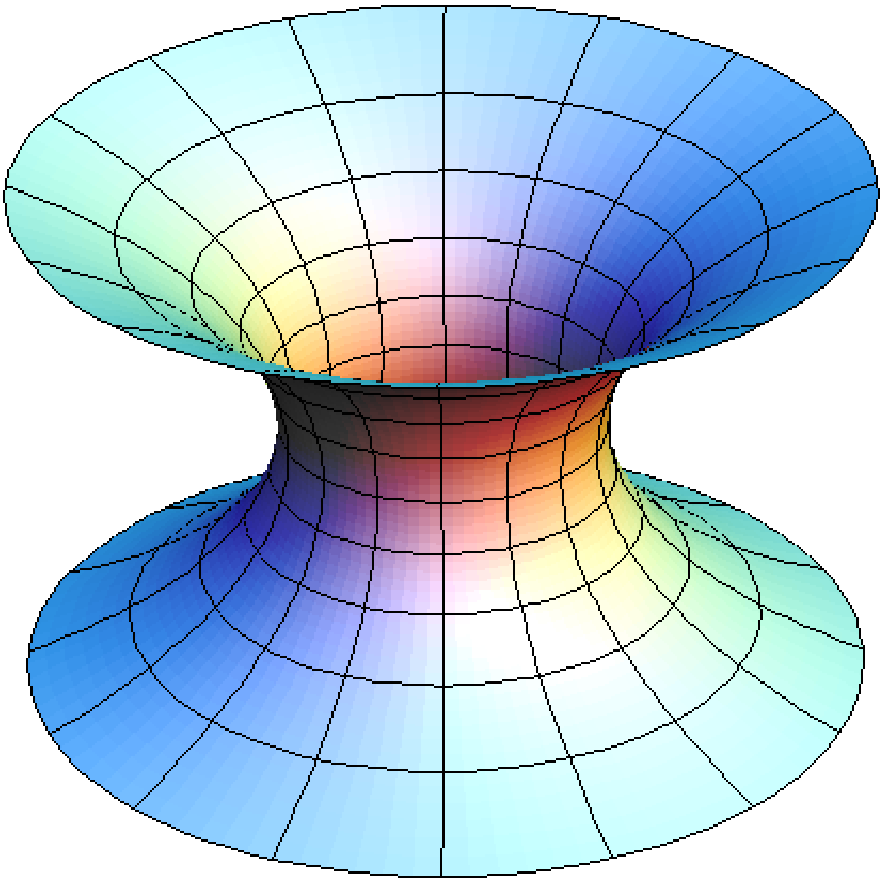
Further Pure A-Level Mathematics
Compulsory Course Component
Core 2

HYPERBOLIC FUNCTIONS



The Catenoid is an example of a minimal surface.
It occupies the least area between the two boundary circles,
one at the top and one at the bottom.
It is formed by rotation of the hyperbolic cosine curve about the y-axis.

HYPERBOLIC FUNCTIONS

Lesson 1

Further A-Level Pure Mathematics, Core 2
Hyperbolic Functions

1.1 Building With Exponentials

When a chain is hung between two fixed points the curve that naturally occurs due to gravity is called a catenary. Mathematicians call this curve “hyperbolic cosine” and as a function it has the abbreviation *cosh*

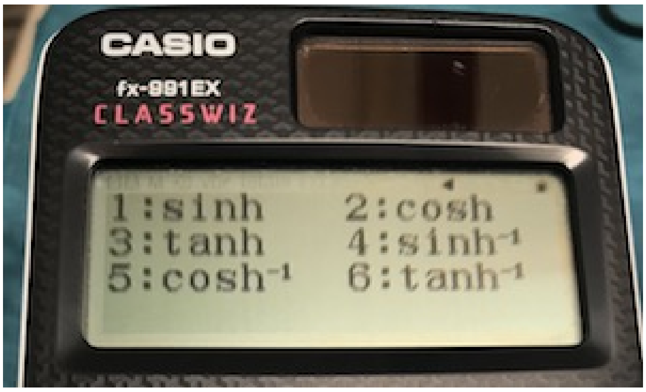
Photograph by Martin Hansen

The catenary is minimising the forces within and upon it's individual parts which is clearly going to give it some interesting features. Surprisingly, it turns out that the *cosh* function is built from a couple of exponential functions.

