3.1 The Method of Differences

Thus far, when the Method of Differences has been employed, the differences have occurred somewhat accidentally as a result of applying partial fractions to problems. The concept can be more purposefully set up and deployed to powerful effect.

First, however, a formal statement of exactly what the method of differences is:

The Method of Differences

If the general term, u_r, of a series can be expressed in the form

$$f(r+1) - f(r)$$

then $$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \big(f(r+1) - f(r)\big)$$

In consequence,

$$u_1 = f(2) - f(1)$$
$$u_2 = f(3) - f(2)$$
$$u_3 = f(4) - f(3)$$
$$\ldots$$
$$u_n = f(n+1) - f(n)$$
$$\therefore \quad \sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

3.2 Example

Use the method of differences to find a formula for,

$$\sum_{r=1}^{n} r(r+1)(r+2)$$

Begin by showing that,

$$r(r+1)(r+2) = \frac{1}{4}\left(r(r+1)(r+2)(r+3) - (r-1)r(r+1)(r+2)\right)$$

Teaching Video : http://www.NumberWonder.co.uk/v9097/3.mp4

[6 marks]

3.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

(a) Prove that $r(r + 1) - (r - 1)r = 2r$

[2 mark]

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$$

[4 marks]

Question 2

Use the facts that

- $(r+1)^3 - r^3 = 3r^2 + 3r + 1$
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (Proven in Question 1)
- $\sum_{r=1}^{n} 1 = n$ (Obvious !)

to construct a proof, using the method of differences, that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

[6 marks]

Question 3

Further A-Level Examination Question from June 2016, FP2, Q2

(a) Show that,

$$r^2(r+1)^2 - (r-1)^2 r^2 \equiv 4r^3$$

[3 marks]

Given that $\sum_{r=1}^{n} r = \frac{1}{2} n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$\left(1^3 + 2^3 + 3^3 + \ldots + n^3\right) = \left(1 + 2 + 3 + \ldots + n\right)^2$$

[4 marks]

Question 4

This question is about finding a formula for the series,

$$\sum_{r=1}^{n} (r \times r!) = 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n!$$

(a) Prove that $\left((r + 1)! - 1 \right) - (r! - 1) = r \times r!$

[2 marks]

(b) Use the part (a) answer and the method of differences to find a formula for $\sum_{r=1}^{n} (r \times r!)$

[4 marks]

(c) Evaluate $\sum_{r=1}^{6} (r \times r!)$

[1 mark]

Question 5

This question is about finding a formula for the series,

$$\sum_{r=1}^{n} \left(r^2 + 1 \right) r! = 2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + \left(n^2 + 1 \right) n!$$

(a) Prove that $\left((r+2)! - (r+1)!\right) - 2\left((r+1)! - r!\right) = \left(r^2 + 1\right) r!$

[4 marks]

(b) Use the part (a) answer and the method of differences to find a formula for $\sum_{r=1}^{n} \left(r^2 + 1 \right) r!$

[6 marks]

Question 6

Further A-Level Examination Question from June 2011, FP2, Q4

Given that,

$$(2r + 1)^3 = Ar^3 + Br^2 + Cr + 1$$

(a) find the values of the constants A, B and C

[2 marks]

(b) Show that,

$$(2r + 1)^3 - (2r - 1)^3 = 24r^2 + 2$$

[2 marks]

(c) Using the result in part (b) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1)$$

[5 marks]