## 4.1 Osborn's Rule

There are obvious similarities between identities involving the circular trigonometric functions (*cos, sin, tan, sec, csc* and *cot*) and the hyperbolic functions (*cosh, sinh, tanh, sech, csch, coth*).

Previously† the following where proven,

| Hyperbolic | Circular |
|---|---|
| $\cosh^2 x - \sinh^2 x = 1$ | $\cos^2 x + \sin^2 x = 1$ |
| $\cosh^2 x + \sinh^2 x = \cosh 2x$ | $\cos^2 x - \sin^2 x = \cos 2x$ |
| $2 \sinh x \cosh x = \sinh 2x$ | $2 \sin x \cos x = \sin 2x$ |
| $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$ | $\cos(A + B) = \cos A \cos B - \sin A \sin B$ |
| $\cosh 3A = 4\cosh^3 A - 3\cosh A$ | $\cos 3A = 4\cos^3 A - 3\cos A$ |
| $\sinh A - \sinh B = 2\sinh\left(\frac{A-B}{2}\right)\cosh\left(\frac{A+B}{2}\right)$ | $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$ |

Given that the circular trigonometric identities are already known, it would be ideal if the hyperbolic identities could be obtained from them. In some of the comparisons in the above table the hyperbolic identity is identical when *cos* is replaced with *cosh* and *sin* replaced with *sinh*. However, in other comparisons when the same swap is made, the odd sign needs changing. Exactly when a sign needs changing is given by Osborn's Rule. It allows any circular trigonometric identity to be taken and the corresponding hyperbolic identity easily written down.

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**Osborn's Rule**

- Replace *cos* by *cosh* : $\cos A \to \cosh A$
- Replace *sin* by *sinh* : $\sin A \to \sinh A$

  However ...
- Replace any product (or implied product) of two *sin* terms by minus the product of two *sinh* terms

  e.g. $\sin A \sin B \to -\sinh A \sinh B$

  $$\tan^2 A \to -\tanh^2 A \qquad \left(\text{an implied product as } \tan^2 A = \frac{\sin^2 A}{\cos^2 A}\right)$$

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† In Lesson 2, Example 2.2 and Exercise 2.3

## 4.2 Exercise

*Any solution based entirely on graphical or numerical methods is not acceptable*

Marks Available : 30

**Question 1**

**( i )** Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$1 + tan^2 x = sec^2 x$$

**[ 2 marks ]**

**( ii )** Prove that your part (i) answer is correct using the facts that,

$$tanh\,x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{and} \quad sech\,x = \frac{2}{e^x + e^{-x}}$$

**[ 6 marks ]**

**Question 2**

Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$cos\,2x = 1 - 2\,sin^2 x$$

**[ 2 marks ]**

**Question 3**

*Further A-Level Examination Question from June 2010, FP3, Q3 (Edexcel)*

**( a )** Starting from the definitions of *sinh x* and *cosh x* in terms of exponentials, prove that,

$$\cosh 2x = 1 + 2\sinh^2 x$$

**[ 3 marks ]**

**( b )** Solve the equation

$$\cosh 2x - 3\sinh x = 15$$

giving your answer as exact logarithms

**[ 5 marks ]**

**Question 4**

Use Osborn's Rule to write down the hyperbolic identity corresponding to,

$$tan(A - B) = \frac{tan A - tan B}{1 + tan A \, tan B}$$

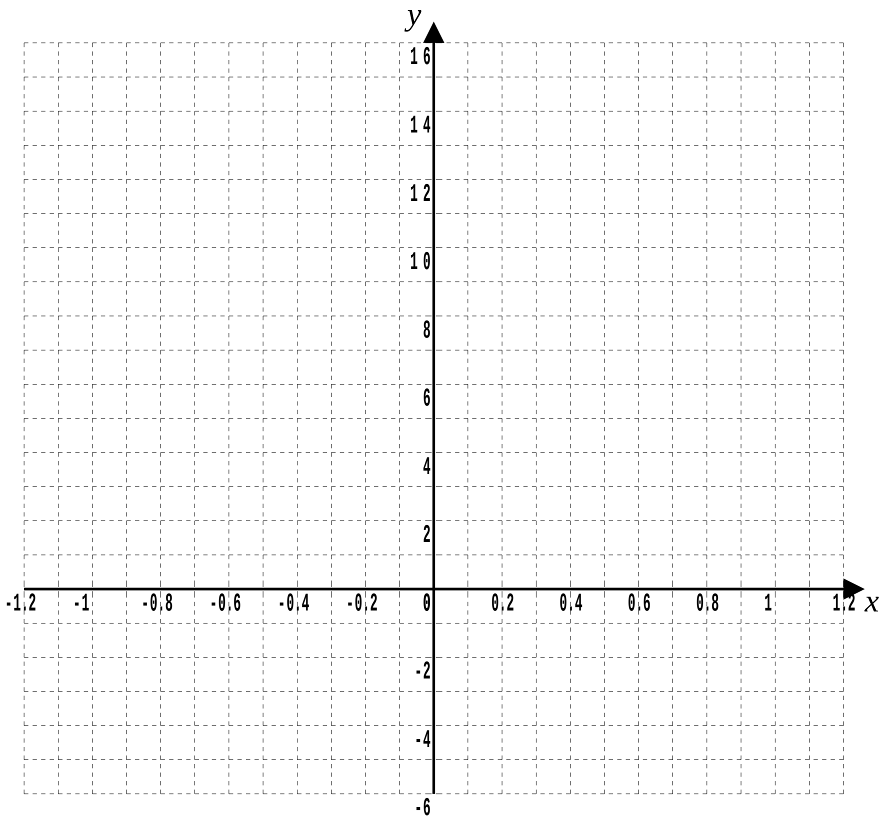
**[ 3 marks ]**

**Question 5**

*Further A-Level Examination Question from June 2011, FP3, Q5 (Edexcel)*

Curve $C_1$ has equation $y = 3 \, sinh \, 2x$, and curve $C_2$ has equation $y = 13 - 3e^{2x}$

**( a )** Sketch the graphs of the curves $C_1$ and $C_2$ on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes




**[ 4 marks ]**

**( b )** Solve the equation $3\,sinh\,2x = 13 - 3\,e^{2x}$ giving your answer in the form $\frac{1}{2}\,ln\,k$, where $k$ is an integer

**[ 5 marks ]**