4.1 Key Polar Curves I

In much the same way that it is useful to know the graph that goes with a few Cartesian equations, it's useful to know the graph that goes with a few Polar equations. The interest is in trying to get “a lot from a little”, and so the desire is to find simple Polar equations that yield graphs that would need formidable algebra to be described in Cartesian form. In this lesson the focus is on graphs that are familiar, but recast in polar form.

4.1 The Line

(i) Generalised Lines

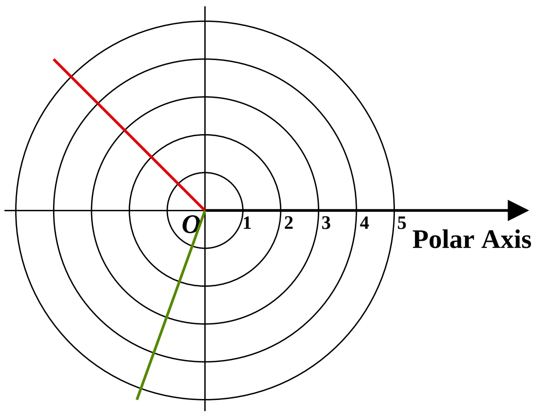
: Cartesian $y = mx + c$: $ax + by = c$

: Polar $r = \dfrac{c}{\sin\theta - m\cos\theta}$: $r = \dfrac{c}{a\cos\theta + b\sin\theta}$

(ii) Half Lines through the pole at an angle of ϕ to the polar axis

: Cartesian $y = \tan(\phi)\,x$ (Whole line)

: Polar $\theta = \phi$ $r \geqslant 0$ (Half line)



Half lines $\theta = 135°$ (red) and $\theta = 250°$ (green)

(iii) Vertical Lines

: Cartesian $x = a$

: Polar $r = a\sec\theta$

(iv) Horizontal Lines

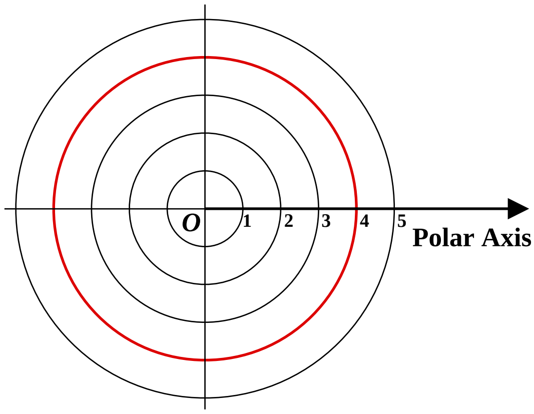
: Cartesian $y = b$

: Polar $r = b\csc\theta$

4.2 The Circle

(i) Centre origin, radius R : Cartesian $x^2 + y^2 = R^2$

: Polar : $r = R$



A circle centre (0, 0) and radius 4 has polar equation, $r = 4$ (red)

(ii) Centre (a, 0), radius a ($\therefore$ passing through the pole)

: Cartesian $(x - a)^2 + y^2 = r^2$ with $r = a$

: Polar $r = 2a\cos\theta$

(iii) Centre (0, b), radius b ($\therefore$ passing through the pole)

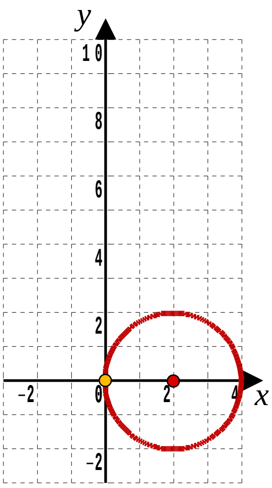
: Cartesian $x^2 + (y - b)^2 = r^2$ with $r = b$

: Polar $r = 2b\sin\theta$

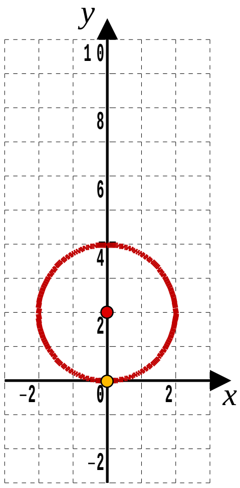
(iv) Centre (a, b), radius r ($\therefore$ passing through the pole)

