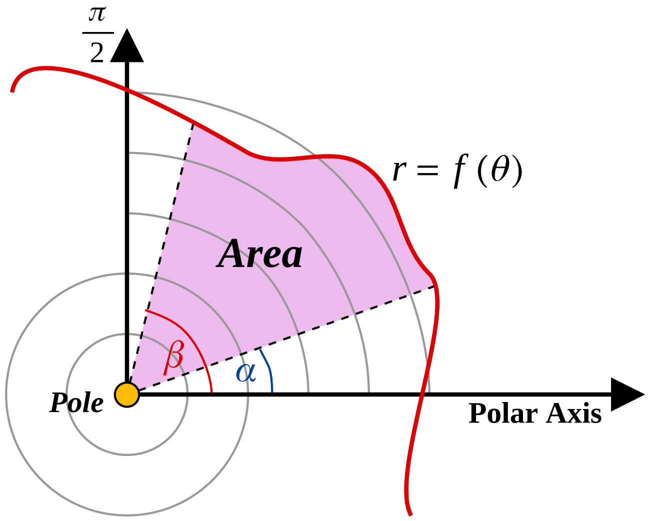
5.1 Areas and Polar Curves

Integration can be used to find the area bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$ radiating out from the pole to the curve. A diagram to keep firmly in mind and the formula to apply are given below;

Polar Area Formula



$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Notice the importance of the pole in the areas being found, and also the similarity between the Polar Area Formula and the formula for the Area of a Sector.
Here is a reminder of that previous formula from the Year 2 course;

Arc Length and Sector Area

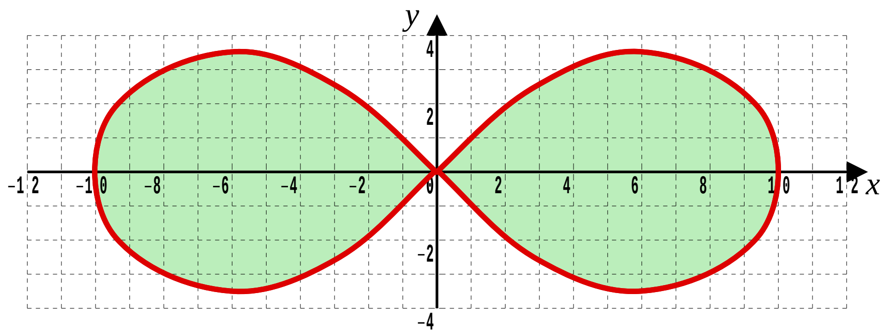
$$Arc\ length = r\,\theta^c$$

$$Sector\ area = \frac{1}{2} r^2 \theta^c$$

Given that integration is a summation process, it should not come as a surprise that what is being summed by the Polar Area Formula is an infinite number of infinitely thin sectors, all radiating out from the pole.
Hence the similarity of the Polar Area Formula with the Sector Area formula.
It goes without saying that angles α and β need to be in radians.

5.2 Example

Previously† it was shown that the famous curve graphed below, The Lemniscate of Bernoulli, had the polar equation $r^2 = 100\, cos\,(2\theta)$



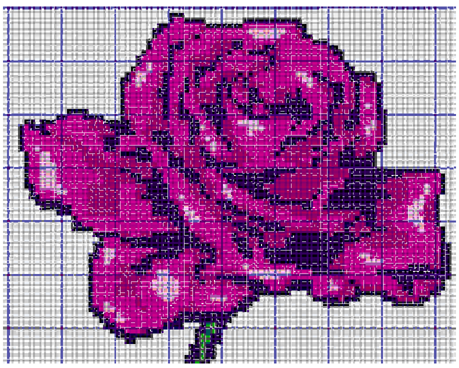
What is the total area of the region enclosed by the two loops ?