Definition of *cosh x*

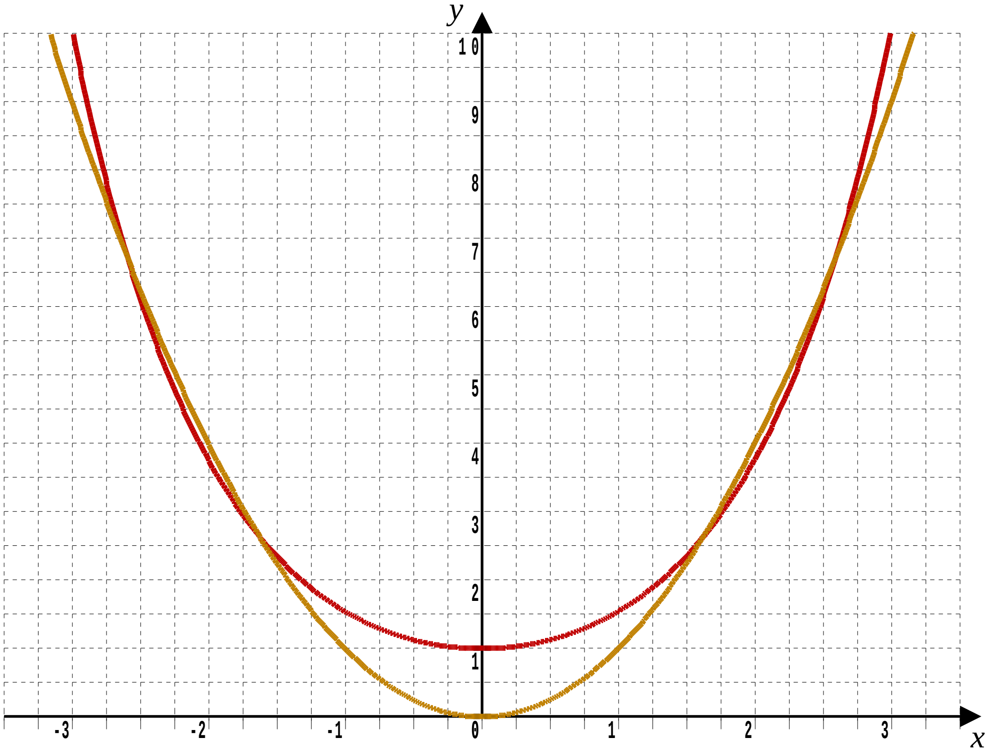
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Some calculators have buttons for the hyperbolic functions, others tuck them away in a menu. On the CASIO fx-991EX Classwiz, access them via the OPTN button.



Photograph by Martin Hansen

Although superficially similar to a parabola, the catenary is not the same curve.



In red, the catenary : $y = cosh\, x$ In gold, the parabola : $y = x^2$

When inverted, a catenary arch is formed which has been used in the construction of bridges and buildings for hundreds of years. It's of interest to architects because of an ability to withstand the weight of the material from which it is constructed. Catenary arches feature in Barcelona's famous Masia Freixa, built in 1896.

Catenary arches and Parabolic arches feature in the Gaudi inspired Masia Freixa

An interior door of the Masia Freixa with a catenary arched door inside a parabolic arched frame

Definition : The Six Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad x \in \mathbb{R}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad x \in \mathbb{R}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad x \in \mathbb{R}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad x \in \mathbb{R}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad x \in \mathbb{R},\ x \neq 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \qquad x \in \mathbb{R},\ x \neq 0$$

1.2 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Use your calculator to solve these equations, correct to 3 decimal places,

(i) $\cosh 2 = x$

[1 mark]

(ii) $\sinh (0.7) = 2x$

[1 mark]

(iii) $\tanh (-1) = x$

[1 mark]

(iv) $\cosh x = 4$

[1 mark]

(v) $\sinh x = 5$

[1 mark]

(vi) $\tanh 3x = 0.9$

[1 mark]

Question 2

Prove the following;

(i) That $\cosh x$ is an even function

[2 marks]

(ii) That $\sinh x$ is an odd function

[2 marks]

(iii) That $\tanh x$ is an odd function

[2 marks]