: Cartesian $(x - a)^2 + (y - b)^2 = r^2$ with $r^2 = a^2 + b^2$

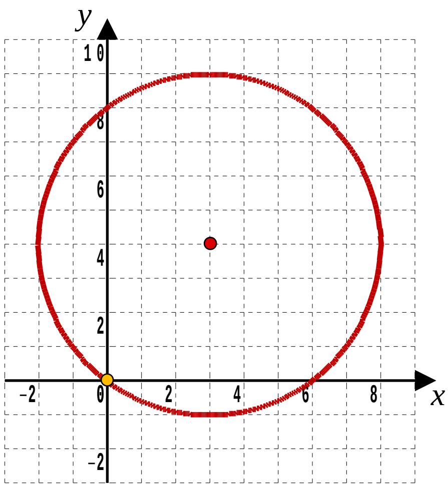
: Polar $r = 2a\cos\theta + 2b\sin\theta$



$r = 4\cos\theta$



$r = 4\sin\theta$



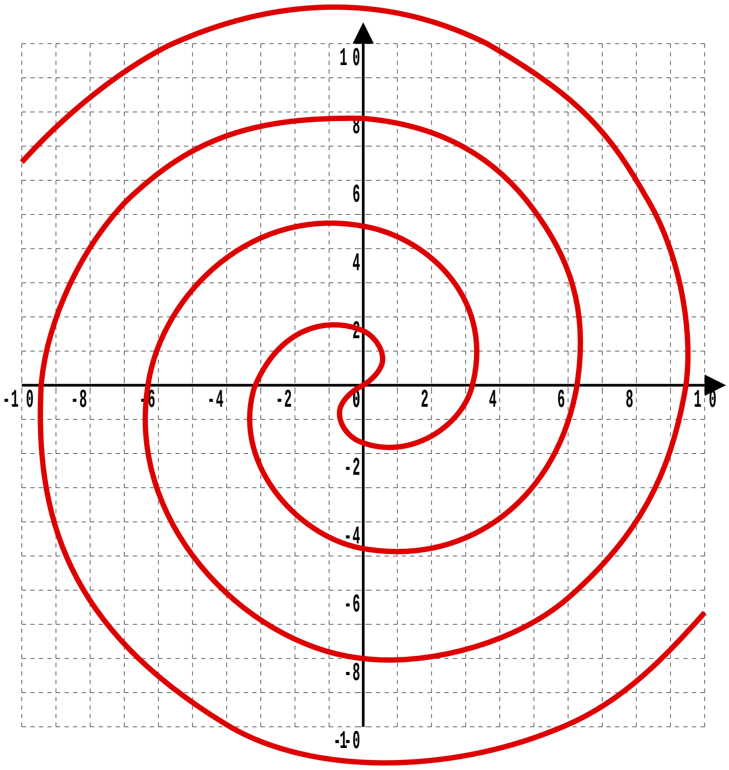
$r = 6\cos\theta + 8\sin\theta$

4.3 The Spiral

(i) The Archimedean Spiral

: Cartesian $y = x\tan\left(\sqrt{x^2 + y^2}\right)$

: Polar $r = \theta, \quad r \geqslant 0$



The Double Archimedian Spiral $r = \pm\theta$ with θ in radians

The triskelion, a symbol dating back over 6000 years of three interlocking Archimedian spirals, often associated with Ireland, where it adorns many graves

(ii) The Equiangular Spiral (also called Logarithmic Spiral, Bernoulli's Spiral)

: Cartesian $y = x\tan\left(ln\left(\sqrt{x^2 + y^2}\right)\right)$

: Polar $r = e^{\theta}$

3.4 Exercise

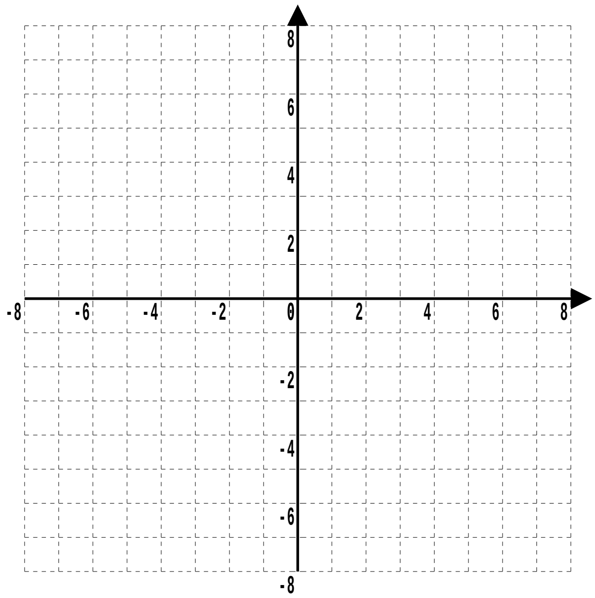
Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 50