Teaching Video: http://www.NumberWonder.co.uk/v9101/5.mp4

[6 marks]

5.3 The Rose Graphs

The Lemniscate of Bernoulli is an example of a "Rose Graph" with two petals. Also called "flower graphs", they increasingly resemble a flower as the multiple of θ is increased and more and more petals occur.
Another rose graphs is explored in the exercise (Question 4)



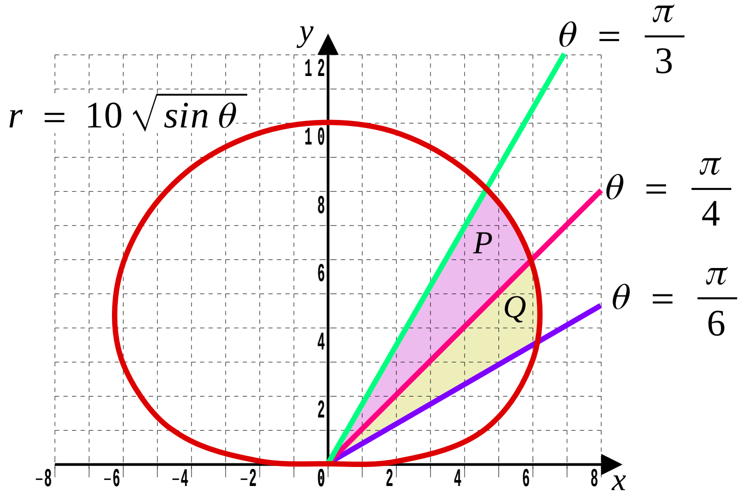
† In Lesson 2, Exercise 2.3, Question 1

5.3 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 70

Question 1

The graph is of the curve with polar equation $r = 10\sqrt{\sin\theta}$ and the three half-lines with polar equations, $\theta = \dfrac{\pi}{6}$, $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{3}$



(i) Calculate the exact value of the area of P by means of the integration;

$$Area\ P = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(10\sqrt{\sin\theta}\right)^2 d\theta$$

[4 marks]

(ii) In a similar manner, show that,

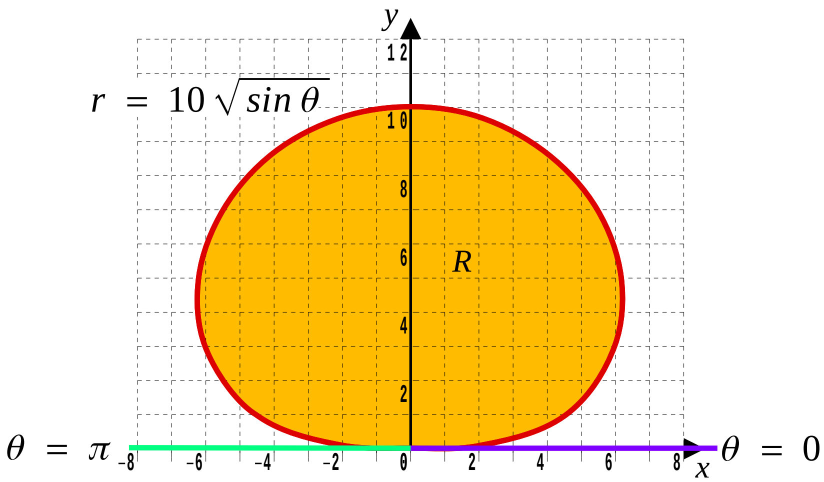
$$Area\ Q = 25\left(\sqrt{3} - \sqrt{2}\right)$$

[4 marks]

(iii) Explain why the domain for θ is $0 \leqslant \theta \leqslant \pi$ rather than $0 \leqslant \theta \leqslant 2\pi$

[2 marks]

(iv)



Keeping in mind the domain for θ identified in part (iii), find the exact total area within the loop, marked R in the above graph.

[3 marks]

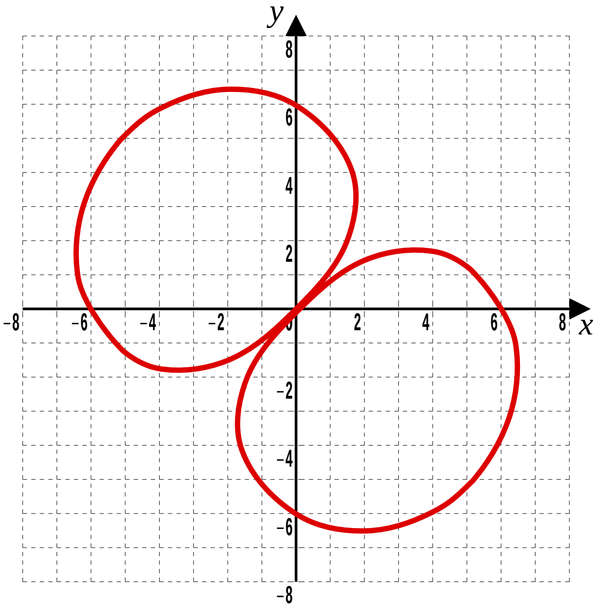
(v) Show that the Cartesian equation for the curve is,

$$\left(x^2 + y^2\right)^3 = 10000\,y^2, \qquad y \geqslant 0$$

[3 marks]

