8.1 Tangents

A curve in polar form is typically given as, $r = f(\theta)$

Using the relationships that $x = r\cos\theta$ and $y = r\sin\theta$ such a polar curve can immediately be recast as a parametric curve in Cartesian form,

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta$$

All of our existing knowledge about parametric curves can then be applied.
For example, the gradient of the curve can be found using the chain rule written in the form,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

Of particular use is the following immediate consequence;

Parallel and Perpendicular Tangents Theorem

- To find a tangent parallel to the polar axis, solve $\frac{dy}{d\theta} = 0$
- To find a tangent perpendicular to the polar axis, solve $\frac{dx}{d\theta} = 0$

It may help to think of of $\frac{dy}{d\theta} = 0$ as "no change in height" (as θ changes) and, similarly, $\frac{dx}{d\theta} = 0$ as "only change in height" (as θ changes) where there's a fudge in which the polar axis is being viewed as a horizontal axis.

Note that *the polar axis* is also often called *the initial line.*

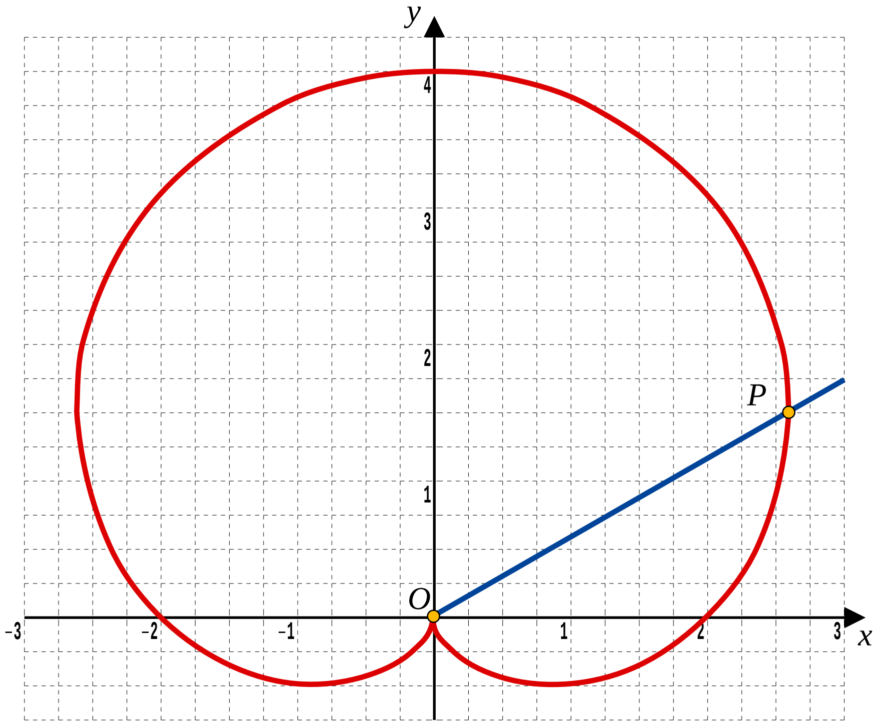
8.2 Polar Curve Sketching

When asked to sketch a polar curve that's not immediately recognised, working out points where there is a tangent parallel or perpendicular to the polar axis may helpful especially if it's quick and easy to do so.
Another short cuts is to spotting if the curve is an odd or an even function.

Points where $\frac{dr}{d\theta} = 0$ correspond to a "stationary point" possibly indicating the tip of a loop where the distance from the pole has stopped increasing and is about to start decreasing (or vice versa).

8.3 Example

The graph shows the polar equation $r = 2(1 + \sin\theta)$, $0 \leqslant \theta < 2\pi$, and a half-line that meets the curve at the pole, O, and at the point P



The tangent to the curve at P is perpendicular to the polar axis.
The polar coordinates of P are (R, ϕ). Determine the exact values of R and ϕ

[6 marks]

The Solution :

$$
\begin{aligned}
x &= r\cos\theta \\
&= 2(1 + \sin\theta)\cos\theta && \text{As } r = 2(1 + \sin\theta) \\
&= 2\cos\theta + 2\sin\theta\cos\theta \\
&= 2\cos\theta + \sin 2\theta && \text{Trigonometric Identity}
\end{aligned}
$$

$$\frac{dx}{d\theta} = -2\sin\theta + 2\cos 2\theta$$

But $\frac{dx}{d\theta} = 0$ for a tangent to be perpendicular to the polar axis

$$
\begin{aligned}
\therefore\ \cos 2\theta - \sin\theta &= 0 && \text{After dividing through by 2} \\
1 - 2\sin^2\theta - \sin\theta &= 0 && \text{Trigonometric Identity} \\
2\sin^2\theta + \sin\theta - 1 &= 0
\end{aligned}
$$

$(2\sin\theta - 1)(\sin + 1) = 0 \Rightarrow$ P corresponds to $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

Substitution this back into $r = 2(1 + \sin\theta)$ gives $P\left(3, \frac{\pi}{6}\right)$