Question 1

(i) On the graph, sketch the polar equations,

$$r = 6 \qquad \text{and} \qquad r = 3\sqrt{3}\ sec\ \theta$$



[2 marks]

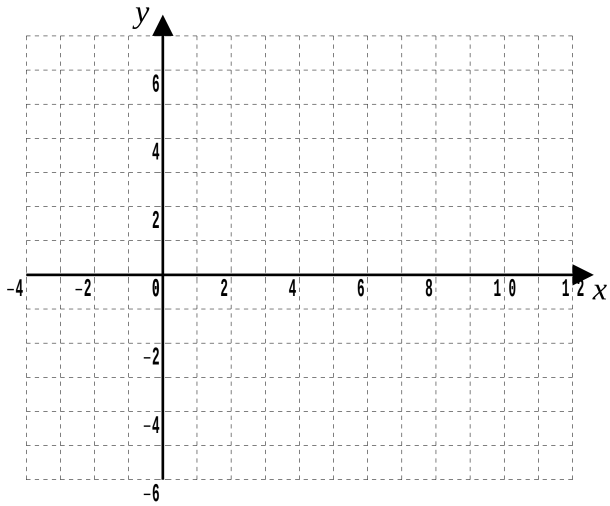
(ii) Use algebra to find the polar coordinates of the points of intersection of the two graphs sketched in part (i)

[4 marks]

Question 2

(i) On the graph, sketch the polar equations,

$$r_1 = 6\,\sin\theta_1 \quad \text{and} \quad r_2 = 6\sqrt{3}\,\cos\theta_2$$



[2 marks]

(ii) Use algebra to find the polar coordinates of the point of intersection of the two graphs sketched in part (i)

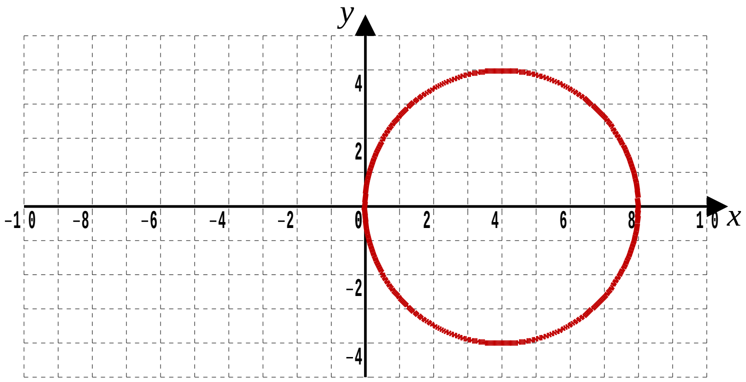
Is the pole when $r_1 = r_2 = 0$ an additional point of intersection ?

[4 marks]

Question 3

On the graph is plotted the polar equation,

$$r = 8\cos\theta$$



(i) To the graph add the points with the following polar coordinates;

$A(4\sqrt{2},\ 45°)$, $B(8,\ 0°)$, $C(4\sqrt{2},\ 315°)$

[3 marks]

(iii) Add the circle $r = 4$ to the graph.

[1 mark]

(iv) Use algebra to find both points of intersection of the two graphs. Give your answer in polar coordinates.

[4 marks]

Question 4

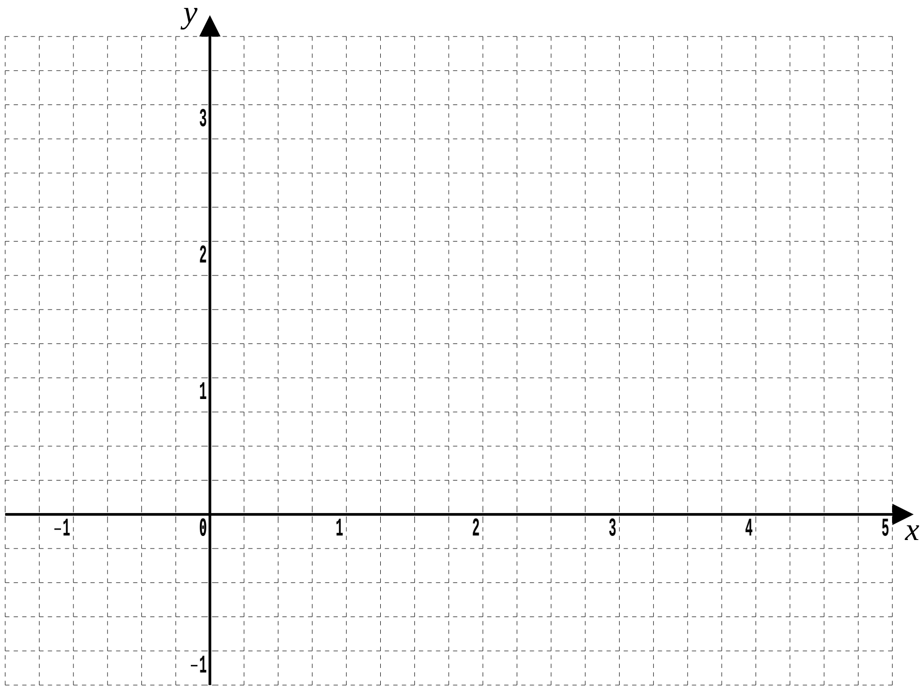
Consider the curve with polar equation $r = 4\,cos\left(\theta - 30°\right)$