Question 2

The graph is of the curve with polar equation $r^2 = 36\ (\ \sin\theta - \cos\theta\ )$



(i) Show that if $I = \dfrac{1}{2}\displaystyle\int_0^{2\pi} r^2\, d\theta$ then $I = 0$

[4 marks]

Stay Positive !

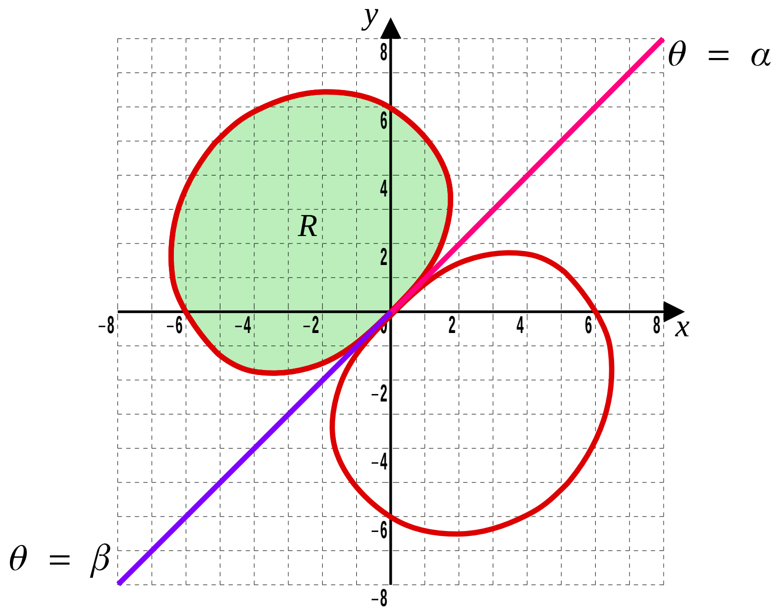
The reason for the failure of I to be the total area within the two loops is because the upper left loop corresponds to r being positive yielding a positive area, and the loop lower right corresponds to r being negative, yielding a negative area. Notice that, unlike with Cartesian integration, the quadrant the curve is in is not what determines whether or not the integral is positive or negative; from the polar point of view there is no x-axis for a curve to be below and thus negative.

The A-Level course recommends that limits should be chosen carefully such that $r \geqslant 0$ for all values within the domain of the integral.

(ii) Let α and β be the solutions to the equation $r^2 = 0$ with $\alpha < \beta$.
Find the value of α and the value of β for $0 \leqslant \alpha < 2\pi,\ 0 \leqslant \beta < 2\pi$

[3 marks]

(iii) Use your part (ii) values to find the exact area on the upper left loop, R.



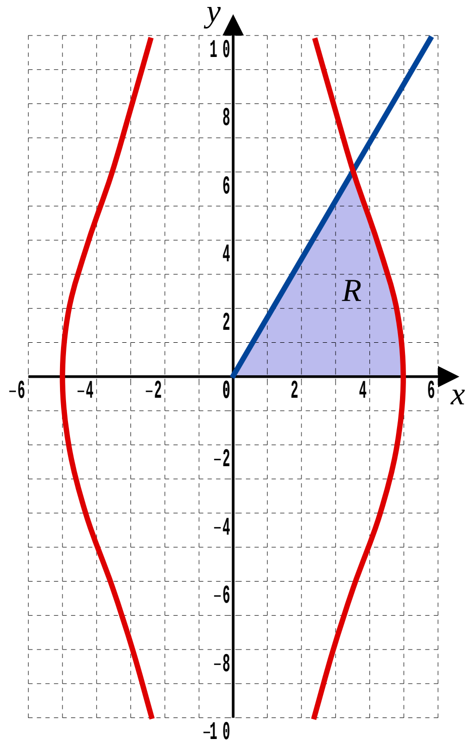
[4 marks]