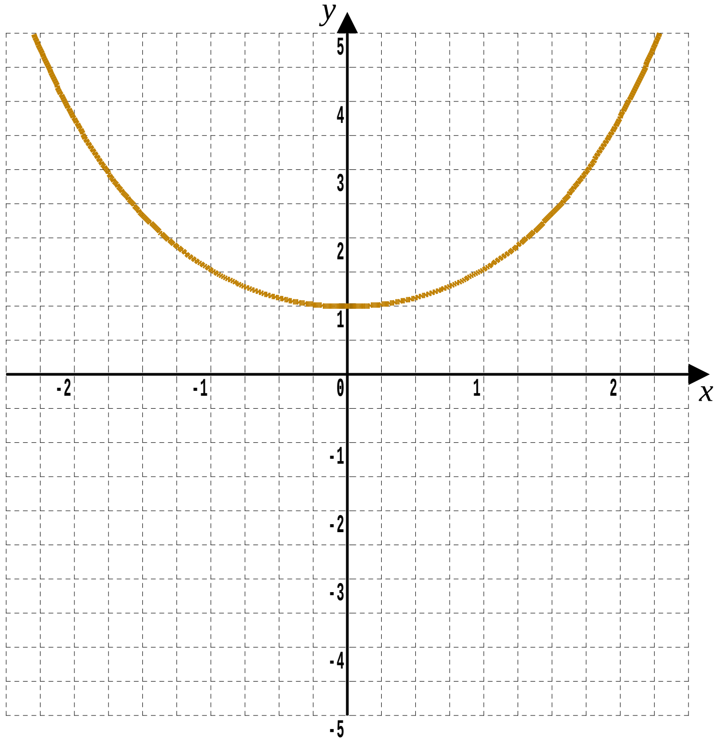
Question 3

(i) Complete the following table;

x	0	0.5	1	1.5	2	2.5
sinh x						

[3 marks]

(ii) Taking care over the different x-axis and y-axis scales, add a plot of $y = \sinh x$ to the graph below of $y = \cosh x$ using the results from the part (i) table and the fact that $y = \sinh x$ is an odd function



[3 marks]

Question 4

(i) Using the definitions of *cosh x* and *sinh x* explain why, for large positive values of x, the graph of $y = \sinh x$ is effectively the same as $y = \cosh x$

[2 marks]

(ii) Using the definitions of *cosh x* and *sinh x* explain why, for large negative values of x, the graph of $y = \sinh x$ is effectively a reflection in the x-axis of $y = \cosh x$

[3 marks]

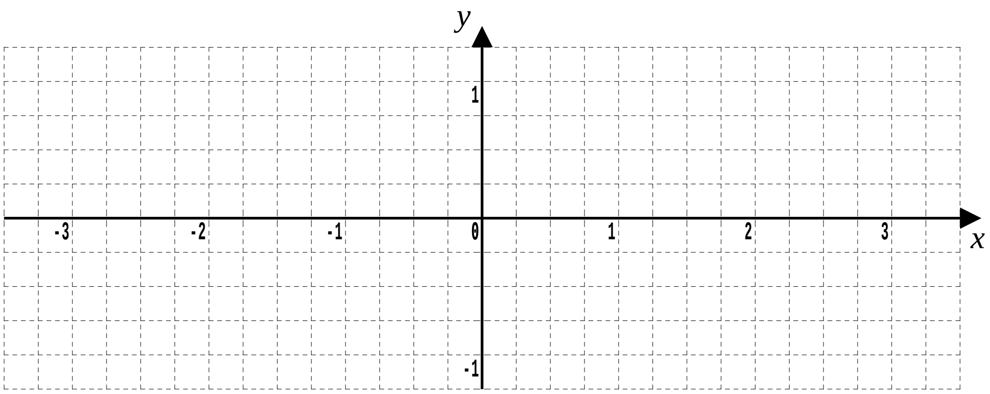
Question 5

(i) Working to two decimal places, complete the following table;

x	0	0.5	1	1.5	2	2.5	3	3.5
tanh x								

[3 marks]

(ii) Plot $y = tanh\ x$ on the graph below using the results from the part (i) table and the fact that $y = tanh\ x$ is an odd function



[2 marks]

Question 6

Using the definition of *tanh x* to explain why, for large positive values of *x*, the graph of $y = tanh\ x$ is effectively the same as $y = 1$

[2 marks]