(i) Prove that this equation can be written as,

$$r = 2\sqrt{3}\,cos\,\theta + 2\,sin\,\theta$$

[2 marks]

(ii) Hence, sketch the curve $r = 4\,cos\left(\theta - 30°\right)$



[3 marks]

(iii) Write down the Cartesian equation of the curve.

[2 marks]

(iv) What transformation would result from replacing θ in the polar equation equation $r = 4\,cos\,\theta$ with $\left(\theta - 30°\right)$?

[2 marks]

Question 5

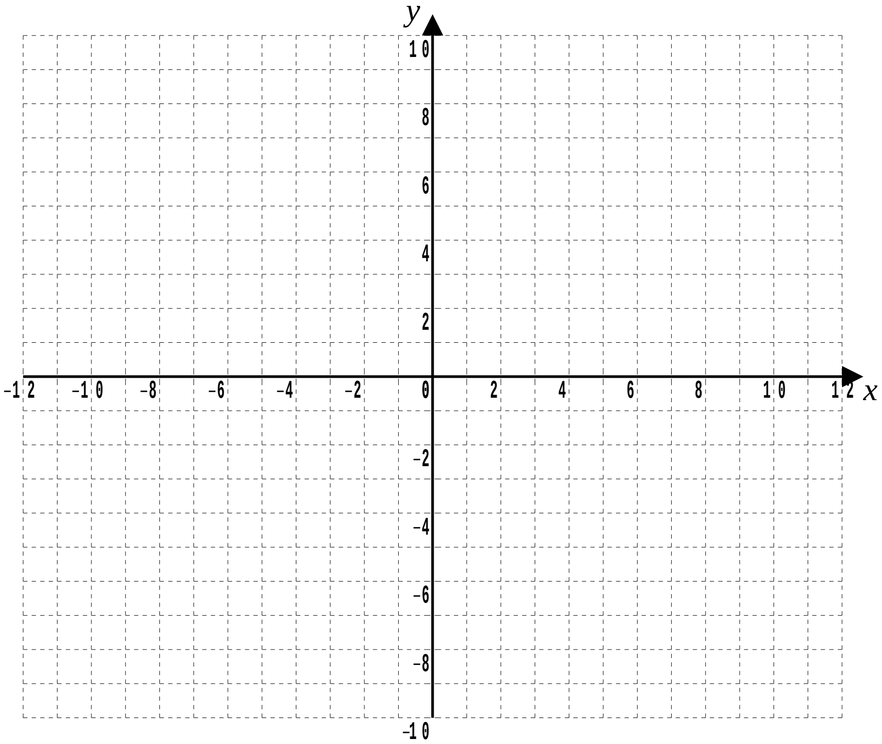
Consider the polar equation $r = 5\,sec\left(\theta - 60°\right)$

(i) Prove that this equation can be written as,

$$r = \frac{10}{cos\,\theta + \sqrt{3}\,sin\,\theta}$$

[2 marks]

(ii) Hence, sketch the graph of $r = 5\,sec\left(\theta - 60°\right)$



[3 marks]

(iii) Write down the Cartesian equation of the curve.

[2 marks]

(iv) What transformation would result from replacing θ in the polar equation equation $r = 5\,sec\,\theta$ with $\left(\theta - 60°\right)$?

[2 marks]

Question 6

Starting with the Cartesian equation of a circle, centre (5, 12), radius 13, and using the relations

- $x = r \cos\theta$
- $y = r \sin\theta$
- $r^2 = x^2 + y^2$

prove using algebra that the polar equation of the circle will be

$$r = 10 \cos\theta + 24 \sin\theta$$

[6 marks]

Question 7

Starting with the Cartesian equation of the Archimedian spiral, prove that its polar equation is $r = \theta$

[3 marks]