(iv) Hence state the total exact area enclosed within the two loops

[1 mark]

Question 3

The graph is of the polar curve, $r^2 = 25 \sec\theta$ and the half-line $\theta = \frac{\pi}{3}$



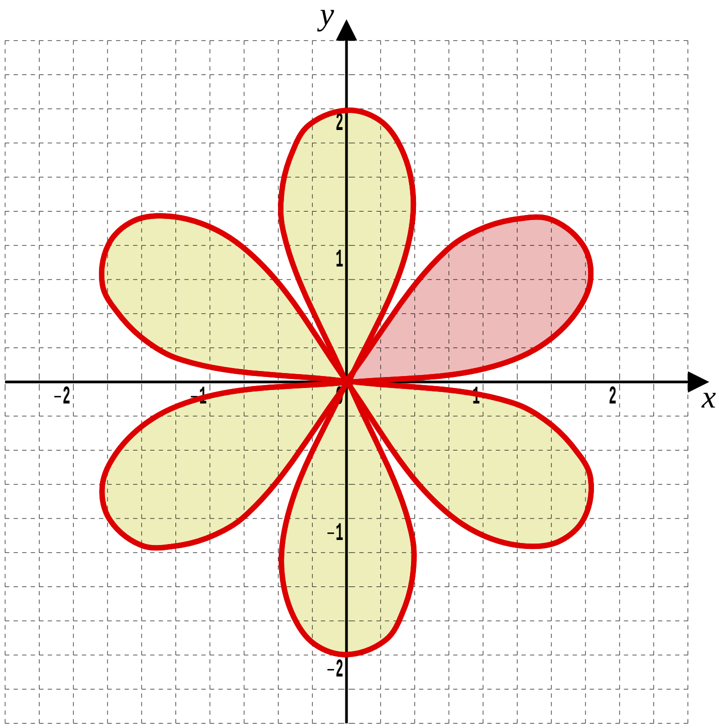
(i) Find the exact value of the area of the region R, shown shaded.

[7 marks]

(ii) Find a Cartesian equation for the curve $r^2 = 25 \sec\theta$

[4 marks]

Question 4



(i) By first finding the area of one petal of the polar curve $r^2 = 4\,sin\,(3\theta)$ find the total area enclosed by the six petals of this rose.

[6 marks]

(ii) Prove that,

$$sin\left(3\theta\right) \;=\; 3\,sin\,\theta \;-\; 4\,sin^{3}\,\theta$$

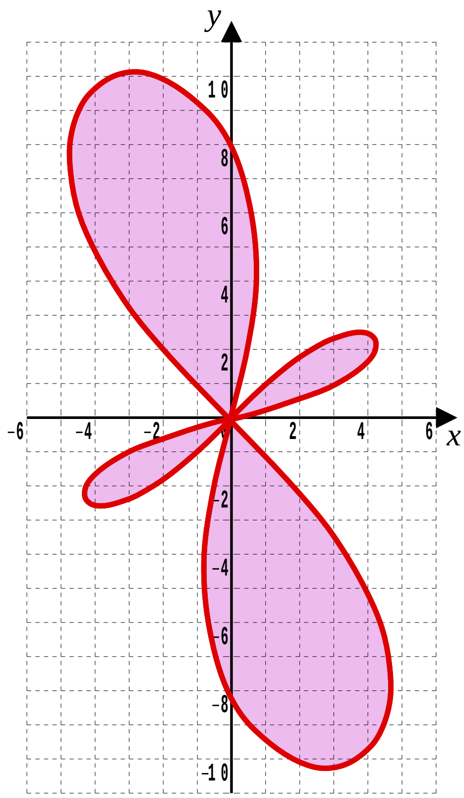
[3 marks]

(iii) Hence, or otherwise, find a Cartesian equation for the rose curve.

[4 marks]

Question 5

The graph is of the polar equation $r^2 = 64(\sin(4\theta) - \cos(2\theta))$



(i) Find the area enclosed by the four loops.

(ii) Find a Cartesian equation for the curve.

[18 marks]