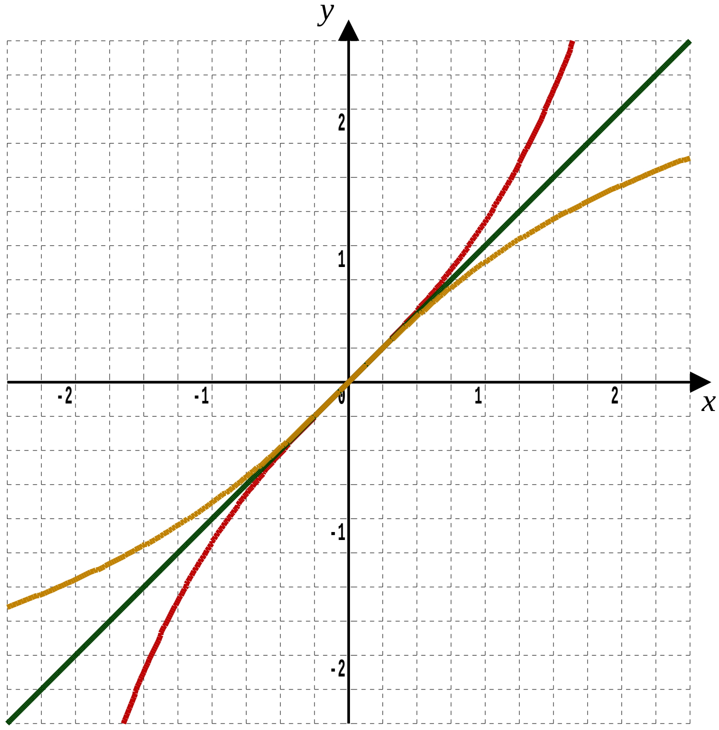
Lesson 3

3.1 *arsinh x*

Any one-one function, such as *sinh x*, has an inverse that, graphically, is a reflection in the line $y = x$. The inverse of *sinh x* is called *arsinh x*.



Red : $y = \sinh x$ Green : $y = x$ Gold : $y = \text{arsinh}\, x$

Given that *sinh x* is defined in terms of exponentials, the expectation would be that the inverse function, *arsinh x*, involves logarithms and this is the case.

The Inverse Of *sinh x* : *arsinh x*

$$\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

A proof of this is written out on the next page which the following eight minute excellent video from *Exam Solutions* will talk through.

Teaching Video: http://www.NumberWonder.co.uk/v9102/3.mp4

3.2 The Proof

$$y = arsinh\, x$$

$$\therefore x = sinh\, y$$

$$= \frac{e^y - e^{-y}}{2} \qquad \text{From the definition of } sinh$$

$$2x = e^y - e^{-y}$$

$$2x\, e^y = (e^y)^2 - 1 \qquad \text{From multiplying through by } e^y$$

$$(e^y)^2 - 2x(e^y) - 1 = 0 \qquad \text{Which is a “quadratic in disguise”}$$

$$e^y = \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= \frac{2x \pm \sqrt{4}\sqrt{x^2 + 1}}{2}$$

$$= x \pm \sqrt{x^2 + 1}$$

Now, $e^y > 0$ since $\sqrt{x^2 + 1} > x$

$$\therefore e^y = x + \sqrt{x^2 + 1}$$

$$y = ln\left(x + \sqrt{x^2 + 1}\right)$$

That is, $arsinh\, x = ln\left(x + \sqrt{x^2 + 1}\right)$ □

3.3 Example

Determine the exact value of

(i) $arsinh(2)$ **(ii)** $arsinh(-3)$

[2 marks]

Solution :

(i) $arsinh(2) = ln\left(2 + \sqrt{5}\right)$ About 1.44

(ii) $arsinh(-3) = ln\left(-3 + \sqrt{10}\right)$ About -1.81

This example is also covered in the teaching video.

3.4 Exercise

Any solution based entirely on graphical or numerical methods is not acceptable

Marks Available : 30

Question 1

Express each of the following as a natural logarithm,

(i) $arsinh(1)$

[1 mark]

(ii) $arsinh(\sqrt{3})$

[1 mark]

(iii) $arsinh(2\sqrt{2})$

[2 marks]

Question 2

Find the exact value of $arsinh\left(\frac{3}{4}\right)$ in as simple a form as possible

[2 marks]

Question 3

Find the exact value of $arsinh\left(\frac{1}{\sqrt{3}}\right)$ in as simple a form as possible

[2 marks]

Question 4

With the aid of the identity $cosh^2 x - sinh^2 x = 1$ solve the following equation, giving exact answers as natural logarithms.

$$2\,cosh^2 x - 5\,sinh\,x = 5$$

[6 marks]

Question 5

Further A-Level Examination Question from June 2015, FP3, Q1 (Edexcel)

Solve the equation

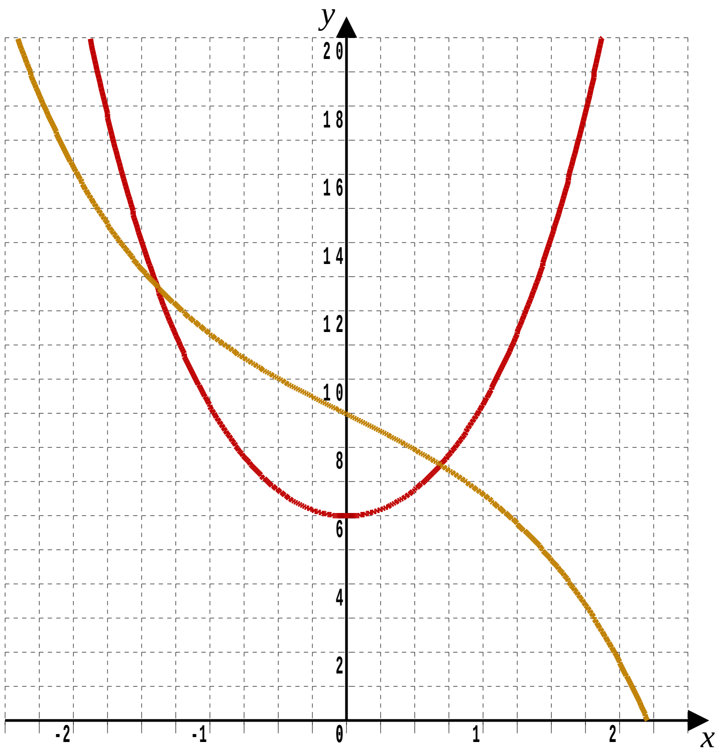
$$2\,cosh^2 x - 3\,sinh\,x = 1$$

giving your answers in terms of natural logarithms

[6 marks]

Question 6

Further A-Level Examination Question from June 2013, FP3(R), Q7(a) (Edexcel)



The curves have equations, $y = 6\cosh x$ and $y = 9 - 2\sinh x$

Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x, find exact values for the x-coordinates of the two points where the curves intersect.

[6 marks]

Question 7

Further A-Level Examination Question from June 2002, P6, Q1 (Edexcel)

Prove that $sinh\left(i\pi\ -\ \theta\right)\ =\ sinh\ \theta$

[4 marks]