9.1 Integration of Quadratic Denominators

The adsorption of the hyperbolic functions into the body of knowledge with which we are familiar has put us in a position to be able to integrate any function that is the reciprocal of a quadratic, even when that quadratic is square rooted, That is, functions of the forms,

$$f(x) = \frac{1}{a x^2 + bx + c}, \qquad g(x) = \frac{1}{\sqrt{ax^2 + bx + c}}$$

This has become about because the hyperbolics have allowed us to have the following “complete list” of standard integrations.

$f(x)$	$f'(x)$	In Formula Book ?
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}} \quad \lvert x \rvert < 1$	Yes
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}} \quad \lvert x \rvert < 1$	Yes
$\arctan x$	$\frac{1}{1+x^2}$	Yes
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$	Yes
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}} \quad x > 1$	Yes
$\operatorname{artanh} x$	$\frac{1}{1-x^2} \quad \lvert x \rvert < 1$	Yes

9.2 The Better Version “Completed List”

In fact, the examination formulae book provides an even better list of standard integrations that may be quoted without proof. Proving these results is reasonably straight forward, should you be asked to do so, using “integration by substitution” where the substitution involves a trigonometric function.

$f(x)$	$\int f(x)\,dx$ (Constant of integration omitted)	In Formula Book ?
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad \lvert x \rvert < a$	Yes
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$	Yes
$\frac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\frac{x}{a}\right),\ \ln\left\{x + \sqrt{x^2 - a^2}\right\} \quad x > a$	Yes
$\frac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\frac{x}{a}\right),\ \ln\left\{x + \sqrt{x^2 + a^2}\right\}$	Yes
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right) \quad \lvert x \rvert < a$	Yes
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	Yes

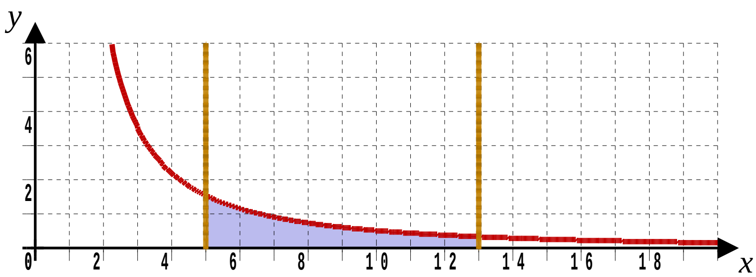
9.3 Example #1

Find $\displaystyle\int \frac{1}{\sqrt{2x^2 + 12x}}\,dx$

The strategy is to, in the denominator, pull out a 2, then complete the square, ahead of using one of the standard results from the previous table.

[4 marks]

9.4 Example #2



Find the area indicated by evaluating $\displaystyle\int_5^{13} \frac{100}{x^2 + 10x - 11}\,dx$

[6 marks]

9.5 Exercise

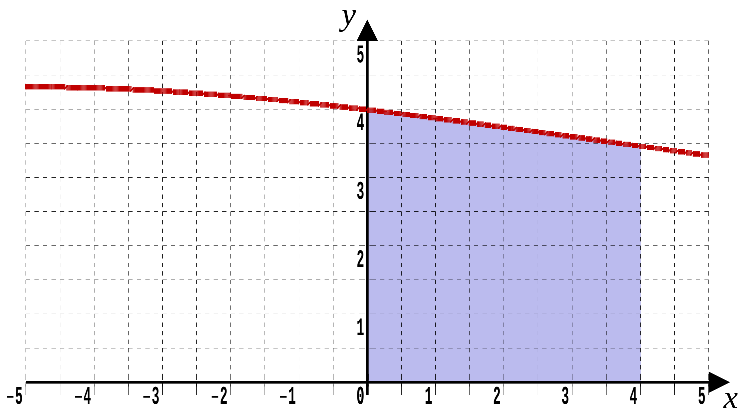
Any solution based entirely on graphical
or numerical methods is not acceptable
Marks Available : 50

Question 1

Find $\displaystyle\int \frac{1}{\sqrt{x^2 - 2x + 10}}\, dx$ giving an answer in terms of the *arsinh* function.

[2 marks]

Question 2



Find the area indicated by evaluating $\displaystyle\int_0^4 \frac{52}{\sqrt{x^2 + 10x + 169}}\, dx$

Give your answer in term of a natural logarithm.

[7 marks]

Question 3

Evaluate $\int_1^3 \frac{1}{\sqrt{x^2 - 2x + 2}}\, dx$, giving your answer as a single natural logarithm.

[5 marks]

Question 4

(i) Write $4x^2 - 12x - 7$ in the form $(ax - b)^2 - c^2$, $a, b, c \in \mathbb{Z}^+$

[3 marks]

(ii) Hence, or otherwise, find $\int \frac{1}{\sqrt{4x^2 - 12x - 7}}\, dx$

[4 marks]

Question 5

(i) Write $16x^2 - 40x + 9$ in the form $(ax - b)^2 - c^2, \quad a, b, c \in \mathbb{Z}^+$

[3 marks]

(ii) Hence, or otherwise, find the exact value of $\int_{-0.25}^{0} \frac{1}{16x^2 - 40x + 9}\, dx$

[5 marks]

Question 6

FM A-Level Examination Question from June 2011, Paper FP3, Q3 (Edexcel)

Show that,

(a) $\displaystyle\int_5^8 \frac{1}{x^2 - 10x + 34}\,dx = k\pi,$ giving the value of the fraction k

[5 marks]

(b) $\displaystyle\int_5^8 \frac{1}{\sqrt{x^2 - 10x + 34}}\,dx = ln\left(A + \sqrt{n}\right)$ giving the values of the integers A and n

[4 marks]

Question 7

FM A-Level Examination Question from June 2016, Paper FP3, Q4 (Edexcel)

(i) Find, without using a calculator, $\int_3^5 \frac{1}{\sqrt{15 + 2x - x^2}}\, dx$

giving your answer as a multiple of π

[5 marks]

(ii) **(a)** Show that $5\cosh x - 4\sinh x = \dfrac{e^{2x} + 9}{2e^x}$

[3 marks]

(b) Hence, using the substitution $u = e^x$ or otherwise, find

$$\int \frac{1}{5\cosh x - 4\sinh x}\, dx$$

[4 marks]