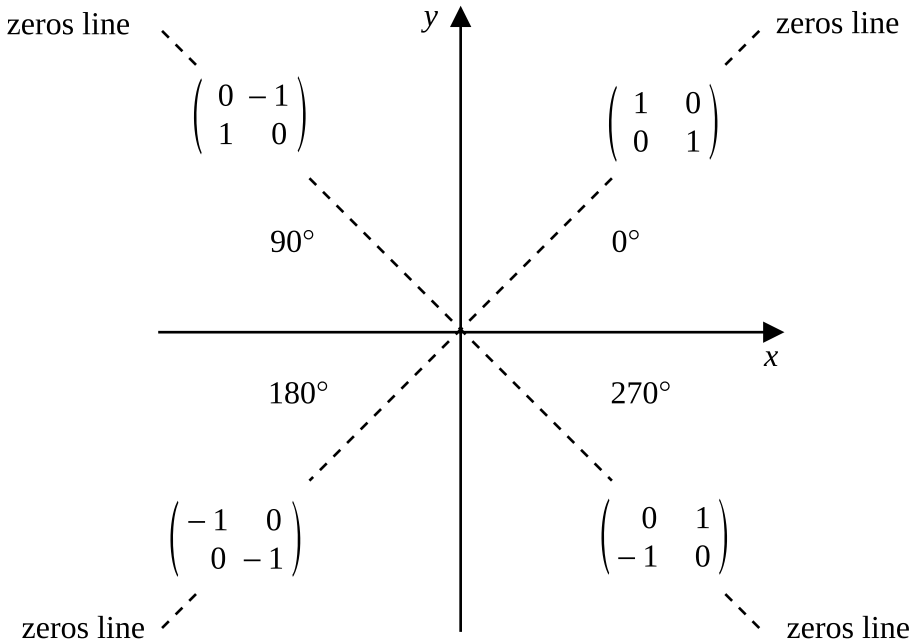
9.1 Rotations In Two and Three Dimensions

Consider the following diagram. It is an aid to remembering the two dimensional rotation matrices for rotations (anticlockwise) of 0°, 90°, 180° and 270°.



$$\mathbf{I} = R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{R}_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{R}_{180} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{R}_{270} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

These matrices are special cases of the following general result,

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Rotation through angle θ about (0, 0)

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

This generalised result was proven in *Matrix Transformations*, Lesson 6.

Proof: https://www.NumberWonder.co.uk/Download/PD9090/9090(6).pdf

The two dimensional rotation matrix, has a centre of rotation about the origin. To rotate about an other point would involve translating the other point to the origin, doing the rotation there, and then translating the origin back to the other point. In three dimensions, the rotation is naturally about either the x, the y or the z axis. For each of the three axes, we'll now give the special rotations of 0°, 90°, 180° and 270° about the axis, followed by the general rotation matrix for that axis. Keep in mind the generalised matrix for two dimensional rotation because, in moving up a dimension, it is this 2 x 2 matrix that is embedded with in the more general 3 x 3 rotation matrix.

- Rotation by θ about the x-axis;

$$\mathbf{R}_{x,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{x,90} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{R}_{x,180} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{R}_{x,270} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{R}_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

- Rotation by θ about the y-axis;

$$\mathbf{R}_{y,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{y,90} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{y,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \mathbf{R}_{y.270} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{y,\theta} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

- Rotation by θ about the z-axis;

$$\mathbf{R}_{z,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{z,90} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_{z,180} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{R}_{z,270} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.2 Example

Further A-Level Examination Question from June 2020, Paper 2, Q9 (AQA)

The matrix $\mathbf{C} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where *a* and *b* are positive real numbers, and

$$\mathbf{C}^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Use $\mathbf{C}$ to show that $cos\left(\frac{\pi}{12}\right)$ can be written in the form $\frac{\sqrt{\sqrt{m} + n}}{2}$, where m and n are integers.

[7 marks]

9.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Further AS-Level Examination Question from November 2021, Q3

The matrix **M** represents a rotation about the x-axis.

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & \frac{\sqrt{3}}{2} \\ 0 & b & -\frac{1}{2} \end{pmatrix}$$

Determine the value of a and the value of b

[2 marks]

Question 2

Further AS-Level Examination Question from November 2020, Q6

Anna has been asked to describe the transformation given by the matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

She writes her answer as follows:

The transformation is a rotation about the x-axis through an angle of θ, where

$$sin\,\theta = \frac{1}{2} \quad \text{and} \quad -\ sin\,\theta = -\frac{1}{2}$$

$$\theta = 30°$$

Identify and correct the error in Anna's work.

[2 marks]

Question 3

Further AS-Level Examination Question from June 2018, Q5 (AQA)

Describe fully the transformation given by the matrix $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[3 marks]

Question 4

Further A-Level Examination Question from June 2019, Paper 1, Q7 (AQA)

Three non-singular square matrices, **A**, **B** and **R** are such that

$$\mathbf{AR} = \mathbf{B}$$

The matrix **R** represents a rotation about the *z*-axis through an angle θ and

$$\mathbf{B} = \begin{pmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Show that **A** is independent of the value of θ

[3 marks]

(b) Give a full description of the single transformation represented **A**

[1 mark]

Question 5

$$\mathbf{M} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Describe the transformation represented by **M**

[3 marks]

(b) Find the image of the point with coordinates $(-1, -2, 1)$ under the transformation represented by **M**

[2 marks]

Question 6

P is the matrix representing a rotation of 120° anticlockwise about the z-axis.

(a) Write down the matrix **P**

[1 mark]

(b) A point $Q = (3, -1, 0)$ is transformed using the matrix **P**. Find the coordinates of the image of Q.

[1 mark]

(c) A point $R = (k, 0, k)$ is transformed using matrix **P**. Find, in terms of k, the exact coordinates of the image of R.

[3 marks]

Question 7

The matrix $\mathbf{C} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where a and b are positive real numbers, and

$$\mathbf{C}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Use $\mathbf{C}$ to show that $cos\left(\frac{\pi}{8}\right)$ can be written in the form $\frac{\sqrt{\sqrt{m} + n}}{2}$, where m and n are integers.

[7 marks]

Question 8

$$\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

(a) Find the transformation represented by **M**

[3 marks]

(b) The point with coordinates (k, $-k$. 0) is transformed using **M**.
Find, in terms of k, the exact coordinates of the image of this point.

[3 marks]

Question 9

The matrix $\mathbf{M} = \begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$ represents a rotation followed by an enlargement.

(a) Find the scale factor of the enlargement.

[2 marks]

(b) Find the angle of rotation.

[3 marks]

A point P is mapped onto a point P' under **M**.
Given that the coordinates of P' are (a, b),

(c) find, in terms of a and b, the coordinates of P.

[4 marks]

Question 10

The formula for the volume of a tetrahedron is,

$$V_{TET} = \frac{1}{3} \times \textit{base area} \times \textit{height}$$

(a) Write down the matrix representing a rotation of 315° anticlockwise about the *y*-axis.

[2 marks]

A tetrahedron *T* has vertices at $(1,\ 0,\ -1)$, $(1,\ 1,\ -1)$, $(3,\ 2,\ 3)$ and $(0,\ 0,\ 0)$.

(b) Find the images of the vertices of the tetrahedron under the transformation described in part (a).

[3 marks]

(c) Hence find the volume of *T*.

[2 marks]