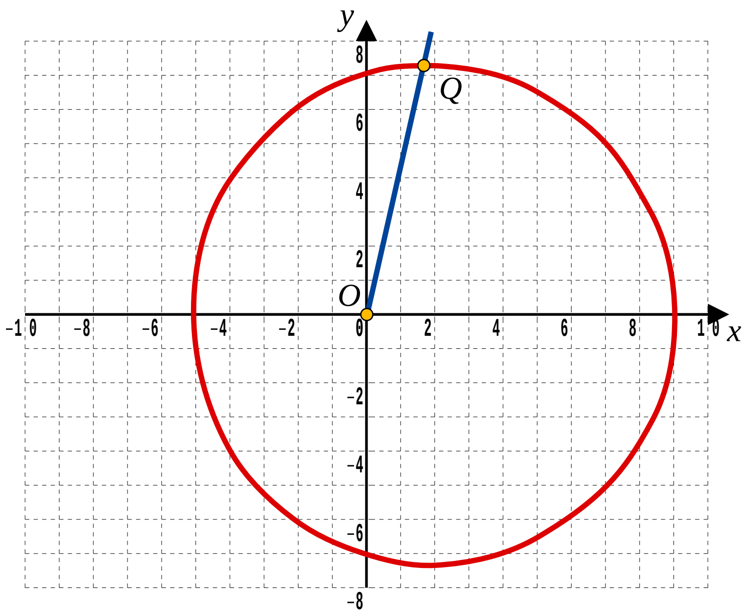
[6 marks]

8.4 Exercise

Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 21

Question 1

The graph shows the polar equation $r = 7 + 2\cos\theta, \quad 0 \leqslant \theta < 2\pi$,
and a half-line that meets the curve at the pole, O, and at the point Q.



The tangent to the curve at Q is parallel to the polar axis.
The polar coordinates of Q are (R, ϕ)

(i) Show that $\cos\phi = \dfrac{1}{4}$

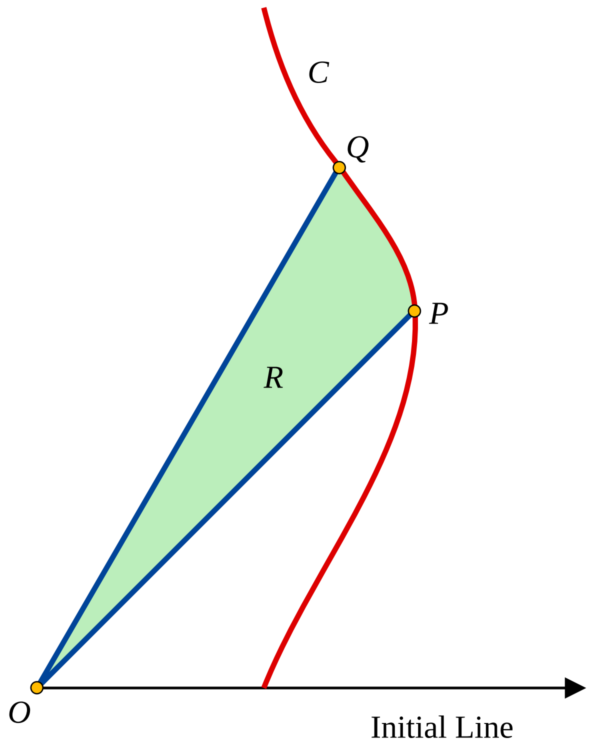
[5 marks]

(ii) Find the exact value of R

[1 mark]

Question 2

Further A-Level Examination Question from June 2014, FP2, Q8 (Edexcel)



The sketch is of part of the curve C with polar equation

$$r = 1 + \tan\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$$

The tangent to the curve C at the point P is perpendicular to the initial line

(a) Find the polar coordinates of the point P

[5 marks]

The point Q lies on the curve C, where $\theta = \dfrac{\pi}{3}$

The shaded region R is bounded by OP, OQ and the curve C, as shown in the sketch.

(b) Find the exact area of R, giving your answer in the form

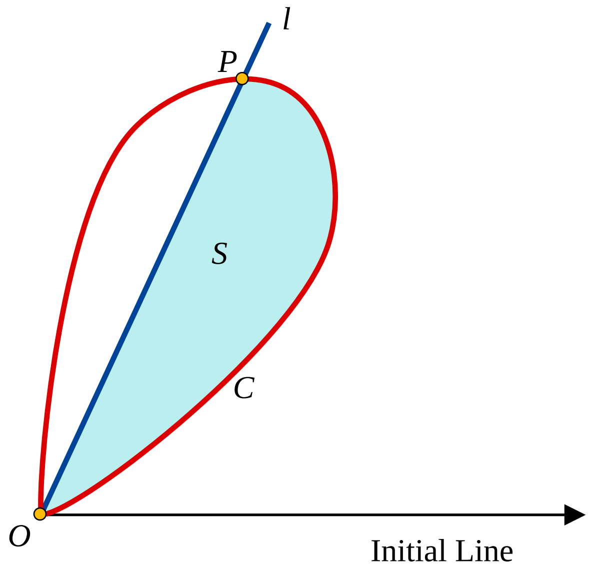
$$\frac{1}{2}\left(\ln p + \sqrt{q} + r \right)$$

where p, q and r are integers to be found

[7 marks]

Question 3

Further A-Level Examination Question from June 2013, FP2, Q8 (Edexcel)



The diagram shows a curve C with polar equation $r = a \sin 2\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

and a half-line l. The half-line l meets C at the pole O and at the point P.
The tangent to C at P is parallel to the initial line.
The polar coordinates of P are (R, ϕ)

(a) Show that $\cos \phi = \dfrac{1}{\sqrt{3}}$

[6 marks]

(b) Find the exact value of R

[2 marks]

The region S, shown shaded, is bounded by C and l.

(c) Use calculus to show that the exact area of S is

$$\frac{1}{36}a^2\left(9\arccos\left(\frac{1}{\sqrt{3}}\right)+\sqrt{2}\right)$$

[7 marks]